On the Symplectomorphism Group of Product of Complex Projective Spaces¹

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Abstract

Seidel proved in [1] that the homomorphism $\beta_k : \pi_k(\text{Symp}(M, \omega)) \rightarrow \pi_k(\text{Diff}(M))$ induced by the inclusion is not surjective for $(M, \omega) = (\mathbb{C}P^{N_1} \times \mathbb{C}P^{N_2}, \omega_{N_1} \times \omega_{N_2})$ and each odd $k \leq \max\{2N_1 - 1, 2N_2 - 1\}$, where ω_n is the standard symplectic form on $\mathbb{C}P^n$ with $\int_{\mathbb{C}P^1} \omega_n = 1$. In this note, this profound result is generalized to the case $(M, \omega) = (\mathbb{C}P^{N_1} \times \cdots \times \mathbb{C}P^{N_m}, \omega_{N_1} \times \cdots \times \omega_{N_m})$ with arbitrary $m \geq 2$.

Mathematics Subject Classification: 53C15, 53D35, 37J05

Keywords: Complex projective space, symplectomorphism group, diffeomorphism group, parameterized Gromov-Witten invariants

1 Introduction

Let ω_n be the standard symplectic form on $\mathbb{C}P^n$, and let $(P_{mn}, \eta_{mn}) = (\mathbb{C}P^m \times \mathbb{C}P^n, \omega_m \times \omega_n)$. For the homomorphisms $\beta_k : \pi_k(\text{Symp}(P_{mn}, \eta_{mn})) \rightarrow \pi_k(\text{Diff}(P_{mn}))$ induced by inclusion $\text{Symp}(P_{mn}, \eta_{mn}) \hookrightarrow \text{Diff}(P_{mn})$, using the parameterized Gromov-Witten invariants (cf.[2]) Seidel [1] proved the following estimate:

$$\begin{aligned} \text{rank}(\text{coker} \beta_k) &\geq b_{2m+1-k}(P_{mn}) - b_{2m+1-k}(\mathbb{C}P^m) + \\ &\quad + b_{2n+1-k}(P_{mn}) - b_{2n+1-k}(\mathbb{C}P^n) > 0, \end{aligned}$$

¹Partially supported by the NNSF 10971014 and 11271044 of China, and PCSIRT.

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for each odd $k \leq \max\{2m-1, 2n-1\}$, where b_* are the Betti numbers. In particular, β_k is not surjective. This is the first example of symplectic manifolds in which the homeomorphism $\beta_k : \pi_k(\text{Symp}(M, \omega)) \rightarrow \pi_k(\text{Diff}(M))$ is not surjective for dimension greater than 4. The case that $m = n = k = 1$ can be derived from a result of Gromov [3, 0.3.C]. This note shows that Seidel's method can still deal with symplectic manifolds $(M, \omega) = (\mathbb{C}P^{N_1} \times \cdots \times \mathbb{C}P^{N_m}, \omega_{N_1} \times \cdots \times \omega_{N_m})$ for each $m \geq 2$. The main result, Theorem 3.2, gives the estimate of $\text{rank}(\text{coker} \beta_{d-1})$ via the Betti numbers for every even $d \geq 2$. In particular, Corollary 3.4 claims $\text{rank}(\text{coker} \beta_k) \geq m C_{N+m-\frac{k+1}{2}}^{m-1} - m$ for each odd $k \in [1, 2N-1]$ if $N_1 = \cdots = N_m = N$, and $\text{rank}(\text{coker} \beta_1) \geq C_{N_1+m-1}^{m-1} + (m-1) \sum_{k=0}^{N_1} C_{N+m-k-1}^{m-1} - m$ if $N_1 < N_2 = \cdots = N_m = N$.

2 Deformations of the $\mathbb{Q}$ -algebra $H^*(M)$

For reader's convenience we review some necessary definitions and properties in [1]. All algebras are finite-dimensional, commutative (in the graded sense), and have a unit.

Definition 2.1 Let R be a graded $\mathbb{F}$ -algebra over a field $\mathbb{F}$. A **deformation** of R of dimension $d > 0$ consists of: (i) a graded $\mathbb{F}$ -algebra $\tilde{R}$, (ii) an element $t \in \tilde{R}^d$ with $t^2 = 0$, such that the sequence $0 \rightarrow \tilde{R}/t\tilde{R} \rightarrow \tilde{R} \rightarrow \tilde{R}/t\tilde{R} \rightarrow 0$ is exact, where the second arrow is $\times t$, (iii) a homomorphism of graded algebras $j : \tilde{R} \rightarrow R$ which is surjective with kernel $t\tilde{R}$.

One can also define the notion of isomorphisms between d -dimensional deformations of R . Denote by $\text{Def}_d(R)$ the set of isomorphism classes of all d -dimensional deformations of R . It has a natural structure of an $\mathbb{F}$ -vector space. In Remark 2.2 of [1], the author also gave the following equivalent definition of $\text{Def}_d(R)$.

Definition 2.2 If $Z_d(R)$ denotes the space of $\mathbb{F}$ -bilinear maps $\psi : R \times R \rightarrow R$ of degree $-d$ which are graded symmetry and satisfy

$$(-1)^{d \deg(x)} x\psi(y, z) - \psi(xy, z) + \psi(x, yz) - \psi(x, y)z = 0,$$

and $B_d(R) \subset Z_d(R)$ is the subset of maps of the form

$$\psi(x, y) = (-1)^{d \deg(x)} x\xi(y) - \xi(xy) + \xi(x)y,$$

where $\xi : R \rightarrow R$ is some linear map of degree $-d$, then $\text{Def}_d(R) = Z_d(R)/B_d(R)$.

So if odd dimension parts of R are 0, then $\text{Def}_d(R) = 0$ for all odd d .

Definition 2.3 ([1, Sec.6]) Let R_1 and R_2 be two graded $\mathbb{F}$ -algebras and let $\tilde{R}_1$ and $\tilde{R}_2$ be their deformations of the same dimension, respectively. For the graded tensor product of R_1 and R_2 , $R = R_1 \otimes R_2$, define a deformation

$(\tilde{R}, t, j)$ of R by $\tilde{R} := \tilde{R}_1 \otimes_{F[\epsilon]/\epsilon^2} \tilde{R}_2$, where ϵ acts on $\tilde{R}_i$ by multiplication with t_i , $t := t_1 \otimes 1 = 1 \otimes t_2$ and $j := j_1 \otimes j_2$. The triple $(\tilde{R}, t, j)$ is called the **exterior product** of $\tilde{R}_1$ and $\tilde{R}_2$.

This notion can be generalized to the case of finitely many deformations of the same dimension.

Definition 2.4 ([1, Sec.6]) A deformation of R is called **split** if it is isomorphic to an exterior product of finitely many deformations of the same dimension.

Denote by $\text{Def}_d^s(R)$ the set of all spit deformations of R .

Definition 2.5 (semi-split) Following the notations in Definition 2.3, we say a deformation $(\tilde{R}, t, j)$ of R to be **semi-split** with respect to R_s ($s = 1$ or 2) if there is a deformation $(\tilde{R}_s, t_s, j_s)$ of R_s and a morphism $\tilde{f}_s : \tilde{R}_s \rightarrow \tilde{R}$ which lie over the natural inclusion $f_s : R_s \rightarrow \tilde{R}$.

Theorem 2.6 ([1, Lemma 2.3]) Let M be a compact smooth manifold, and let $\pi : E \rightarrow S^d$ be a smooth fibre bundle with fibre M . For some fixed $b_0 \in S^d$ let $i : M \rightarrow E_{b_0}$ be a diffeomorphism. Let ϵ denote the generator of $H^d(S^d)$ and $t := \pi^*(\epsilon)$. If i^* is onto, then $(H^*(E; \mathbb{F}), t, i^*)$ is a d -dimensional deformation of $H^*(M; \mathbb{F})$.

According to [page 138, 4], if i^* is onto then M is said to be *totally non-homologous to zero* in E with respect to the field $\mathbb{F}$.

Lemma 2.7 Let $\pi : E \rightarrow S^d$ be as in Theorem 2.6. Suppose that $d > 0$ is even and that the cohomology $H^*(M; \mathbb{F})$ has no nontrivial element of odd degree. Then i^* is onto.

Proof. This easily follows from spectral sequences. In fact, the second page of the spectral sequence is zero except $(2n, d)$ position. Then all elements are not changed under the differential d_r for $r \neq d + 1$. But on the $(d + 1)$ -th page, the differential $d_{d+1} = 0$ since the cohomology lies in even degree. As in [Theorem 5.8, 4] we may express i^* by d_r , and deduce that i^* is onto. $\square$

In the following we always assume $M = \mathbb{C}P^{N_1} \times \cdots \times \mathbb{C}P^{N_m}$ with $m \geq 2$. Then for each even $d > 0$ and a smooth fibre bundle $\pi : E \rightarrow S^d$ with fibre M Theorem 2.6 and Lemma 2.7 show that $(H^*(E; \mathbb{Q}), t, i^*)$ is a d -dimensional deformation of

$$R := H^*(\mathbb{C}P^{N_1} \times \cdots \times \mathbb{C}P^{N_m}; \mathbb{Q}) = H^*(\mathbb{C}P^{N_1}; \mathbb{Q}) \otimes \cdots \otimes H^*(\mathbb{C}P^{N_m}; \mathbb{Q}).$$

With the similar arguments to Sections 2, 3 in [1], we may get the following theorem, which classifies deformations of R .

Theorem 2.8 For each even $d \leq 2 \max_s \{N_s + 1\}$, $\text{Def}_d(R)$ is isomorphic to

$$R^{2N_1+2-d} \bigoplus \cdots \bigoplus R^{2N_m+2-d}$$

if $d > 2$ (where $R^{2N_s+2-d} = 0$ if $d > 2N_s + 2$), and isomorphic to

$$R^{2N_1}/\mathbb{Q}(N_1 + 1)u^{N_1} \bigoplus \cdots \bigoplus R^{2N_m}/\mathbb{Q}(N_m + 1)u^{N_m}$$

if $d = 2$.

Proof. Step 1. Each group of elements $a_s \in R^{2N_s+2-d}$, $s = 1, \dots, m$, determines a deformation of R . In fact, let u_s be the generators of $H^2(\mathbb{C}P^{N_s}; \mathbb{Q})$, $s = 1, \dots, m$. Then $R = \mathbb{Q}[u_1, \dots, u_m]/(u_1^{N_1+1}, \dots, u_m^{N_m+1})$. Since $d \leq 2N_s+2$ for some s , for each such s we may write a_s as

$$a_s = \sum_{i_1+\dots+i_m=N_s+1-d/2} b_{s,i_1\dots i_m} u_1^{i_1} u_2^{i_2} \cdots u_m^{i_m}, \quad (2.1)$$

where $b_{s,i_1\dots i_m} \in \mathbb{Q}$ and $i_1, \dots, i_m$ are nonnegative integers. Denote by $\tilde{R}_{a_1,\dots,a_m}$ the $\mathbb{Q}$ -algebra with generators $\tilde{u}_1, \dots, \tilde{u}_m, t$ of degrees $2, \dots, 2, d$, and relations

$$\left. \begin{aligned} &\tilde{u}_s^{N_s+1} + t \sum_{i_1+\dots+i_m=N_s+1-d/2} b_{s,i_1\dots i_m} \tilde{u}_1^{i_1} \tilde{u}_2^{i_2} \cdots \tilde{u}_m^{i_m} = 0 \\ &\text{for each } s \text{ with } 2N_s+2 \geq d, \quad \text{and} \quad t^2 = 0. \end{aligned} \right\} \quad (2.2)$$

Define the homomorphism $j : \tilde{R}_{a_1,\dots,a_m} \rightarrow R$ by $j(\tilde{u}_s) = u_s$, $s = 1, \dots, m$ and $j(t) = 0$. It is easy to check that $(\tilde{R}_{a_1,\dots,a_m}, j, t)$ is a deformation of R .

Step 2. Each deformation $(\tilde{R}, \bar{j}, \bar{t})$ of R is isomorphic to some $(\tilde{R}_{a_1,\dots,a_m}, j, t)$ as in Step 1. Let $\tilde{v}_s$ be the lifts of u_s in $\tilde{R}$, i.e. $\bar{j}(\tilde{v}_s) = u_s$, $s = 1, \dots, m$. Since $\bar{j}(\tilde{v}_s^{N_s+1}) = u_s^{N_s+1} = 0$, by Definition 2.1(iii) we have some $\tilde{a}_s \in \tilde{R}$ such that $\tilde{v}_s^{N_s+1} = -\bar{t}\tilde{a}_s$ for $s = 1, \dots, m$. Set $a_s := \bar{j}(\tilde{a}_s)$, $s = 1, \dots, m$. Then $a_s \in R^{2N_s+2-d}$ (by comparing their degrees) and hence can be written as the form in (2.1) if $2N_s+2 \geq d$. It follows from these and Definition 2.1(ii) that there exist the following relations as in (2.2) for each s with $2N_s+2 \geq d$,

$$\tilde{v}_s^{N_s+1} + \bar{t} \sum_{i_1+\dots+i_m=N_s+1-d/2} b_{s,i_1\dots i_m} \tilde{v}_1^{i_1} \cdots \tilde{v}_m^{i_m} = 0 \quad \text{and} \quad \bar{t}^2 = 0.$$

Using these it is easy to construct a homomorphism $\tilde{f}$ from $\tilde{R}_{a_1,\dots,a_m}$ to $\tilde{R}$ by sending $\tilde{u}_s$ to $\tilde{v}_s$, $s = 1, \dots, m$ and t to $\bar{t}$. Actually, it is a morphism from the deformations $(\tilde{R}_{a_1,\dots,a_m}, j, t)$ to $(\tilde{R}, \bar{j}, \bar{t})$ since $\tilde{f}(t) = \bar{t}$ and $\bar{j} \circ \tilde{f} = j$. By the claim below [Definition 2.1, 1] $(\tilde{R}, \bar{j}, \bar{t})$ and $(\tilde{R}_{a_1,\dots,a_m}, j, t)$ are isomorphic.

Step 3. Let $(\tilde{R}_{a_1,\dots,a_m}, t, j)$ and $(\tilde{R}_{a'_1,\dots,a'_m}, t', j')$ be two deformations of R as in Step 1, and let ψ be an isomorphism from the former to the latter. We claim:

- If $d > 2$, then $\tilde{R}_{a_1,\dots,a_m}$ is isomorphic to $\tilde{R}_{a'_1,\dots,a'_m}$ if and only if $(a_1, \dots, a_m) = (a'_1, \dots, a'_m)$.
- If $d = 2$, then $\tilde{R}_{a_1,\dots,a_m}$ is isomorphic to $\tilde{R}_{a'_1,\dots,a'_m}$ if and only if $a_s - a'_s$ is a multiple of $(N_s+1)u_s^{N_s}$, $s = 1, \dots, m$.

Indeed, suppose that $\tilde{u}'_1, \dots, \tilde{u}'_m, t'$ are generators of $\tilde{R}_{a'_1,\dots,a'_m}$ of degrees $2, \dots, 2, d$ as in Step 1. Then $j(\tilde{u}_s) = j'(\tilde{u}'_s) = u_s$, $s = 1, \dots, m$. Since $j = j' \circ \psi$, we have $j'(\psi(\tilde{u}_s) - \tilde{u}'_s) = 0$ and hence $\psi(\tilde{u}_s) - \tilde{u}'_s = t'\tilde{r}_s$ for some $\tilde{r}_s \in \tilde{R}_{a'_1,\dots,a'_m}$ by Definition 2.1(iii), $s = 1, \dots, m$.

If $d > 2$ we have $\tilde{r}_s = 0$ and hence

$$\psi(\tilde{u}_s) = \tilde{u}_s', \quad s = 1, \dots, m. \quad (2.3)$$

For each s with $2N_s + 2 \geq d$ we may write

$$\begin{aligned} \tilde{a}_s &= \sum_{i_1 + \dots + i_m = N_s + 1 - d/2} b_{s, i_1, \dots, i_m} \tilde{u}_1^{i_1} \tilde{u}_2^{i_2} \dots \tilde{u}_m^{i_m}, \\ \tilde{a}_s' &= \sum_{i_1 + \dots + i_m = N_s + 1 - d/2} b'_{s, i_1, \dots, i_m} (\tilde{u}_1')^{i_1} (\tilde{u}_2')^{i_2} \dots (\tilde{u}_m')^{i_m}. \end{aligned}$$

By (2.2) we have $(\tilde{u}_s)^{N_s+1} + t\tilde{a}_s = 0$ and $(\tilde{u}_s')^{N_s+1} + t'\tilde{a}_s' = 0$, $s = 1, \dots, m$. Since $\psi(t) = t'$, it follows from these and (2.3) that $t'(\psi(\tilde{a}_s) - \tilde{a}_s') = 0$. By Definition 2.1(ii) this leads to $\psi(\tilde{a}_s) - \tilde{a}_s' = 0$. Moreover, since $a_s = j(\tilde{a}_s)$, $a_s' = j'(\tilde{a}_s')$, $s = 1, \dots, m$, and $j' \circ \psi = j$, we arrive at $a_s = j(\tilde{a}_s) = j' \circ \psi(\tilde{a}_s) = j'(\tilde{a}_s') = a_s'$ for $s = 1, \dots, m$ because $a_s = a_s' = 0$ for s with $2N_s + 2 < d$. The first claim is proved.

If $d = 2$, we may assume $\psi(\tilde{u}_s) - \tilde{u}_s' = t'\beta_s = \beta_s t'$ for some $\beta_s \in \tilde{R}_{a_1', \dots, a_m'}^0$, $s = 1, \dots, m$. Since $\tilde{u}_s^{N_s+1} + t\tilde{a}_s = 0$ and $(\tilde{u}_s')^{N_s+1} + t'\tilde{a}_s' = 0$, we get

$$\begin{aligned} 0 &= \psi(\tilde{u}_s^{N_s+1} + t\tilde{a}_s) \\ &= (\psi(\tilde{u}_s))^{N_s+1} + \psi(t)\psi(\tilde{a}_s) \\ &= (\tilde{u}_s' + \beta_s t')^{N_s+1} + \psi(t)\psi(\tilde{a}_s) \\ &= (\tilde{u}_s')^{N_s+1} + (N_s + 1)\beta_s t'(\tilde{u}_s')^{N_s} + \psi(t)\psi(\tilde{a}_s) \\ &= -t'\tilde{a}_s' + (N_s + 1)\beta_s t'(\tilde{u}_s')^{N_s} + \psi(t)\psi(\tilde{a}_s) \\ &= t'(\psi(\tilde{a}_s) - \tilde{a}_s' + (N_s + 1)\beta_s(\tilde{u}_s')^{N_s}) \end{aligned}$$

because $\psi(t) = t'$, $s = 1, \dots, m$. By the exact sequence in Definition 2.1(ii) we deduce that $\psi(\tilde{a}_s) - \tilde{a}_s' + (N_s + 1)\beta_s(\tilde{u}_s')^{N_s} = 0$ and hence

$$\begin{aligned} 0 &= j'(\psi(\tilde{a}_s) - \tilde{a}_s' + (N_s + 1)\beta_s(\tilde{u}_s')^{N_s}) \\ &= j' \circ \psi(\tilde{a}_s) - j'(\tilde{a}_s') + (N_s + 1)j'(\beta_s)(j'(\tilde{u}_s'))^{N_s} \\ &= a_s - a_s' + (N_s + 1)j'(\beta_s)(u_s)^{N_s} \end{aligned}$$

for $s = 1, \dots, m$. Theorem 2.8 is proved. $\square$

Theorem 2.9([Prop.4.2, 1]) *For a compact manifold (M, ω) suppose that there exists a spherical homology class $A \in H_2^s(M; \mathbb{Z})$ such that $\omega(A)$ is positive and generates the period group $\omega(H_2^s(M; \mathbb{Z})) \subset \mathbb{R}$. The classical method can define the Gromov-Witten invariant $\mathcal{GW}_{A,0,3}^{(M,\omega)} : H^*(M; \mathbb{Q})^{\otimes} \rightarrow \mathbb{Q}$, which can be written as a (graded) symmetric, bilinear map $\psi_A : H^*(M; \mathbb{Q}) \times H^*(M; \mathbb{Q}) \rightarrow H^*(M; \mathbb{Q})$ of degree $-2c_1(A)$. Let (E, π, i, Ω) be a symplectic fibre bundle over*

S^d ($d \geq 2$) with fibre (M, ω) , such that i^* is onto. Then the deformation of $R := H^*(M; \mathbb{Q})$ determined by $H^*(E; \mathbb{Q})$ admits an extension of ψ_A in the sense of [Def.4.1, 1].

The proof of this result was completed with the parametrized Gromov-Witten invariants. Using this theorem Seidel obtained:

Theorem 2.10 ([Prop.6.5, 1]) *Suppose that (M', ω') is a compact manifold whose cohomology ring $H^*(M'; \mathbb{Q})$ is generated by $H^2(M'; \mathbb{Q})$, and that $\omega'(H_2^s(M'; \mathbb{Z})) \subset \mathbb{Z}$. Let (E, π, i, Ω) be a symplectic fibre bundle with fibre $(M, \omega) = (M', \omega') \times (\mathbb{C}P^n, \omega_n)$ over S^d for some even $d > 0$. Then the deformation of $R = H^*(M; \mathbb{Q})$ given by $H^*(E; \mathbb{Q})$ is semi-split with respect to $H^*(M'; \mathbb{Q})$.*

Corresponding to split theorem [Prop.6.2, 1] we have

Theorem 2.11 *Let (E, π, i, Ω) be a symplectic fibre bundle over S^d with fibre $(M, \omega) = (\mathbb{C}P^{N_1} \times \cdots \times \mathbb{C}P^{N_m}, \omega_{N_1} \times \cdots \times \omega_{N_m})$ for some even $d > 0$. Then the deformation determined by $H^*(E; \mathbb{Q})$ is split, i.e. it is the exterior product of certain deformations of $H^*(\mathbb{C}P^{N_s}; \mathbb{Q})$, $s = 1, \dots, m$.*

Proof. For each $s = 1, \dots, m$, we can write $(M, \omega) = (M'_s, \omega'_s) \times (\mathbb{C}P^{N_s}, \omega_{N_s})$, where $(M'_s, \omega'_s) = (\mathbb{C}P^{N_1} \times \cdots \times \widehat{\mathbb{C}P^{N_s}} \times \cdots \times \mathbb{C}P^{N_m}, \omega_{N_1} \times \cdots \times \widehat{\omega_{N_s}} \times \cdots \times \omega_{N_m})$. Clearly, $H^2(M'_s; \mathbb{Q})$ generates the ring $H^*(M'_s; \mathbb{Q})$, and $\omega'_s(H_2^s(M'_s; \mathbb{Z})) \subset \mathbb{Z}$. By Theorem 2.6 and Lemma 2.7 both $H^*(M; \mathbb{Q})$ and $H^*(M'_s; \mathbb{Q})$ have d -dimension deformations. In particular, $(H^*(E; \mathbb{Q}), t, j = i^*)$ is a d -dimension deformation of $H^*(M; \mathbb{Q})$. Now Theorem 2.10 tells us that $H^*(E; \mathbb{Q})$ is semi-split with respect to $R_s := H^*(M'_s; \mathbb{Q})$. Hence there exists a deformation $(\tilde{R}_s, t_s, j_s)$ and a morphism $\tilde{f}_s : \tilde{R}_s \rightarrow H^*(E; \mathbb{Q})$ covering the inclusion $f_s : R_s \rightarrow H^*(M; \mathbb{Q})$.

Suppose that $\tilde{f}_s(\tilde{r}_s) = 0$. Since f_s is injective and $j\tilde{f}_s = f_s j_s$, we have $j_s(\tilde{r}_s) = 0$. It follows from Definition 2.1(iii) that $\tilde{r}_s = t_s \tilde{r}'_s$ for some $\tilde{r}'_s \in \tilde{R}_s$. Note that $t\tilde{f}_s(\tilde{r}'_s) = \tilde{f}_s(t_s) \tilde{f}_s(\tilde{r}'_s) = \tilde{f}_s(t_s \tilde{r}'_s) = \tilde{f}_s(\tilde{r}_s) = 0$. By the exact sequence in Definition 2.1(ii) we deduce that $\tilde{f}_s(\tilde{r}'_s) = 0$. This implies that $\tilde{r}'_s = t_s \tilde{\gamma}_s$ for some $\tilde{\gamma}_s \in \tilde{R}_s$ again. Hence $\tilde{r}_s = t_s \tilde{r}'_s = (t_s)^2 \tilde{\gamma}_s = 0$, and $\tilde{f}_s$ is injective.

Identifying $\tilde{R}_s$ and its image via $\tilde{f}_s$ and checking the proof of Proposition 6.5 in [1], the deformation can be chosen good enough, for example, we can require that $\tilde{R}_s$ is generated by $t, \tilde{u}_1, \dots, \widehat{\tilde{u}_s}, \dots, \tilde{u}_m$. Then $W_s := \bigcap_{j \neq s} \tilde{R}_j = \text{span}\{t, \tilde{u}_s\}$ is a d -dimension deformation of $H^*(\mathbb{C}P^{N_s}; \mathbb{Q})$ which covers the inclusion $H^*(\mathbb{C}P^{N_s}; \mathbb{Q}) \rightarrow H^*(E; \mathbb{Q})$. It follows that $H^*(E; \mathbb{Q})$ is the wedge product of W_s , $s = 1, \dots, m$. $\square$

From the proof of this theorem one may see that the conditions $\int_{\mathbb{C}P^1} \omega_{N_s} = 1$, $s = 1, \dots, m$, are necessary for us using Theorem 2.10.

3 Results and proofs

Denote by $\mathcal{F}(S^d, M)$ the set of all isomorphism classes of smooth fibre bundles over S^d with fibre M . It is a group under the operation of fibre connected sum, and is isomorphic to $\pi_{d-1}(\text{Diff}(M))$. Fix a base point $b_0 \in S^d$. We have an explicit isomorphism obtained by the clutching construction,

$$\mathcal{I}_d : \pi_{d-1}(\text{Diff}(M)) \rightarrow \mathcal{F}(S^d, M), [\phi] \mapsto [E_\phi, \pi_\phi, i_\phi],$$

where $\phi : (S^{d-1}, b_0) \rightarrow (\text{Diff}(M), id)$ and $i_\phi : M \rightarrow (E_\phi)_{b_0}$ is a diffeomorphism. Consider the subgroup $G_{d-1} = \{[\phi] \in \pi_{d-1}(\text{Diff}(M)) \mid i_\phi^* \text{ is onto}\}$. By Lemma 2.7 it is equal to $\pi_{d-1}(\text{Diff}(M))$ provided that $d > 0$ is even and that $H^*(M; \mathbb{F})$ has no nontrivial element of odd degree. Theorem 2.6 defines a homeomorphism

$$\alpha_{d-1} : G_{d-1} \rightarrow \text{Def}_d(H^*(M; \mathbb{Q})).$$

Corresponding to [Prop.3.1, 1] we have:

Theorem 3.1 *For an even $0 < d \leq 2 \max_s \{N_s + 1\}$ and $(M, \omega) = (\mathbb{C}P^{N_1} \times \cdots \times \mathbb{C}P^{N_m}, \omega_{N_1} \times \cdots \times \omega_{N_m})$, the homomorphism*

$$\alpha_{d-1} \bigotimes id_{\mathbb{Q}} : \pi_{d-1}(\text{Diff}(M)) \otimes \mathbb{Q} = G_{d-1} \otimes \mathbb{Q} \rightarrow \text{Def}_d(R)$$

is surjective.

Proof. We only need to prove that each $0 \oplus 0 \cdots \oplus R^{2N_j+2-d} \oplus 0 \cdots \oplus 0$ sits in the image of $\alpha_{d-1} \bigotimes id_{\mathbb{Q}}$ for each j with $2N_j + 2 > d$. Since the proofs are similar, without loss of generality we may assume $j = 1$. We also write $p = d/2$ for conveniences. By Theorem 2.8

$$R^{2N_1+2-d} = \sum_{\substack{k_1 + \cdots + k_m = N_1 + 1 - p \\ \& k_s \leq N_s \ \forall s}} \mathbb{Q} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}.$$

Fix a group of nonnegative integers $i_1, \dots, i_m$ such that $i_1 + \cdots + i_m = N_1 + 1 - p$ and $i_s \leq N_s \ \forall s \leq m$. It suffices to prove

Claim A *The deformation $\tilde{R}_{a0\dots 0}$ associated with element*

$$a = u_1^{i_1} u_2^{i_2} \cdots u_m^{i_m} \tag{3.1}$$

is contained in the image of $\alpha_{d-1} \bigotimes id_{\mathbb{Q}}$.

Since $i_2 + p \in [p, N_2 + p]$, by [Lemma 3.2, 1] we can find a complex vector bundle of rank $i_2 + p$, $\eta_{i_2} \rightarrow \mathbb{C}P^{N_2} \times S^d$, such that $\eta_{i_2}|_{\mathbb{C}P^{N_2} \times b_0}$ is trivial and that $c_s(\eta_{i_2}) \neq 0$ with $s = i_2 + p$. Clearly, $c_s(\eta_{i_2}) = 0$ for $1 \leq s < p$ (if $p > 1$). Moreover, the first condition implies that the Chern classes of it can be written as

$$c_s(\eta_{i_2}) = \gamma_s(u_2^{s-p}) \otimes \epsilon, \quad \gamma_s \in \mathbb{Z}, \ \forall s \geq p,$$

where u_2 and ϵ are the generators of $H^2(\mathbb{C}P^{N_2}; \mathbb{Q})$ and $H^d(S^d; \mathbb{Q})$, respectively.

Pull η_{i_2} back to $\mathbb{C}P^{N_2} \times \cdots \times \mathbb{C}P^{N_m} \times S^d$ via the obvious projection to $\mathbb{C}P^{N_2} \times S^d$, and denote it by $\tilde{\eta}_{i_2}$. Then we have also

$$\left. \begin{aligned} c_s(\tilde{\eta}_{i_2}) &= 0 \quad \forall 0 < s < p \text{ (if } p > 1), \\ c_s(\tilde{\eta}_{i_2}) &= \gamma_s(u_2^{s-p}) \otimes \epsilon, \quad \gamma_s \in \mathbb{Z}, \quad \forall s \geq p. \end{aligned} \right\} \quad (3.2)$$

For a $s \in \{3, \dots, m\}$, if $i_s \geq 1$ we may use the obstruction theory to choose a complex line bundle $L_s \rightarrow \mathbb{C}P^{N_s}$ such that $c_1(L_s) = u_s$, and then set ξ_{i_s} to be the sum of i_s line bundles L_s . We get that $c_j(\xi_{i_s}) = \tau_j u_s^j$ with $\tau_j \in \mathbb{Z} \forall j$, and specially

$$c_{i_s}(\xi_{i_s}) = u_s^{i_s}. \quad (3.3)$$

Pull ξ_{i_s} back to $\mathbb{C}P^{N_2} \times \cdots \times \mathbb{C}P^{N_m} \times S^d$, and denote it by $\tilde{\xi}_{i_s}$. Let ε denote the trivial complex line bundle over $\mathbb{C}P^{N_2} \times \cdots \times \mathbb{C}P^{N_m} \times S^d$. Define

$$\zeta := \begin{cases} \bigoplus_{s=3, i_s > 0}^m \tilde{\xi}_{i_s} \oplus \tilde{\eta}_{i_2} & \text{if } i_1 = 0, \\ \bigoplus_{s=3, i_s > 0}^m \tilde{\xi}_{i_s} \oplus \tilde{\eta}_{i_2} \oplus \varepsilon^{i_1} & \text{if } i_1 > 0 \end{cases}$$

and $i := \sum_{s=2}^m i_s + p$. Then ζ is a complex vector bundle over $\mathbb{C}P^{N_2} \times \cdots \times \mathbb{C}P^{N_m} \times S^d$ of rank $\sum_{s=1}^m i_s + p = N_1 + 1$ with the Chern classes

$$c_i(\zeta) = \gamma_{i_2+p} u_2^{i_2} u_3^{i_3} \cdots u_m^{i_m} \otimes \epsilon, \quad c_0(\zeta) = 1 \quad \text{and} \quad c_j(\zeta) = 0 \text{ for other } j$$

by (3.2) and (3.3), where γ_{i_2+p} is as in (3.2). Note that we can always write

$$c_s(\zeta) = \sum_{\substack{j_2 + \cdots + j_m = s \\ \& j_2 \geq p}} \gamma_{s, j_2 \cdots j_m} u_2^{j_2-p} u_3^{j_3} \cdots u_m^{j_m} \otimes \epsilon \quad (3.4)$$

with $\gamma_{s, j_2 \cdots j_m} \in \mathbb{Z}$ for $s = 1, \dots, i$.

Let $\pi : \mathbb{P}(\zeta) \rightarrow \mathbb{C}P^{N_2} \times \cdots \times \mathbb{C}P^{N_m} \times S^d$ be the projectivized bundle of the bundle ζ . By $\mathbb{P}(\zeta) \rightarrow \mathbb{C}P^{N_2} \times \cdots \times \mathbb{C}P^{N_m} \times S^d \rightarrow S^d$ we may view $\mathbb{P}(\zeta)$ as a bundle over S^d with fiber $\mathbb{C}P^{N_1} \times \cdots \times \mathbb{C}P^{N_m}$. So the deformation of R determined by $\mathbb{P}(\zeta)$ as in Theorem 2.6 sits in the image of $\alpha_{d-1} \otimes id_{\mathbb{Q}}$. From the Leary-Hirsch theorem we know that $H^*(\mathbb{P}(\zeta); \mathbb{Q})$ is a free module over $H^*(\mathbb{C}P^{N_2} \times \cdots \times \mathbb{C}P^{N_m} \times S^d; \mathbb{Q})$ with base $\{1, v, \dots, v^{N_1}\}$. Here the class $v \in H^2(\mathbb{P}(\zeta); \mathbb{Q})$ satisfies a relation

$$v^{N_1+1} + \sum_{s=1}^{N_1+1} \pi^* c_s(\zeta) v^{N_1+1-s} = v^{N_1+1} + \sum_{s=p}^i \pi^* c_s(\zeta) v^{N_1+1-s} = 0. \quad (3.5)$$

(Actually, v restricts to a generator of $H^2(\mathbb{P}(\zeta)_x; \mathbb{Q})$ on the fiber $\mathbb{P}(\zeta)_x \equiv \mathbb{C}P^{N_1}$ at $x \in \mathbb{C}P^{N_2} \times \cdots \times \mathbb{C}P^{N_m} \times S^d$). Take $\tilde{u}_1 = v$, $\tilde{u}_s = \pi^* u_s$, $s = 2, \dots, m$, and $t = \pi^* \epsilon$. It follows from these, (3.4) and (3.5) that

$$\tilde{u}_1^{N_1+1} + t \sum_{s=p}^i \sum_{\substack{j_2 + \cdots + j_m = s \\ \& j_2 \geq p}} \gamma_{s,j_2 \dots j_m} \tilde{u}_1^{N_1+1-s} \tilde{u}_2^{j_2-p} \tilde{u}_3^{j_3} \cdots \tilde{u}_m^{j_m} = 0.$$

This shows that the deformation of R determined by $\mathbb{P}(\zeta)$ as in Theorem 2.6 is isomorphic $\tilde{R}_{b,0,\dots,0}$, where

$$b = \sum_{s=p}^i \sum_{\substack{j_2 + \cdots + j_m = s \\ \& j_2 \geq p}} \gamma_{s,j_2 \dots j_m} u_1^{N_1+1-s} u_2^{j_2-p} u_3^{j_3} \cdots u_m^{j_m} \quad (3.6)$$

by (3.4). **Hence $\tilde{R}_{b,0,\dots,0}$ is in the image of $\alpha_{d-1} \otimes id_{\mathbb{Q}}$.**

Let us write $b = b_p + b_{p+1} + \cdots + b_i$, where

$$\begin{aligned} b_p &= \gamma_{p,p0\dots 0} u_1^{N_1+1-p}, \\ b_{p+1} &= \sum_{\substack{j_2 + \cdots + j_m = p+1 \\ \& j_2 \geq p}} \gamma_{p+1,j_2 \dots j_m} u_1^{N_1-p} u_2^{j_2-p} u_3^{j_3} \cdots u_m^{j_m} + b_p, \\ &\vdots \\ b_{i-1} &= \sum_{\substack{j_2 + \cdots + j_m = i-1 \\ \& j_2 \geq p}} \gamma_{i-1,j_2 \dots j_m} u_1^{N_1+2-i} u_2^{j_2-p} u_3^{j_3} \cdots u_m^{j_m} + b_{i-2}, \\ b &= \gamma_{i_2+p} u_1^{N_1+1-i} u_2^{i_2} u_3^{i_3} \cdots u_m^{i_m} + b_{i-1}. \end{aligned}$$

We shall prove that $\tilde{R}_{b_k,0,\dots,0}$, $k = p, \dots, i-1$ (if $i > p$), are in the image of $\alpha_{d-1} \otimes id_{\mathbb{Q}}$, which lead to Claim A.

Step 1. $\tilde{R}_{a,0,\dots,0}$ is in the image of $\alpha_{d-1} \otimes id_{\mathbb{Q}}$ with $a = u_1^{N_1+1-p}$.

Let $\tilde{\eta}_0$ be obtained by taking $i_2 = 0$ in (3.2). Then $c_s(\tilde{\eta}_0) = 0$ for $s \neq p$, and $c_p(\tilde{\eta}_0) = \gamma_p \epsilon$ with $\gamma_p \in \mathbb{Z} \setminus \{0\}$. Set $\zeta = \tilde{\eta}_0 \oplus \epsilon^{N_1+1-p}$. It is a complex vector bundle of rank $N_1 + 1$, and $c_s(\zeta) = c_s(\tilde{\eta}_0) \forall s$. As above we have $u_1^{N_1+1} + c_p(\zeta) u_1^{N_1+1-p} = 0$. So the deformation of R determined by $\mathbb{P}(\zeta)$ is $\tilde{R}_{a_p,0,\dots,0}$, where $a_p = \gamma_p u_1^{N_1+1-p}$. The desired conclusion follows immediately.

If $i > p$ we proceed with the following:

Step 2. Given nonnegative integers $j_2, \dots, j_m$ satisfying $j_2 + \cdots + j_m = p+1$ and $j_2 \geq p$, $\tilde{R}_{a,0,\dots,0}$ is in the image of $\alpha_{d-1} \otimes id_{\mathbb{Q}}$ with $a = u_1^{N_1-p} u_2^{j_2-p} u_3^{j_3} \cdots u_m^{j_m}$.

As in the above construction of b in (3.6) we may obtain an element

$$a_{p+1} = \alpha_{p+1,j_2 \dots j_m} u_1^{N_1-p} u_2^{j_2-p} u_3^{j_3} \cdots u_m^{j_m} + \alpha_p u_1^{N_1+1-p}$$

with $\alpha_{p+1,j_2\cdots j_m} \in \mathbb{Z}$ and $\alpha_p \in \mathbb{Z}$, such that $\tilde{R}_{a_{p+1},0,\dots,0}$ is in the image of $\alpha_{d-1} \otimes id_{\mathbb{Q}}$. This and Step 1 give rise to the expected result. In particular we deduce that $\tilde{R}_{b_{p+1},0,\dots,0}$ is in the image of $\alpha_{d-1} \otimes id_{\mathbb{Q}}$.

If $i > p+1$, we proceed with the next step. Finally, we obtain that $\tilde{R}_{a,0,\dots,0}$ is in the image of $\alpha_{d-1} \otimes id_{\mathbb{Q}}$ with $a = u_1^{N_1+1-i} u_2^{i_2} u_3^{i_3} \cdots u_m^{i_m}$. Claim A is proved. $\square$

With the same arguments as in [Section 7, 1] Theorem 2.11 and Theorem 3.1 may lead to the following (3.7) and (3.8), respectively.

Theorem 3.2 *For $(M, \omega) = \mathbb{C}P^{N_1} \times \cdots \times \mathbb{C}P^{N_m}$ and any positive even $d \leq \max\{2N_1, \dots, 2N_m\}$ let $\beta_{d-1} : \pi_{d-1}(\text{Symp}(M, \omega)) \rightarrow \pi_{d-1}(\text{Diff}(M))$ be the inclusion, and let i denote the natural inclusion $\text{Def}_d^s(R) \hookrightarrow \text{Def}_d(R)$, where $R := H^*(M; \mathbb{Q})$. Then*

$$(\alpha_{d-1} \otimes id_{\mathbb{Q}}) \circ (\beta_{d-1} \otimes id_{\mathbb{Q}}) = i \circ (\alpha_{d-1} \otimes id_{\mathbb{Q}}|_{\pi_{d-1}(\text{Symp}(M, \omega)) \otimes \mathbb{Q}}), \quad (3.7)$$

$$\text{rank}(\text{coker} \beta_{d-1}) \geq \dim \text{Def}_d(R) - \dim \text{Def}_d^s(R). \quad (3.8)$$

In particular, for $d > 2$, $\text{Def}_d^s(R) = \bigoplus_{i=1}^m H^{2N_i+2-d}(\mathbb{C}P^{N_i})$ and hence

$$\text{rank}(\text{coker} \beta_{d-1}) \geq \sum_{i=1}^m b_{2N_i+2-d}(M) - \sum_{i=1}^m b_{2N_i+2-d}(\mathbb{C}P^{2N_i}); \quad (3.9)$$

for $d = 2$, $\text{Def}_2^s(R) = 0$ and so

$$\text{rank}(\text{coker} \beta_1) \geq \dim \text{Def}_2(R) = \sum_{i=1}^m \dim R^{2N_i} - m. \quad (3.10)$$

By (3.9), (3.10) and Kunneth formula, we know that β_{d-1} is not surjective for any even $d \in [2, \max_i\{2N_i + 2\}]$. Using the following lemma some precise estimates can be derived from Theorem 3.2.

Lemma 3.3 *For any positive integers $m \geq 2, n \geq 0$ let $A_{k_1, \dots, k_m} = 1$ for all nonnegative integers $k_1, \dots, k_m$ with $k_1 + \cdots + k_m = n$. Then*

$$\sum_{k_1 + \cdots + k_m = n} A_{k_1, \dots, k_m} = C_{n+m-1}^{m-1}.$$

Since $b_{2q}(\mathbb{C}P^n) = 1$ for $q = 0, 1, \dots, n$, and $b_q(\mathbb{C}P^n) = 0$ for other q ,

$$\dim R^{2N} = \dim H^{2N}(M) = \sum_{k_1 + \cdots + k_m = N} b_{2k_1}(\mathbb{C}P^{2N_1}) \cdots b_{2k_m}(\mathbb{C}P^{2N_m}).$$

Let $N_1 = \cdots = N_m = N$ and $2 < d = 2l \leq 2N + 2$. Clearly, $A_{k_1, \dots, k_m} := b_{2k_1}(\mathbb{C}P^{2N}) \cdots b_{2k_m}(\mathbb{C}P^{2N}) = 1$ for any nonnegative integers $k_1, \dots, k_m$ with

$k_1 + \cdots + k_m = N + 1 - l$. So the right side of (3.9) becomes

$$\begin{aligned}
 & \sum_{i=1}^m b_{2N_i+2-d}(M) - \sum_{i=1}^m b_{2N_i+2-d}(\mathbb{C}P^{2N_i}) \\
 &= m b_{2N+2-2l}(M) - m = m \dim H^{2N+2-2l}(M) - m \\
 &= m \sum_{k_1+\cdots+k_m=N+1-l} b_{2k_1}(\mathbb{C}P^{2N}) \cdots b_{2k_m}(\mathbb{C}P^{2N}) - m \\
 &= m \sum_{k_1+\cdots+k_m=N+1-l} A_{k_1 \cdots k_m} - m = m C_{N+m-l}^{m-1} - m
 \end{aligned}$$

because of Lemma 3.3.

Similarly, for $d = 2$, $N_1 = \cdots = N_m = N$ the right side of (3.10) becomes

$$\sum_{i=1}^m \dim R^{2N_i} - m = m \dim R^{2N} - m = m C_{N+m-1}^{m-1} - m.$$

Hence we get the following (3.11).

Corollary 3.4 *If $N_1 = \cdots = N_m = N$ and $d = 2l \in [2, 2N + 2]$ is even then*

$$\text{rank}(\text{coker} \beta_{d-1}) \geq m C_{N+m-l}^{m-1} - m. \quad (3.11)$$

If $N_1 < N_2 = \cdots = N_m = N$ then

$$\text{rank}(\text{coker} \beta_1) \geq C_{N_1+m-1}^{m-1} + (m-1) \sum_{k=0}^{N_1} C_{N+m-k-1}^{m-1} - m. \quad (3.12)$$

For the proof of (3.12), since $N_1 < N_2 = \cdots = N_m = N$ we have

$$\dim R^{2N_1} = \sum_{k_1+\cdots+k_m=N_1} b_{2k_1}(\mathbb{C}P^{2N_1}) b_{2k_2}(\mathbb{C}P^{2N}) \cdots b_{2k_m}(\mathbb{C}P^{2N}) = C_{N_1+m-1}^{m-1}.$$

Note that $b_{2k_1}(\mathbb{C}P^{2N_1}) = 0$ for $k_1 > N_1$. Using Lemma 3.3 we deduce that

$$\begin{aligned}
 R^{2N} &= \sum_{k_1+\cdots+k_m=N} b_{2k_1}(\mathbb{C}P^{2N_1}) b_{2k_2}(\mathbb{C}P^{2N}) \cdots b_{2k_m}(\mathbb{C}P^{2N}) \\
 &= \sum_{k_1=0}^{N_1} \sum_{k_2+\cdots+k_m=N-k_1} b_{2k_2}(\mathbb{C}P^{2N}) \cdots b_{2k_m}(\mathbb{C}P^{2N}) = \sum_{k_1=0}^{N_1} C_{N+m-k_1-1}^{m-1}.
 \end{aligned}$$

Hence in this case the right side of (3.10) becomes

$$\sum_{i=1}^m \dim R^{2N_i} - m = C_{N_1+m-1}^{m-1} + (m-1) \sum_{k=0}^{N_1} C_{N+m-k-1}^{m-1} - m.$$

In general, one can also give the estimate of $\text{coker}(\beta_{d-1})$ via (3.9) and (3.10) by tediously long and insipid computations. They are left to the interesting readers.

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Received: April 15, 2013