The q -log-Concavity of the q -Stirling Polynomials of the Second Kind

Miguel Angel Alvarez

Instituto Politécnico Nacional, ESIME, UPA
Av. de las Granjas 682, Col. Santa Catarina, Atzacapotzalco
México D.F., 02550 México

Jaime Navarro

Universidad Autónoma Metropolitana
Departamento de Ciencias Básicas
P. O. Box 16-306, México City, 02000 México
jnfu@correo.azc.uam.mx

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Abstract. The q -log-concavity of the q -Stirling polynomials $S_q[m+k, m]$ of the second kind is proved by using an explicit expression of the coefficients of such a polynomial, when $k = 1, 2$.

Mathematics Subject Classification: 35G05, 35G16, 35G35

Keywords: q -Stirling polynomial, q -log-concavity, 0 – 1 tableaux

1. INTRODUCTION

The q -log-concavity of the q -Stirling Polynomials of the second kind $S_q[m, k]$ was conjectured by M. Wachs and D. White in [6]. They used a computer program to prove this conjecture for $m \leq 20$. Also, in [3], the q -log-concavity of the q -Stirling numbers of the second kind was studied by using combinatorial interpretations in terms of “0 – 1 tableaux”.

The log-concave property has been also studied for F.Z. Zhao in [7]. In Theorem 2.5, Zhao proved that for a given sequence $a = (a_0, a_1, \dots, a_n, \dots)$ of real numbers, the generalized Stirling numbers $S_a(n, k)$ of the second kind are log-concave.

2. NOTATIONS AND DEFINITIONS

In this section, by following Pierre Lroux notations [3], see also [2] and [6], we define the q -Stirling polynomials of the second kind.

Definition 1. For a real variable $q \in \mathbb{R} \setminus \{1\}$ and m in $\mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, the q -Stirling polynomial in the variable q of the second kind is defined as

$$(1) \quad S_q[m, k] = S_q[m-1, k-1] + [k]_q S_q[m-1, k],$$

where

$$(2) \quad S_q[0, 0] = 1, \quad S_q[m, k] = 0 \quad \text{if } k \leq 0 \text{ or } k > m,$$

and where

$$(3) \quad [m]_q := 1 + q + q^2 + \dots + q^{m-1}, \quad \text{with } [0]_q = 0$$

Definition 2. The q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[k]_q! [n-k]_q!} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k < 0 \text{ or } k > n, \end{cases}$$

where

$$[m]_q! := [1]_q [2]_q \cdots [m]_q, \quad \text{with } [0]_q! = 1$$

In [1], L. M. Butler proved the q -log-concavity in k of the q -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$, and announced the conjecture that the q -Stirling polynomial of the second kind $S_q[m, k]$ is q -log-concave in k . Also, in [5], B. E. Sagan proved that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is strongly q -log concave. Later, many mathematicians settled this conjecture and related open questions concerning the q -Stirling numbers, see [6]. Different methods of proof were proposed, see for example [4].

3. PARTIAL RESULTS

In this paper we prove that the q -Stirling polynomial of the second kind are log-concave. For this purpose, we have some partial results. From Definition 1, we deduce the following **five** Claims, where the proofs of Claims 1, 2 and 4 are based on the mathematical induction method. The proof of Claim 3 comes from Claim 2, and the proof of Claim 5 comes from Claim 4.

Claim 1. For any m in $\mathbb{N}$, we have $\mathbf{S}_q[m, m] = 1$.

Claim 2. For any m in $\mathbb{N}$,

$$(4) \quad \mathbf{S}_q[m+1, m] = \sum_{i=1}^m [i]_q$$

Claim 3. For any $m \in \mathbb{N}$, the polynomial $\mathbf{S}_q[m+1, m]$ has degree $m-1$.

Claim 4. For any m in $\mathbb{N}$,

$$(5) \quad \mathbf{S}_q[m+2, m] = \sum_{i=1}^m [i]_q \mathbf{S}_q[i+1, i]$$

Claim 5. For any m in $\mathbb{N}$, the polynomial $\mathbf{S}_q[m+2, m]$ has degree $2m-2$.

Note 1. In this paper, we will consider only the q -Stirling polynomials of the second kind $S_q[m+k, m]$, where $k = 1, 2$.

Now, we give the following definitions:

Definition 2. A q -polynomial of degree t , $p(q) = \alpha_0 + \alpha_1 q + \alpha_2 q^2 + \cdots + \alpha_t q^t$ is said to be *log-concave*, see [3], if

$$(6) \quad \alpha_{i-1} \alpha_{i+1} \leq \alpha_i^2, \quad \text{for } 1 \leq i \leq t-1.$$

Note 2. The log-concavity property for sequences is found in combinatorial analysis and other branches of Mathematics. For example, in [4], B. E. Sagan proved the log-concavity in k for:

The binomial coefficients $\binom{m}{k}$, the Stirling numbers of the first kind $c(n, k)$, and the Stirling numbers of the second kind $S(n, k)$.

4. MAIN RESULTS

The main results of this paper are now given. That is, we will prove that the q -Stirling Polynomials of the second kind $\mathbf{S}_q[m+k, m]$ are *log-concave* for $k = 1, 2$.

Theorem 1. The q -Stirling Polynomial of the second kind $\mathbf{S}_q[m+1, m]$ is log-concave for any m in $\mathbb{N}$.

Proof. By Claim 3, since $\mathbf{S}_q[m+1, m] = \sum_{i=0}^{m-1} (m-i) q^i$, it follows that

$$\alpha_{i-1} \alpha_{i+1} = (m-(i-1))(m-(i+1)) = (m-i)^2 - 1, \quad \text{and} \quad \alpha_i^2 = (m-i)^2.$$

Hence $\alpha_{i-1} \alpha_{i+1} \leq \alpha_i^2$.

This proves Theorem 1. □

Theorem 2. *The q -Stirling Polynomial of the second kind*

$$S_q[m+2, m] = (\alpha_0 + \alpha_1 q) + (\alpha_2 q^2 + \cdots + \alpha_{m-1} q^{m-1}) + (\alpha_m q^m + \cdots + \alpha_{2m-2} q^{2m-2})$$

is log-concave for any m in $\mathbb{N}$.

Proof.

In order to prove the log-concavity, we will consider the following **six** cases:

- 1) $\alpha_0 \alpha_2 \leq \alpha_1^2$
- 2) $\alpha_1 \alpha_3 \leq \alpha_2^2$,
- 3) $\alpha_j \alpha_{j+2} \leq \alpha_{j+1}^2$, for j even and odd, where $2 \leq j \leq m-3$,
- 4) $\alpha_{m-2} \alpha_m \leq \alpha_{m-1}^2$, for m even and odd,
- 5) $\alpha_{m-1} \alpha_{m+1} \leq \alpha_m^2$, for m even and odd, and
- 6) $\alpha_j \alpha_{j+2} \leq \alpha_{j+1}^2$, for j even and odd, where $m \leq j \leq 2m-4$.

For this purpose, we will find an explicit formula for the coefficients α_i , with $i = 0, 2, \dots, 2m-2$, which is given in the following Lemma, and whose proof is given in the Appendix.

Lemma 1. *The q -Stirling polynomial $S_q[m+2, m]$ can be written as:*

$$S_q[m+2, m] = (\alpha_0 + \alpha_1 q) + (\alpha_2 q^2 + \cdots + \alpha_{m-1} q^{m-1}) + (\alpha_m q^m + \cdots + \alpha_{2m-2} q^{2m-2}),$$

where

a) For $j = 0, 1$,

$$\alpha_j = \frac{(j+1)}{2} m^2 + \frac{(1-j^2)}{2} m - \frac{j(j+1)}{2}$$

b) For $j = 2, 3, \dots, m-1$,

$$\alpha_j = \begin{cases} \frac{(j+1)}{2} m^2 + \frac{(1-j^2)}{2} m + \frac{j^3}{12} - \frac{3}{8} j^2 - \frac{7}{12} j & \text{if } j \text{ is even} \\ \frac{(j+1)}{2} m^2 + \frac{(1-j^2)}{2} m + \frac{j^3}{12} - \frac{3}{8} j^2 - \frac{7}{12} j - \frac{1}{8} & \text{if } j \text{ is odd} \end{cases}$$

c) For $j = m, m+1, \dots, 2m-2$,

$$\alpha_j = \begin{cases} \frac{(2m-j)^3}{12} + \frac{(2m-j)^2}{8} - \frac{(2m-j)}{12} & \text{if } j \text{ is even} \\ \frac{(2m-j)^3}{12} + \frac{(2m-j)^2}{8} - \frac{(2m-j)}{12} - \frac{1}{8} & \text{if } j \text{ is odd} \end{cases}$$

Back to the proof of Theorem 2, we will consider the following cases:

Case 1. From Lemma 1,

$$\begin{aligned} \alpha_1^2 - \alpha_0\alpha_2 &= (m^2 - 1)^2 \\ &- \left(\frac{m(m+1)}{2} \right) \left[\frac{8}{12} - \left(\frac{4m-3}{8} \right) 4 + \left(\frac{6m^2-7}{12} \right) 2 + \frac{m^2+m}{2} \right] \\ &= \frac{1}{4}m^4 - \frac{1}{4}m^2 + m + 1 = \frac{1}{4}(m+1)(m^3 - m^2 + 4) = \frac{1}{4}(m+1) [m^2(m-1) + 4]. \end{aligned}$$

Since $m \in \mathbb{N}$, it follows that

$$\boxed{\alpha_1^2 - \alpha_0\alpha_2 \geq 0.}$$

Case 2. From Lemma 1,

$$\begin{aligned} \alpha_2^2 - \alpha_1\alpha_3 &= \left[\frac{8}{12} - \left(\frac{4m+3}{8} \right) 4 + \left(\frac{6m^2-7}{12} \right) 2 + \frac{m^2+m}{2} \right]^2 \\ &- [m^2 - 1] \left[\frac{27}{12} - \left(\frac{4m+3}{8} \right) 9 + \left(\frac{6m^2-7}{12} \right) 3 + \frac{m^2+m}{2} - \frac{1}{8} \right] \\ &= \frac{1}{4}m^4 - \frac{1}{2}m^3 + \frac{5}{4}m^2 + 2m + 1 = \frac{1}{4}(m^4 - 2m^3 + 5m^2 + 8m + 4) \\ &= \frac{1}{4}[m^3(m-2) + 5m^2 + 8m + 4]. \end{aligned}$$

Note that if $m = 1$, $m^3(m-2) + 5m^2 + 8m + 4 = -1 + 5 + 8 + 4 = 16 > 0$.
Then for any $m \in \mathbb{N}$, we have

$$\boxed{\alpha_2^2 - \alpha_1\alpha_3 \geq 0.}$$

Case 3. Consider $2 \leq j \leq m-3$.

3a) Suppose that j is **even**. Then $j+1$ is odd and $j+2$ is even. Thus, from part b) of Lemma 1, if we arrange over the powers of j , the coefficient α_j are:

$$\alpha_j = \begin{cases} \frac{1}{12}j^3 - \left(\frac{4m+3}{8} \right) j^2 + \left(\frac{6m^2-7}{12} \right) j + \frac{m^2+m}{2} & \text{if } j \text{ is even} \\ \frac{1}{12}j^3 - \left(\frac{4m+3}{8} \right) j^2 + \left(\frac{6m^2-7}{12} \right) j + \frac{m^2+m}{2} - \frac{1}{8} & \text{if } j \text{ is odd} \end{cases}$$

Hence, by (7),

$$\begin{aligned}
& \alpha_{j+1}^2 - \alpha_j \alpha_{j+2} \\
&= \left[\frac{1}{12}(j+1)^3 - \left(\frac{4m+3}{8} \right) (j+1)^2 + \left(\frac{6m^2-7}{12} \right) (j+1) + \frac{m^2+m}{2} - \frac{1}{8} \right]^2 \\
&\quad - \left[\frac{1}{12}j^3 - \left(\frac{4m+3}{8} \right) j^2 + \left(\frac{6m^2-7}{12} \right) j + \frac{m^2+m}{2} \right] \\
&\quad \cdot \left[\frac{1}{12}(j+2)^3 - \left(\frac{4m+3}{8} \right) (j+2)^2 + \left(\frac{6m^2-7}{12} \right) (j+2) + \frac{m^2+m}{2} \right] \\
&= 12m^4 - 24m^3j + (24j^2 - 12 + 12j) m^2 + (18j^2 + 80j - 8j^3 + 48) m \\
&\quad + 48 - 3j^3 + 2j^2 + 48j + j^4 \\
&= (12m^4 - 24m^3j + 12m^2j^2) + (12m^2j^2 - 8mj^3) + (12m^2j - 12m^2) \\
&\quad + (18mj^2 - 3j^3) + (80mj + 48m + 48 + 2j^2 + 48j + j^4)
\end{aligned}$$

Now, since $2 \leq j \leq m-3 < m$, it follows that

$$\begin{aligned}
12m^4 - 24m^3j + 12m^2j^2 &= 12m^2(m^2 - 2mj + j^2) = 12m^2(m-j)^2 \geq 0, \\
12m^2j^2 - 8mj^3 &= mj^2(12m - 8j) = mj^2[8(m-j) + 4m] \geq 0, \\
12m^2j - 12m^2 &= 12m^2(j-1) \geq 0, \\
18mj^2 - 3j^3 &= 3j^2(6m-j) = 3j^2[(m-j) + 5m] \geq 0,
\end{aligned}$$

and since $80mj + 48m + 48 + 2j^2 + 48j + j^4 \geq 0$, it follows that

$$\boxed{\alpha_{j+1}^2 - \alpha_j \alpha_{j+2} \geq 0.}$$

3b). Suppose now that j is **odd**. Then $j+1$ is even and $j+2$ is odd. Thus, from (7),

$$\begin{aligned}
& \alpha_{j+1}^2 - \alpha_j \alpha_{j+2} \\
&= \left[\frac{1}{12}(j+1)^3 - \left(\frac{4m+3}{8} \right) (j+1)^2 + \left(\frac{6m^2-7}{12} \right) (j+1) + \frac{m^2+m}{2} \right]^2 \\
&\quad - \left[\frac{1}{12}j^3 - \left(\frac{4m+3}{8} \right) j^2 + \left(\frac{6m^2-7}{12} \right) j + \frac{m^2+m}{2} - \frac{1}{8} \right] \\
&\quad \cdot \left[\frac{1}{12}(j+2)^3 - \left(\frac{4m+3}{8} \right) (j+2)^2 + \left(\frac{6m^2-7}{12} \right) (j+2) + \frac{m^2+m}{2} - \frac{1}{8} \right] \\
&= 12m^4 - 24m^3j + (24j^2 + 12 + 24j) m^2 + (6j^2 + 56j - 8j^3 + 42) m \\
&\quad + 24 - j^3 - j^2 + 25j + j^4 \\
&= (12m^4 - 24m^3j + 12m^2j^2) + (12m^2j^2 - 8mj^3) + (6mj^2 - j^3 - j^2) \\
&\quad + (12m^2 + 24m^2j^2 + 56mj + 42m + 24 + 25j + j^4)
\end{aligned}$$

Again, since $2 \leq j \leq m-3 < m$, it follows that

$$12m^4 - 24m^3j + 12m^2j^2 = 12m^2(m^2 - 2mj + j^2) = 12m^2(m - j)^2 \geq 0,$$

$$12m^2j^2 - 8mj^3 = mj^2(12m - 8j) = mj^2[8(m - j) + 4m] \geq 0,$$

$$6mj^2 - j^3 - j^2 = j^2(6m - j - 1) = j^2[(m - j) + (m - 1) + 4m] \geq 0,$$

and since $12m^2 + 24m^2j^2 + 56mj + 42m + 24 + 25j + j^4 \geq 0$, it follows that

$$\boxed{\alpha_{j+1}^2 - \alpha_j\alpha_{j+2} \geq 0.}$$

Case 4.

4a). Suppose that m is **even**. The $m - 2$ is even and $m - 1$ is odd. Then from Lemma 1,

$$\begin{aligned} & \alpha_{m-1}^2 - \alpha_{m-2}\alpha_m \\ &= \left[\frac{1}{12}(m-1)^3 - \left(\frac{4m+3}{8} \right) (m-1)^2 + \left(\frac{6m^2-7}{12} \right) (m-11) + \frac{m^2+m}{2} - \frac{1}{8} \right]^2 \\ & - \left[\frac{1}{12}(m-2)^3 - \left(\frac{4m+3}{8} \right) (m-2)^2 + \left(\frac{6m^2-7}{12} \right) (m-2) + \frac{m^2+m}{2} \right] \\ & \cdot \left[\frac{1}{12}m^3 + \frac{1}{8}m^2 - \frac{1}{12}m \right] = \frac{5}{48}m^4 + \frac{19}{48}m^3 + \frac{1}{3}m^2 - \frac{1}{12}m \\ &= \frac{1}{48}m(5m-1)(m+2)^2 \geq 0, \quad \text{for any } m \text{ in } \mathbb{N}. \end{aligned}$$

4b). Suppose that m is **odd**. Then $m - 2$ is odd and $m - 1$ is even. Again, from Lemma 1,

$$\begin{aligned} & \alpha_{m-1}^2 - \alpha_{m-2}\alpha_m \\ &= \left[\frac{1}{12}(m-1)^3 - \left(\frac{4m+3}{8} \right) (m-1)^2 + \left(\frac{6m^2-7}{12} \right) (m-11) + \frac{m^2+m}{2} \right]^2 \\ & - \left[\frac{1}{12}(m-2)^3 - \left(\frac{4m+3}{8} \right) (m-2)^2 + \left(\frac{6m^2-7}{12} \right) (m-2) + \frac{m^2+m}{2} - \frac{1}{8} \right] \\ & \cdot \left[\frac{1}{12}m^3 + \frac{1}{8}m^2 - \frac{1}{12}m - \frac{1}{8} \right] = \frac{5}{48}m^4 + \frac{7}{16}m^3 + \frac{25}{48}m^2 + \frac{1}{1m}m - \frac{1}{18} \\ &= \frac{1}{48}(2+m)(m+1)(5m^2+6m-3) \geq 0, \quad \text{for any } m \text{ in } \mathbb{N}. \end{aligned}$$

Thus, from 4a) and 4b) we have

$$\boxed{\alpha_{m-1}^2 - \alpha_{m-2}\alpha_m \geq 0.}$$

Case 5.

5a). Suppose that m is **even**. Then $m - 1$ and $m + 1$ are odd. Thus from Lemma 1,

$$\begin{aligned} \alpha_m^2 - \alpha_{m-1}\alpha_{m+1} &= \left[\frac{1}{12}m^3 + \frac{1}{8}m^2 - \frac{1}{12}m \right]^2 \\ &- \left[\frac{1}{12}(m-1)^3 - \left(\frac{4m+3}{8} \right) (m-1)^2 + \left(\frac{6m^2-7}{12} \right) (m-1) + \frac{m^2+m}{2} - \frac{1}{8} \right] \\ &\cdot \left[\frac{1}{12}(m-1)^3 + \frac{1}{8}(m-1)^2 - \frac{1}{12}(m-1) - \frac{1}{8} \right] = \frac{1}{48}m^4 + \frac{1}{16}m^3 - \frac{1}{24}m^2 \\ &= \frac{1}{48}m^2(2+m)(m+1) \geq 0, \quad \text{for any } m \text{ in } \mathbb{N}. \end{aligned}$$

5b). Suppose that m is **odd**. Then $m - 1$ and $m + 1$ are even. Thus from Lemma 1,

$$\begin{aligned} \alpha_m^2 - \alpha_{m-1}\alpha_{m+1} &= \left[\frac{1}{12}m^3 + \frac{1}{8}m^2 - \frac{1}{12}m - \frac{1}{8} \right]^2 \\ &- \left[\frac{1}{12}(m-1)^3 - \left(\frac{4m+3}{8} \right) (m-1)^2 + \left(\frac{6m^2-7}{12} \right) (m-1) + \frac{m^2+m}{2} \right] \\ &\cdot \left[\frac{1}{12}(m-1)^3 + \frac{1}{8}(m-1)^2 - \frac{1}{12}(m-1) \right] = \frac{1}{48}m^4 + \frac{1}{48}m^3 - \frac{1}{48}m^2 - \frac{1}{48}m \\ &= \frac{1}{48}m(m-1)(m+1)^2 \geq 0, \quad \text{for any } m \text{ in } \mathbb{N}. \end{aligned}$$

Thus, from 5a) and 5b) we have

$$\boxed{\alpha_m^2 - \alpha_{m-1}\alpha_{m+1} \geq 0.}$$

Case 6. Consider $m \leq j \leq 2m - 4$.

6a). Suppose that j is **even**. Then from Lemma 1,

$$\begin{aligned} \alpha_{j+1}^2 - \alpha_j\alpha_{j+2} &= \left[\frac{(2m-(j+1))^3}{12} + \frac{(2m-(j+1))^2}{8} - \frac{(2m-(j+1))}{12} - \frac{1}{8} \right]^2 \\ &- \left[\frac{(2m-j)^3}{12} + \frac{(2m-j)^2}{8} - \frac{(2m-j)}{12} \right] \\ &\cdot \left[\frac{(2m-(j+2))^3}{12} + \frac{(2m-(j+2))^2}{8} - \frac{(2m-(j+2))}{12} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3}m^4 + \left(-\frac{2}{3}j - \frac{1}{2}\right)m^3 + \left(\frac{1}{2}j^2 + \frac{3}{4}j + \frac{1}{6}\right)m^2 \\
&+ \left(-\frac{1}{6}j^3 - \frac{3}{8}j^2 - \frac{1}{6}j\right)m + \left(\frac{1}{16}j^3 + \frac{1}{48}j^4 + \frac{1}{24}j^2\right) \\
&= \frac{1}{48}(2m-j-1)(2m-j-2)(2m-j)^2.
\end{aligned}$$

Now, since $0 < m \leq j \leq 2m-4 \leq 2m-2$, it follows that $0 \leq 2m-j-2 \leq 2m-j-1$. Then $(2m-j-1)(2m-j-2) \geq 0$. Therefore,

$$\boxed{\alpha_{j+1}^2 - \alpha_j \alpha_{j+2} \geq 0.}$$

6b). Suppose now that j is odd. Then $j+1$ is even and $j+2$ is odd. Then by Lemma 1,

$$\begin{aligned}
&\alpha_{j+1}^2 - \alpha_j \alpha_{j+2} \\
&= \left[\frac{(2m-(j+1))^3}{12} + \frac{(2m-(j+1))^2}{8} - \frac{(2m-(j+1))}{12} \right]^2 \\
&- \left[\frac{(2m-j)^3}{12} + \frac{(2m-j)^2}{8} - \frac{(2m-j)}{12} - \frac{1}{8} \right] \\
&\cdot \left[\frac{(2m-(j+2))^3}{12} + \frac{(2m-(j+2))^2}{8} - \frac{(2m-(j+2))}{12} - \frac{1}{8} \right] \\
&= \frac{1}{48}(2m-j+1)(2m-j)(2m-j-1)^2.
\end{aligned}$$

Since $0 < m \leq j \leq 2m-4 \leq 2m-2 < 2m$, it follows that $0 < 2m-j < 2m-j+1$. Then $(2m-j+1)(2m-j) \geq 0$. Therefore

$$\boxed{\alpha_{j+1}^2 - \alpha_j \alpha_{j+2} \geq 0.}$$

This completes the proof of Theorem 2. □

5. APPENDIX

In order to prove Lemma 1, we need the following **eight** Lemmas.

Lemma 2. For any $m \in \mathbb{N}$,

$$[m]_q S_q[m+1, m] = \sum_{j=0}^{m-1} \left(\sum_{k=0}^j (m-k) \right) q^j + \sum_{j=m}^{2m-2} \left(\sum_{k=j-m+1}^{m-1} (m-k) \right) q^j$$

Proof. By Claim 3, $[m]_q S_q[m+1, m]$ is a polynomial of degree $2m-2$, and

$$\begin{aligned}
[m]_q S_q[m+1, m] &= [m]_q \sum_{i=0}^{m-1} (m-i)q^i \\
&= (1+q+q^2+\cdots+q^{m-1})(m+(m-1)q+(m-2)q^2+\cdots+q^{m-1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
[m]_q S_q[m+1, m] &= (m+(m-1)q+(m-2)q^2+\cdots+q^{m-1}) \\
&\quad + (mq+(m-1)q^2+(m-2)q^3+\cdots+2q^{m-1}+q^m) \\
&\quad + \cdots + (mq^{m-1}+(m-1)q^m+(m-2)q^{m+1}+\cdots+q^{2m-2}).
\end{aligned}$$

By factoring the powers of q ,

$$\begin{aligned}
[m]_q S_q[m+1, m] &= m+(m+(m-1))q+(m+(m-1)+(m-2))q^2 \\
&\quad + \cdots + (m+(m-1)+(m-2)+\cdots+2+1)q^{m-1} \\
&\quad + (1+2+\cdots+(m-1))q^m+(1+2+\cdots+(m-2))q^{m+1} \\
&\quad + \cdots + (1+2)q^{(2m-2)-1}+1q^{2m-2} \\
&= \sum_{j=0}^{m-1} \left(\sum_{k=0}^j (m-k) \right) q^j + \sum_{j=m}^{2m-2} \left(\sum_{k=j-m+1}^{m-1} (m-k) \right) q^j
\end{aligned}$$

This proves Lemma 2. □

Lemma 3. For any $m \in \mathbb{N}$,

$$S_q[m+2, m] = P + Q,$$

where

$$P = \sum_{i=0}^{m-1} \left(\sum_{j=i}^{m-1} \sum_{k=0}^i (j+1-k) \right) q^i,$$

and

$$Q = \sum_{i=2}^m \left(\sum_{j=0}^{l-1} \sum_{k=1+j}^{i-1-j} (i-j-k) \right) q^i + \sum_{i=m+1}^{2m-2} \left(\sum_{j=i-m}^{l-1} \sum_{k=1+j}^{i-1-j} (i-j-k) \right) q^i,$$

with

$$l = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even} \\ \frac{i-1}{2} & \text{if } i \text{ is odd} \end{cases}$$

Proof. Note that from Claim 4,

$$S_q[m+2, m] = \sum_{i=1}^m [i]_q S_q[i+1, i],$$

and from Lemma 2,

$$S_q[m+2, m] = \sum_{i=1}^m \left[\left(\sum_{j=0}^{i-1} \binom{j}{k=0} (i-k) \right) q^j + \sum_{j=i}^{2i-2} \left(\sum_{k=j-i+1}^{i-1} (i-k) \right) q^j \right].$$

Then by taking the sum over the index i , we get:

$$\begin{aligned} S_q[m+2, m] &= \sum_{j=0}^0 \binom{j}{k=0} (1-k) q^j + \sum_{j=1}^0 \binom{0}{k=j} (1-k) q^j \\ &\quad + \sum_{j=0}^1 \binom{j}{k=0} (2-k) q^j + \sum_{j=2}^2 \binom{1}{k=j-1} (2-k) q^j \\ &\quad + \sum_{j=0}^2 \binom{j}{k=0} (3-k) q^j + \sum_{j=3}^4 \binom{2}{k=j-2} (3-k) q^j \\ &\quad \vdots \\ &\quad + \sum_{j=0}^{m-3} \binom{j}{k=0} (m-2-k) q^j + \sum_{j=m-2}^{2(m-2)-2} \binom{m-2-1}{k=j-(m-2)+1} (m-2-k) q^j \\ &\quad + \sum_{j=0}^{m-2} \binom{j}{k=0} (m-1-k) q^j + \sum_{j=m-1}^{2(m-1)-2} \binom{m-1-1}{k=j-(m-1)+1} (m-1-k) q^j \\ &\quad + \sum_{j=0}^{m-1} \binom{j}{k=0} (m-k) q^j + \sum_{j=m}^{2m-2} \binom{m-1}{k=j-m+1} (m-k) q^j \\ &= P + Q, \end{aligned}$$

where P is the first column of this expression, and Q is the second one. That is,

$$\begin{aligned} P &= \sum_{j=0}^0 \binom{j}{k=0} (1-k) q^j + \sum_{j=0}^1 \binom{j}{k=0} (2-k) q^j + \sum_{j=0}^2 \binom{j}{k=0} (3-k) q^j \\ &\quad + \cdots + \sum_{j=0}^{m-3} \binom{j}{k=0} (m-2-k) q^j + \sum_{j=0}^{m-2} \binom{j}{k=0} (m-1-k) q^j \\ &\quad + \sum_{j=0}^{m-1} \binom{j}{k=0} (m-k) q^j, \end{aligned}$$

and

$$\begin{aligned}
 Q = & \sum_{j=1}^0 \left(\sum_{k=j}^0 (1-k) \right) q^j + \sum_{j=2}^2 \left(\sum_{k=j-1}^1 (2-k) \right) q^j + \sum_{j=3}^4 \left(\sum_{k=j-2}^2 (3-k) \right) q^j \\
 & + \cdots + \sum_{j=m-2}^{2(m-2)-2} \left(\sum_{k=j-(m-2)+1}^{m-2-1} (m-2-k) \right) q^j \\
 & + \sum_{j=m-1}^{2(m-1)-2} \left(\sum_{k=j-(m-1)+1}^{m-1-1} (m-1-k) \right) q^j + \sum_{j=m}^{2m-2} \left(\sum_{k=j-m+1}^{m-1} (m-k) \right) q^j.
 \end{aligned}$$

Now, by considering the sums over the index j , we get for P :

$$\begin{aligned}
 P = & \left(\sum_{k=0}^0 (1-k) q^0 \right) + \left(\sum_{k=0}^0 (2-k) q^0 + \sum_{k=0}^1 (2-k) q^1 \right) \\
 & + \cdots + \left(\sum_{k=0}^0 (m-k) q^0 + \cdots + \sum_{k=0}^{m-2} (m-k) q^{m-2} + \sum_{k=0}^{m-1} (m-k) q^{m-1} \right).
 \end{aligned}$$

By factoring the powers of q ,

$$\begin{aligned}
 P = & \left(\sum_{k=0}^0 (1-k) + \cdots + \sum_{k=0}^0 (m-1-k) + \sum_{k=0}^0 (m-k) \right) q^0 \\
 & + \left(\sum_{k=0}^1 (2-k) + \cdots + \sum_{k=0}^1 (m-1-k) + \sum_{k=0}^1 (m-k) \right) q^1 \\
 & + \left(\sum_{k=0}^2 (3-k) + \cdots + \sum_{k=0}^2 (m-1-k) + \sum_{k=0}^2 (m-k) \right) q^2 \\
 & + \cdots + \left(\sum_{k=0}^{m-3} (m-2-k) + \sum_{k=0}^{m-3} (m-1-k) + \sum_{k=0}^{m-3} (m-k) \right) q^{m-3} \\
 & + \left(\sum_{k=0}^{m-2} (m-1-k) + \sum_{k=0}^{m-2} (m-k) \right) q^{m-2} + \left(\sum_{k=0}^{m-1} (m-k) \right) q^{m-1}.
 \end{aligned}$$

Then,

$$\begin{aligned} P = & \left[\sum_{j=0}^{m-1} \left(\sum_{k=0}^0 (j+1-k) \right) \right] q^0 + \left[\sum_{j=1}^{m-1} \left(\sum_{k=0}^1 (j+1-k) \right) \right] q^1 \\ & + \left[\sum_{j=2}^{m-1} \left(\sum_{k=0}^2 (j+1-k) \right) \right] q^2 + \cdots + \left[\sum_{j=m-2}^{m-1} \left(\sum_{k=0}^{m-2} (j+1-k) \right) \right] q^{m-2} \\ & + \left[\sum_{j=m-1}^{m-1} \left(\sum_{k=0}^{m-1} (j+1-k) \right) \right] q^{m-1}. \end{aligned}$$

Hence,

$$P = \sum_{i=0}^{m-1} \left(\sum_{j=i}^{m-1} \sum_{k=0}^i (j+1-k) \right) q^i.$$

On the other hand, by considering the sums over the index j , we get for Q :

$$\begin{aligned} Q = & (0q^1) + \left(\sum_{k=1}^1 (2-k)q^2 \right) + \left(\sum_{k=1}^2 (3-k)q^3 + \sum_{k=2}^2 (3-k)q^4 \right) \\ & + \left(\sum_{k=1}^3 (4-k)q^4 + \sum_{k=2}^3 (4-k)q^5 + \sum_{k=3}^3 (4-k)q^6 \right) \\ & \vdots \\ & + \left(\sum_{k=1}^{m-3} (m-2-k)q^{m-2} + \cdots + \sum_{k=m-3}^{m-3} (m-2-k)q^{2m-6} \right) \\ & + \left(\sum_{k=1}^{m-2} (m-1-k)q^{m-1} + \cdots + \sum_{k=m-2}^{m-2} (m-1-k)q^{2m-4} \right) \\ & + \left(\sum_{k=1}^{m-1} (m-k)q^m + \sum_{k=2}^{m-1} (m-k)q^{m+1} + \cdots + \sum_{k=m-1}^{m-1} (m-k)q^{2m-2} \right). \end{aligned}$$

Then by factoring, Q can be written as:

$$\begin{aligned}
Q &= \left[\sum_{k=1}^1 (2-k) \right] q^2 + \left[\sum_{k=1}^2 (3-k) \right] q^3 \\
&+ \left[\sum_{k=2}^2 (3-k) + \sum_{k=1}^3 (4-k) \right] q^4 + \left[\sum_{k=2}^3 (4-k) + \sum_{k=1}^4 (5-k) \right] q^5 \\
&+ \left[\sum_{k=3}^3 (4-k) + \sum_{k=2}^4 (5-k) + \sum_{k=1}^5 (6-k) \right] q^6 \\
&+ \left[\sum_{k=3}^4 (5-k) + \sum_{k=2}^5 (6-k) + \sum_{k=1}^6 (7-k) \right] q^7 \\
&\vdots \\
&+ \begin{cases} [X]q^m & \text{if } m \text{ is even, or} \\ [Y]q^m & \text{if } m \text{ is odd} \end{cases} + \begin{cases} [A]q^{m+1} & \text{if } m+1 \text{ is even, or} \\ [B]q^{m+1} & \text{if } m+1 \text{ is odd} \end{cases} \\
&\vdots \\
&+ \left[\sum_{k=m-2}^{m-1} (m-k) \right] q^{2m-3} + \left[\sum_{k=m-1}^{m-1} (m-k) \right] q^{2m-2},
\end{aligned}$$

where

$$\begin{aligned}
X &= \sum_{k=\frac{m}{2}}^{\frac{m}{2}} \left(\frac{m}{2} + 1 - k \right) + \sum_{k=\frac{m}{2}-1}^{\frac{m}{2}+1} \left(\frac{m}{2} + 2 - k \right) + \cdots + \sum_{k=1}^{m-1} (m-k), \\
Y &= \sum_{k=\frac{m-1}{2}}^{\frac{m-1}{2}+1} \left(\frac{m+1}{2} + 1 - k \right) + \sum_{k=\frac{m-1}{2}-1}^{\frac{m-1}{2}+2} \left(\frac{m+1}{2} + 2 - k \right) \cdots + \sum_{k=1}^{m-1} (m-k), \\
A &= \sum_{k=1+\frac{m+1}{2}-1}^{(m+1)-1-(\frac{m+1}{2}-1)} \left(m+1 - \left(\frac{m+1}{2} - 1 \right) - k \right) \\
&+ \sum_{k=1+\frac{m+1}{2}}^{(m+1)-1-(\frac{m+1}{2})} \left(m+1 - \left(\frac{m+1}{2} \right) - k \right) \\
&+ \cdots + \sum_{k=3}^{(m+1)-1-2} (m+1-2-k) + \sum_{k=2}^{(m+1)-1-1} (m+1-1-k),
\end{aligned}$$

and

$$\begin{aligned}
B = & \sum_{k=1+\frac{m}{2}-1}^{(m+1)-1-(\frac{m}{2}-1)} \left(m+1-\left(\frac{m}{2}-1\right)-k\right) + \sum_{k=1+\frac{m}{2}}^{(m+1)-1-(\frac{m}{2})} \left(m+1-\left(\frac{m}{2}\right)-k\right) \\
& + \cdots + \sum_{k=3}^{(m+1)-1-2} (m+1-2-k) + \sum_{k=2}^{(m+1)-1-1} (m+1-1-k).
\end{aligned}$$

Hence,

$$\begin{aligned}
Q = & \left[\sum_{k=1}^1 (2-k) \right] q^2 + \left[\sum_{k=1}^2 (3-k) \right] q^3 \\
& + \left[\sum_{j=0}^{\frac{4}{2}-1} \left(\sum_{k=1+j}^{4-1-j} (4-j-k) \right) \right] q^4 + \left[\sum_{j=0}^{\frac{5-1}{2}-1} \left(\sum_{k=1+j}^{5-1-j} (5-j-k) \right) \right] q^5 \\
& + \left[\sum_{j=0}^{\frac{6}{2}-1} \left(\sum_{k=1+j}^{6-1-j} (6-j-k) \right) \right] q^6 + \left[\sum_{j=0}^{\frac{7-1}{2}-1} \left(\sum_{k=1+j}^{7-1-j} (7-j-k) \right) \right] q^7 \\
& \vdots \\
& + \left[\sum_{j=0}^{\frac{m}{2}-1} \left(\sum_{k=1+j}^{m-1-j} (m-j-k) \right) \right] q^m \quad \text{if } m \text{ is even, or} \\
& \left[\sum_{j=0}^{\frac{m-1}{2}-1} \left(\sum_{k=1+j}^{m-1-j} (m-j-k) \right) \right] q^m \quad \text{if } m \text{ is odd} \\
& + \left[\sum_{j=1}^{l-1} \left(\sum_{k=1+j}^{(m+1)-1-j} (m+1-j-k) \right) \right] q^{m+1}, \quad l = \begin{cases} \frac{(m+1)-1}{2}, & m+1 \text{ is odd} \\ \frac{m+1}{2}, & m+1 \text{ is even} \end{cases} \\
& + \left[\sum_{j=2}^{l-1} \left(\sum_{k=1+j}^{(m+2)-1-j} (m+2-j-k) \right) \right] q^{m+2}, \quad l = \begin{cases} \frac{(m+2)-1}{2}, & m+2 \text{ is odd} \\ \frac{m+2}{2}, & m+2 \text{ is even} \end{cases} \\
& \vdots \\
& + \left[\sum_{j=m-3}^{\frac{(2m-3)-1}{2}-1} \left(\sum_{k=1+j}^{(2m-3)-1-j} (2m-3-j-k) \right) \right] q^{2m-3} \\
& + \left[\sum_{j=m-2}^{\frac{(2m-2)-1}{2}-1} \left(\sum_{k=1+j}^{(2m-2)-1-j} (2m-2-j-k) \right) \right] q^{2m-2}.
\end{aligned}$$

That is,

$$Q = \sum_{i=2}^m \left(\sum_{j=0}^{l-1} \sum_{k=1+j}^{i-1-j} (i-j-k) \right) q^i + \sum_{i=m+1}^{2m-2} \left(\sum_{j=i-m}^{l-1} \sum_{k=1+j}^{i-1-j} (i-j-k) \right) q^i,$$

with

$$l = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even} \\ \frac{i-1}{2} & \text{if } i \text{ is odd} \end{cases}$$

This proves Lemma 3. □

Lemma 4. *The coefficients of the polynomial*

$$S_q[m+2, m] = \sum_{j=0}^{2m-2} \alpha_j q^j$$

are given by:

a)

$$\alpha_j = \sum_{i=j+1}^m \left((j+1)i - \frac{j(j+1)}{2} \right), \quad \text{for } j = 0, 1$$

b)

$$\alpha_j = \sum_{i=j+1}^m \left((j+1)i - \frac{j(j+1)}{2} \right) + \begin{cases} C & \text{if } j \text{ is even} \\ D & \text{if } j \text{ is odd} \end{cases}$$

for $2 \leq j \leq m-1$, where

$$C = \sum_{k=1}^{j-1} (j-k) + \sum_{k=2}^{j-2} (j-1-k) + \cdots + \sum_{k=\frac{j}{2}}^{\frac{j}{2}} \left(\frac{j}{2} + 1 - k \right),$$

and

$$D = \sum_{k=1}^{j-1} (j-k) + \sum_{k=2}^{j-2} (j-1-k) + \cdots + \sum_{k=\frac{j-1}{2}}^{\frac{j-1}{2}+1} \left(\frac{j-1}{2} + 2 - k \right)$$

c)

$$\alpha_j = \begin{cases} E & \text{if } j \text{ is even} \\ F & \text{if } j \text{ is odd} \end{cases}$$

for $m \leq j \leq 2m-2$, where

$$E = \sum_{k=\frac{j}{2}}^{\frac{j}{2}} \left(\frac{j}{2} + 1 - k \right) + \sum_{k=\frac{j}{2}-1}^{\frac{j}{2}+1} \left(\frac{j}{2} + 2 - k \right) + \cdots + \sum_{k=j+1-m}^{m-1} (m-k),$$

and

$$F = \sum_{k=\frac{j+1}{2}-1}^{\frac{j+1}{2}} \left(\frac{j+1}{2} + 1 - k\right) + \sum_{k=\frac{j+1}{2}-2}^{\frac{j+1}{2}+1} \left(\frac{j+1}{2} + 2 - k\right) + \cdots + \sum_{k=j+1-m}^{m-1} (m - k)$$

Proof. From Lemma 3,

$$\begin{aligned} S_q[m+2, m] &= \left[\sum_{j=0}^{m-1} \sum_{k=0}^0 (j+1-k) \right] q^0 + \left[\sum_{j=1}^{m-1} \sum_{k=0}^1 (j+1-k) \right] q^1 \\ &+ \left[\sum_{j=2}^{m-1} \sum_{k=0}^2 (j+1-k) + \sum_{j=0}^0 \sum_{k=1+j}^{1-j} (2-j-k) \right] q^2 \\ &+ \left[\sum_{j=3}^{m-1} \sum_{k=0}^3 (j+1-k) + \sum_{j=0}^0 \sum_{k=1+j}^{2-j} (3-j-k) \right] q^3 \\ &+ \cdots + [G]q^{m-1}, \text{ where } l = \begin{cases} \frac{m-1}{2}, & \text{if } m-1 \text{ is even} \\ \frac{(m-1)-1}{2}, & \text{if } m-1 \text{ is odd} \end{cases} \\ &+ [H]q^m, \text{ where } l = \begin{cases} \frac{m}{2}, & \text{if } m \text{ is even} \\ \frac{m-1}{2}, & \text{if } m \text{ is odd} \end{cases} \\ &+ [I]q^{m+1}, \text{ where } l = \begin{cases} \frac{m+1}{2}, & \text{if } m+1 \text{ is even} \\ \frac{(m+1)-1}{2}, & \text{if } m+1 \text{ is odd} \end{cases} \\ &+ [J]q^{m+2}, \text{ where } l = \begin{cases} \frac{m+2}{2}, & \text{if } m+2 \text{ is even} \\ \frac{(m+2)-1}{2}, & \text{if } m+2 \text{ is odd} \end{cases} \\ &+ \cdots + \left[\sum_{j=m-2}^{m-2} \sum_{k=1+j}^{2m-3-j} (2m-2-j-k) \right] q^{2m-2}, \end{aligned}$$

where

$$\begin{aligned} G &= \sum_{j=m-1}^{m-1} \sum_{k=0}^{m-1} (j+1-k) + \sum_{j=0}^{l-1} \sum_{k=1+j}^{m-2-j} (m-1-j-k), \\ H &= \sum_{j=0}^{l-1} \sum_{k=1+j}^{m-1-j} (m-j-k), \\ I &= \sum_{j=1}^{l-1} \sum_{k=1+j}^{m-j} (m+1-j-k), \quad \text{and} \\ J &= \sum_{j=2}^{l-1} \sum_{k=1+j}^{m+1-j} (m+2-j-k). \end{aligned}$$

Now, by taking the sum over the index j , we get

$$\begin{aligned}
S_q[m+2, m] &= \left[\sum_{k=0}^0 (1-k) + \sum_{k=0}^0 (2-k) + \cdots + \sum_{k=0}^0 (m-k) \right] q^0 \\
&+ \left[\sum_{k=0}^1 (2-k) + \sum_{k=0}^1 (3-k) + \cdots + \sum_{k=0}^1 (m-k) \right] q^1 \\
&+ \left[\left(\sum_{k=0}^2 (3-k) + \sum_{k=0}^2 (4-k) + \cdots + \sum_{k=0}^2 (m-k) \right) + \left(\sum_{k=1}^1 (2-k) \right) \right] q^2 \\
&+ \left[\left(\sum_{k=0}^3 (4-k) + \sum_{k=0}^3 (5-k) + \cdots + \sum_{k=0}^3 (m-k) \right) + \left(\sum_{k=1}^2 (3-k) \right) \right] q^3 \\
&+ \left[\left(\sum_{k=0}^4 (5-k) + \cdots + \sum_{k=0}^4 (m-k) \right) + \left(\sum_{k=2}^2 (3-k) + \sum_{k=1}^3 (4-k) \right) \right] q^4 \\
&+ \left[\left(\sum_{k=0}^5 (6-k) + \cdots + \sum_{k=0}^5 (m-k) \right) + \left(\sum_{k=2}^3 (4-k) + \sum_{k=1}^4 (5-k) \right) \right] q^5 \\
&\vdots \\
&+ \left[\sum_{k=0}^{m-1} (m-k) + \sum_{k=1}^{m-2} (m-1-k) + \cdots + \sum_{k=1+(l-1)}^{m-2-(l-1)} (m-1-(l-1)-k) \right] q^{m-1} \\
&+ \left[\sum_{k=1}^{m-1} (m-k) + \sum_{k=2}^{m-2} (m-1-k) + \cdots + \sum_{k=1+(l-1)}^{m-1-(l-1)} (m-(l-1)-k) \right] q^m \\
&+ \left[\sum_{k=2}^{m-1} (m-k) + \sum_{k=3}^{m-2} (m-1-k) + \cdots + \sum_{k=1+(l-1)}^{m-(l-1)} (m+1-(l-1)-k) \right] q^{m+1} \\
&+ \left[\sum_{k=3}^{m-1} (m-k) + \sum_{k=4}^{m-2} (m-1-k) + \cdots + \sum_{k=1+(l-1)}^{m+1-(l-1)} (m+2-(l-1)-k) \right] q^{m+2} \\
&\vdots \\
&+ \left[\sum_{k=m-3}^{m-1} (m-k) + \sum_{k=m-2}^{m-2} (m-1-k) \right] q^{2m-4} \\
&+ \left[\sum_{k=m-2}^{m-1} (m-k) \right] q^{2m-3} + \left[\sum_{k=m-1}^{m-1} (m-k) \right] q^{2m-2}.
\end{aligned}$$

Now, by taking the sums over the index k , we get

$$\begin{aligned}
 S_q[m+2, m] &= [1+2+\cdots+m] q^0 \\
 &+ [(2+1)+(3+2)+\cdots+(m+(m-1))] q^1 \\
 &+ [(3+2+1)+(4+3+2)+\cdots+(m+(m-1)+(m-2))+(1)] q^2 \\
 &+ [(4+3+2+1)+\cdots+(m+(m-1)+(m-2)+(m-3))+(2+1)] q^3 \\
 &\vdots \\
 &+ [K]q^{m-1} + [L]q^m + [M]q^{m+1} + [N]q^{m+2} \\
 &+ \cdots + [(3+2+1)+(1)] q^{2m-4} + [(2+1)] q^{2m-3} + [(1)] q^{2m-2},
 \end{aligned}$$

where

$$\begin{aligned}
 K &= [m+(m-1)+(m-2)+\cdots+1] + [(m-2)+(m-3)+\cdots+1] \\
 &+ \cdots + [m-2-2(l-1)] + [m-3-2(l-1)] + \cdots + 1,
 \end{aligned}$$

$$\begin{aligned}
 L &= [(m-1)+(m-2)+\cdots+1] + [(m-3)+(m-4)+\cdots+1] \\
 &+ \cdots + [(m-1)-2(l-1)] + [(m-2)-2(l-1)] + \cdots + 1,
 \end{aligned}$$

$$\begin{aligned}
 M &= [(m-2)+(m-3)+\cdots+1] + [(m-4)+(m-5)+\cdots+1] \\
 &+ \cdots + [(m-2(l-1))] + [(m-1)-2(l-1)] + \cdots + 1, \text{ and}
 \end{aligned}$$

$$\begin{aligned}
 N &= [(m-3)+(m-4)+\cdots+1] + [(m-5)+(m-6)+\cdots+1] \\
 &+ \cdots + [(m+1)-2(l-1)] + (m-2(l-1)) + \cdots + 1.
 \end{aligned}$$

Now, note that the coefficient of q^1 can be written as:

$$2(2+3+4+\cdots+m) - (m-1).$$

The coefficient of q^2 can be written as:

$$3(3+4+5+\cdots+m) - (m-2)(1+2) + 1.$$

The coefficient of q^3 can be written as:

$$4(4+5+6+\cdots+m) - (m-3)(1+2+3) + (2+1).$$

The coefficient of q^4 can be written as:

$$5(5+6+7+\cdots+m) - (m-4)(1+2+3+4) + (1) + (3+2+1).$$

The coefficient of q^5 can be written as:

$$6(6+7+8+\cdots+m) - (m-5)(1+2+3+4+5) + (2+1) + (4+3+2+1).$$

That is, the coefficient of q^{m-1} can be written as:

$$m(m) - (1)[1 + 2 + \cdots + (m-1)] + [(m-2) + (m-3) + \cdots + 1] \\ + [(m-4) + (m-5) + \cdots + 1] + \cdots + [(m-2-2(l-1)) + (m-3-2(l-1)) + \cdots + 1].$$

Thus,

$$\begin{aligned} S_q[m+2, m] &= \left[\sum_{i=1}^m i \right] q^0 + \left[2 \sum_{i=2}^m i - \sum_{i=2}^m 1 \right] q^1 + \left[3 \sum_{i=3}^m i - \sum_{i=3}^m \frac{2(2+1)}{2} + 1 \right] q^2 \\ &+ \left[4 \sum_{i=4}^m i - \sum_{i=4}^m \frac{3(3+1)}{2} + (2+1) \right] q^3 \\ &+ \left[5 \sum_{i=5}^m i - \sum_{i=5}^m \frac{4(4+1)}{2} + (1) + (3+2+1) \right] q^4 \\ &+ \left[6 \sum_{i=6}^m i - \sum_{i=6}^m \frac{5(5+1)}{2} + (2+1) + (4+3+2+1) \right] q^5 \\ &\vdots \\ &+ \left[m \sum_{i=m}^m i - \sum_{i=m}^m \frac{m(m+1)}{2} + b_{m-1} \right] q^{m-1} \\ &+ \left[\sum_{k=1}^{m-1} (m-k) + \sum_{k=2}^{m-2} (m-1-k) + \cdots + \sum_{k=1+(l-1)}^{m-1-(l-1)} (m-(l-1)-k) \right] q^m \\ &+ \left[\sum_{k=2}^{m-1} (m-k) + \sum_{k=3}^{m-2} (m-1-k) + \cdots + \sum_{k=1+(l-1)}^{m-(l-1)} (m+1-(l-1)-k) \right] q^{m+1} \\ &\vdots \\ &+ \left[\sum_{k=m-3}^{m-1} (m-k) + \sum_{k=m-2}^{m-2} (m-1-k) \right] q^{2m-4} \\ &+ \left[\sum_{k=m-2}^{m-1} (m-k) \right] q^{2m-3} + \left[\sum_{k=m-1}^{m-1} (m-k) \right] q^{2m-2}, \end{aligned}$$

where

$$b_{m-1} = \sum_{k=1}^{(m-1)-1} ((m-1)-k) + \sum_{k=2}^{(m-1)-2} ((m-1)-1-k) + \cdots + \sum_{k=\frac{m-1}{2}}^{\frac{m-1}{2}} \left(\frac{m-1}{2} + 1 - k \right)$$

if $m-1$ is even, or

$$b_{m-1} = \sum_{k=1}^{m-2} ((m-1) - k) + \sum_{k=2}^{m-3} ((m-2) - k) + \cdots + \sum_{k=\frac{m-2}{2}}^{\frac{m-2}{2}+1} \left(\frac{m-2}{2} + 2 - k \right)$$

if $m-1$ is odd.

Hence,

$$\begin{aligned} S_q[m+2, m] &= \left[\sum_{i=1}^m i \right] q^0 + \left[\sum_{i=2}^m \left(2i - \frac{1(1+1)}{2} \right) \right] q^1 \\ &+ \left[\sum_{i=3}^m \left(3i - \frac{2(2+1)}{2} \right) + \sum_{k=1}^1 (2-k) \right] q^2 + \left[\sum_{i=4}^m \left(4i - \frac{3(3+1)}{2} \right) + \sum_{k=1}^2 (3-k) \right] q^3 \\ &+ \left[\sum_{i=5}^m \left(5i - \frac{4(4+1)}{2} \right) + \sum_{k=2}^2 (3-k) + \sum_{k=1}^3 (4-k) \right] q^4 \\ &+ \left[\sum_{i=6}^m \left(6i - \frac{5(5+1)}{2} \right) + \sum_{k=2}^3 (4-k) + \sum_{k=1}^4 (5-k) \right] q^5 \\ &\vdots \\ &+ \left[\sum_{i=m}^m \left(mi - \frac{m(m+1)}{2} \right) + b_{m-1} \right] q^{m-1} \\ &+ \left[\sum_{k=1}^{m-1} (m-k) + \sum_{k=2}^{m-2} (m-1-k) + \cdots + \sum_{k=1+(l-1)}^{m-1-(l-1)} (m-(l-1)-k) \right] q^m \\ &+ \left[\sum_{k=2}^{m-1} (m-k) + \sum_{k=3}^{m-2} (m-1-k) + \cdots + \sum_{k=1+(l-1)}^{m-(l-1)} (m+1-(l-1)-k) \right] q^{m+1} \\ &\vdots \\ &+ \left[\sum_{k=m-3}^{m-1} (m-k) + \sum_{k=m-2}^{m-2} (m-1-k) \right] q^{2m-4} \\ &+ \left[\sum_{k=m-2}^{m-1} (m-k) \right] q^{2m-3} + \left[\sum_{k=m-1}^{m-1} (m-k) \right] q^{2m-2}. \end{aligned}$$

Thus, if

$$\begin{aligned} S_q[m+2, m] \\ = (\alpha_0 + \alpha_1 q) + (\alpha_2 q^2 + \cdots + \alpha_{m-1} q^{m-1}) + (\alpha_m q^m + \cdots + \alpha_{2m-2} q^{2m-2}), \end{aligned}$$

then we note that

a)

$$(8) \quad \alpha_0 = \sum_{i=1}^m i, \quad \text{and} \quad \alpha_1 = \sum_{i=2}^m \left(2i - \frac{1(1+1)}{2} \right)$$

b) For $2 \leq j \leq m-1$

$$(9) \quad \alpha_j = \sum_{i=j+1}^m \left((j+1)i - \frac{j(j+1)}{2} \right) + b_j,$$

where

$$(10) \quad b_j = \sum_{k=1}^{j-1} (j-k) + \sum_{k=2}^{j-2} (j-1-k) + \cdots + \sum_{k=\frac{j}{2}}^{\frac{j}{2}} \left(\frac{j}{2} + 1 - k \right) \text{ if } j \text{ is even,}$$

and

$$(11) \quad b_j = \sum_{k=1}^{j-1} (j-k) + \sum_{k=2}^{j-2} (j-1-k) + \cdots + \sum_{k=\frac{j-1}{2}}^{\frac{j-1}{2}+1} \left(\frac{j-1}{2} + 2 - k \right) \text{ if } j \text{ is odd}$$

c) For $m \leq j \leq 2m-2$ we consider two cases:

Case 1. Consider m even. In this case $l = \frac{m}{2}$. Thus,

$$\begin{aligned} \alpha_m &= \sum_{k=1}^{m-1} (m-k) + \sum_{k=2}^{m-2} (m-1-k) + \cdots + \sum_{k=\frac{m}{2}}^{\frac{m}{2}} \left(\frac{m}{2} + 1 - k \right), \\ \alpha_{m+1} &= \sum_{k=2}^{m-1} (m-k) + \sum_{k=3}^{m-2} (m-1-k) + \cdots + \sum_{k=\frac{m}{2}}^{\frac{m+2}{2}} \left(\frac{m}{2} + 2 - k \right), \cdots, \\ \alpha_{2m-4} &= \sum_{k=m-3}^{m-1} (m-k) + \sum_{k=m-2}^{m-2} (m-1-k), \\ \alpha_{2m-3} &= \sum_{k=m-2}^{m-1} (m-k), \quad \text{and} \quad \alpha_{2m-2} = \sum_{k=m-1}^{m-1} (m-k). \end{aligned}$$

Case 2. Consider m odd. In this case, $l = \frac{m-1}{2}$. Then

$$\begin{aligned}\alpha_m &= \sum_{k=1}^{m-1} (m-k) + \sum_{k=2}^{m-2} (m-1-k) + \cdots + \sum_{k=\frac{m-1}{2}} \left(\frac{m+1}{2} + 1 - k \right), \\ \alpha_{m+1} &= \sum_{k=2}^{m-1} (m-k) + \sum_{k=3}^{m-2} (m-1-k) + \cdots + \sum_{k=\frac{m+1}{2}}^{\frac{m-1}{2}+1} \left(\frac{m-1}{2} + 2 - k \right), \cdots, \\ \alpha_{2m-4} &= \sum_{k=m-3}^{m-1} (m-k) + \sum_{k=m-2}^{m-2} (m-1-k), \\ \alpha_{2m-3} &= \sum_{k=m-2}^{m-1} (m-k), \quad \text{and} \quad \alpha_{2m-2} = \sum_{k=m-1}^{m-1} (m-k).\end{aligned}$$

Then from cases 1 and 2, we can write α_j for $m \leq j \leq 2m-2$ as:

$$(12) \quad \alpha_j = \sum_{k=j+1-m}^{m-1} (m-k) + \sum_{k=j+2-m}^{m-2} (m-1-k) + \cdots + \sum_{k=\frac{j}{2}}^{\frac{j}{2}} \left(\frac{j}{2} + 1 - k \right)$$

if j is even, and

$$(13) \quad \alpha_j = \sum_{k=j+1-m}^{m-1} (m-k) + \sum_{k=j+2-m}^{m-2} (m-1-k) + \cdots + \sum_{k=\frac{j+1}{2}-1}^{\frac{j+1}{2}} \left(\frac{j+1}{2} + 1 - k \right)$$

if j is odd.

This proves Lemma 4. □

The proofs of the following five Lemmas are straightforward.

Lemma 5. For $2 \leq j \leq m-1$, the first term in (9) can be written as:

$$\sum_{i=j+1}^m \left((j+1)i - \frac{j(j+1)}{2} \right) = \frac{(j+1)}{2} m^2 + \frac{(1-j^2)}{2} m - \frac{j(j+1)}{2}$$

Lemma 6. For $2 \leq j \leq m-1$, the term given in (10) can be written as:

$$(14) \quad \sum_{k=1}^{j-1} (j-k) + \sum_{k=2}^{j-2} (j-1-k) + \cdots + \sum_{k=\frac{j}{2}}^{\frac{j}{2}} \left(\frac{j}{2} + 1 - k \right) = \sum_{k=1}^{\frac{j}{2}} k(2j+1-k),$$

and the term given in (11) can be written as:

$$(15) \quad \sum_{k=1}^{j-1} (j-k) + \sum_{k=2}^{j-2} (j-1-k) + \cdots + \sum_{k=\frac{j-1}{2}}^{\frac{j-1}{2}+1} \left(\frac{j-1}{2} + 2 - k \right) = \sum_{k=1}^{\frac{j-1}{2}} k(2j+1-4k).$$

Lemma 7. For $2 \leq j \leq m-1$, the right hand side of (14) can be written as:

$$a) \quad \sum_{k=1}^{\frac{j}{2}} k(2j+1-4k) = \frac{j^3}{12} + \frac{j^2}{8} - \frac{j}{12} \quad \text{if } j \text{ is even,}$$

and the right hand side of (15) can be written as:

$$b) \quad \sum_{k=1}^{\frac{j-1}{2}} k(2j+1-4k) = \frac{j^3}{12} + \frac{j^2}{8} - \frac{j}{12} - \frac{1}{8} \quad \text{if } j \text{ is odd.}$$

Lemma 8. For $m \leq j \leq 2m-2$, the term given in (12) can be written as:

$$\begin{aligned} & \sum_{k=j+1-m}^{m-1} (m-k) + \sum_{k=j+2-m}^{m-2} (m-1-k) + \cdots + \sum_{k=\frac{j}{2}}^{\frac{j}{2}} \left(\frac{j}{2} + 1 - k \right) \\ &= \sum_{k=1}^{m-\frac{j}{2}} k(4m-2j+1-4k) \quad \text{if } j \text{ is even.} \end{aligned}$$

and the term given in (13) can be written as:

$$\begin{aligned} & \sum_{k=j+1-m}^{m-1} (m-k) + \sum_{k=j+2-m}^{m-2} (m-1-k) + \cdots + \sum_{k=\frac{j+1}{2}-1}^{\frac{j+1}{2}} \left(\frac{j+1}{2} + 1 - k \right) \\ &= \sum_{k=1}^{m-\frac{j+1}{2}} k(4m-2j+1-4k) \quad \text{if } j \text{ is odd} \end{aligned}$$

Lemma 9. Consider $m \leq j \leq 2m-2$.

a) If j is even, then

$$\begin{aligned} & \sum_{k=1}^{m-\frac{j}{2}} k(4m-2j+1-4k) \\ &= \frac{2}{3}m^3 + \left(\frac{1}{2} - j \right) m^2 + \left(\frac{1}{2}j^2 - \frac{1}{2}j - \frac{1}{6} \right) m - \frac{j^3}{12} + \frac{j^2}{8} + \frac{j}{12} \end{aligned}$$

b) If j is odd, then

$$\begin{aligned} & \sum_{k=1}^{m-\frac{j+1}{2}} k(4m-2j+1-4k) \\ &= \frac{2}{3}m^3 + \left(\frac{1}{2}-j\right)m^2 + \left(\frac{1}{2}j^2 - \frac{1}{2}j - \frac{1}{6}\right)m - \frac{j^3}{12} + \frac{j^2}{8} + \frac{j}{12} - \frac{1}{8} \end{aligned}$$

Proof of Lemma 1.

a) For $j = 0, 1$, the proof comes from (8).

b) Suppose that $2 \leq j \leq m-1$ and j is even. Then from (9), Lemma 4, (10) and Lemma 7,

$$\begin{aligned} \alpha_j &= \frac{(j+1)}{2}m^2 + \frac{(1-j^2)}{2}m - \frac{j(j+1)}{2} + \frac{j^3}{12} + \frac{j^2}{8} - \frac{j}{12} \\ &= \frac{(j+1)}{2}m^2 + \frac{(1-j^2)}{2}m + \left(-\frac{1}{2} + \frac{1}{8}\right)j^2 + \frac{j^3}{12} + \left(-\frac{1}{2} - \frac{1}{12}\right)j \\ &= \frac{(j+1)}{2}m^2 + \frac{(1-j^2)}{2}m - \frac{3}{8}j^2 + \frac{j^3}{12} - \frac{7}{12}j \end{aligned}$$

Suppose now that j is odd. Then again by Lemmas 4, 6 and 7,

$$\begin{aligned} \alpha_j &= \frac{(j+1)}{2}m^2 + \frac{(1-j^2)}{2}m - \frac{j(j+1)}{2} + \frac{j^3}{12} + \frac{j^2}{8} - \frac{j}{12} - \frac{1}{8} \\ &= \frac{(j+1)}{2}m^2 + \frac{(1-j^2)}{2}m - \frac{3}{8}j^2 + \frac{j^3}{12} - \frac{7}{12}j - \frac{1}{8} \end{aligned}$$

c) Suppose that $m \leq j \leq 2m-2$ and j is even. Then by Lemmas 4, 8 and 9,

$$\begin{aligned} \alpha_j &= \frac{2}{3}m^3 + \left(\frac{1}{2}-j\right)m^2 + \left(\frac{1}{2}j^2 - \frac{1}{2}j - \frac{1}{6}\right)m - \frac{1}{12}j^3 + \frac{1}{8}j^2 + \frac{1}{12}j \\ &= \left(\frac{2}{3}m^3 - jm^2 + \frac{1}{2}j^2m - \frac{1}{12}j^3\right) + \left(\frac{1}{2}m^2 - \frac{1}{2}jm + \frac{1}{8}j^2\right) + \left(-\frac{1}{6}m + \frac{1}{12}j\right) \\ &= \left(\frac{8m^3 - 12m^2j + 6mj^2 - j^3}{12}\right) + \left(\frac{4m^2 - 4mj + j^2}{8}\right) + \left(\frac{-2m + j}{12}\right) \\ &= \frac{(2m-j)^3}{12} + \frac{(2m-j)^2}{8} - \frac{(2m-j)}{12} \end{aligned}$$

Suppose now that j is odd. Then again by Lemmas 4, 8 and 9,

$$\alpha_j = \frac{2}{3}m^3 + \left(\frac{1}{2} - j\right)m^2 + \left(\frac{1}{2}j^2 - \frac{1}{2}j - \frac{1}{6}\right)m - \frac{1}{12}j^3 + \frac{1}{8}j^2 + \frac{1}{12}j - \frac{1}{8}$$

Hence,

$$\alpha_j = \frac{(2m-j)^3}{12} + \frac{(2m-j)^2}{8} - \frac{(2m-j)}{12} - \frac{1}{8}$$

This completes the proof of Lemma 1 □

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Received: April 23, 2013