States on Subtraction Algebras

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Abstract

In this paper, we study the Bosbach state on subtraction algebra and show that there exists a Bosbach state via X/θ where θ is a congruence relation induced by an ideal of X .

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1 Introduction

B. M. Schein [3] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition “ $\circ$ ” of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction “ $\backslash$ ” (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]. He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B.

Zelinka [4] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we study the Bosbach state on subtraction algebra and show that there exists a Bosbach state via X/θ where θ is a congruence relation induced by an ideal of X .

2 Subtraction algebras

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation “ $-$ ” that satisfies the following identities: for any $x, y, z \in X$,

$$(S1) \quad x - (y - x) = x;$$

$$(S2) \quad x - (x - y) = y - (y - x);$$

$$(S3) \quad (x - y) - z = (x - z) - y.$$

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X : $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [6], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true: for all $x, y, z \in X$,

$$(p1) \quad (x - y) - y = x - y.$$

$$(p2) \quad x - 0 = x \text{ and } 0 - x = 0.$$

$$(p3) \quad x - y \leq x.$$

$$(p4) \quad x - (x - y) \leq y.$$

$$(p5) \quad (x - y) - (y - x) = x - y.$$

$$(p6) \quad x - (x - (x - y)) = x - y.$$

$$(p7) \quad (x - y) - (z - y) \leq x - z.$$

$$(p8) \quad x \leq y \text{ if and only if } x = y - w \text{ for some } w \in X.$$

- (p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
- (p10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
- (p11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.
- (p12) $(x - y) - z = (x - z) - (y - z)$.

A non-empty subset I of a subtraction algebra X is called a *subalgebra* if $x - y \in I$ for all $x, y \in I$. A mapping f from a subtraction algebra X to a subtraction algebra Y is called a *homomorphism* if $f(x - y) = f(x) - f(y)$ for all $x, y \in X$. A homomorphism f from a subtraction algebra X to itself is called an *endomorphism* of X . If $x \leq y$ implies $f(x) \leq f(y)$, f is called an *isotone map*.

A nonempty subset I of a subtraction algebra X is called an *ideal* of X if it satisfies the following:

- (I1) $0 \in I$,
- (I2) for any $x, y \in X$, $y \in I$ and $x - y \in I$ imply $x \in I$.

Lemma 2.1. Let $(X; -)$ be a subtraction algebra and I a nonempty subset of X . Then I is an ideal of X if and only if it satisfies the following properties:

- (1) $x \in I$ and $y \leq x$ imply $y \in I$,
- (2) if $x, y \in I$ and $x \vee y$ exists, then $x \vee y \in I$.

3 States on subtraction algebras

Definition 3.1. Let X be a subtraction algebra. A *Bosbach state* on X is a function $s : X \rightarrow [0, 1]$ such that the following axioms hold:

- (i) $s(0) = 0$,
- (ii) $s(x) + s(y - x) = s(y) + s(x - y)$ for all $x, y \in X$.

For a Bosbach state $s : X \rightarrow [0, 1]$, define a set $Ker(s)$ by

$$Ker(s) = \{x \in X \mid s(x) = 0\}.$$

A function $s : X \rightarrow [0, 1]$ is called a *trivial Bosbach state* on X if $s(x) = 0$ for all $x \in X$. Also, the function

$$s(x) = \begin{cases} 0 & \text{if } x = 0, \\ t & \text{otherwise.} \end{cases}$$

is called a *constant Bosbach state* for some $t \in [0, 1]$.

Example 3.2. Let $X = \{0, a, b\}$ be a set in which “ $-$ ” is defined by

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a function $s : X \rightarrow [0, 1]$ by

$$s(0) = 0, s(a) = 0.3, s(b) = 0.7.$$

Then s is a Bosbach state on X .

Example 3.3. Let $X = \{0, x, y, 1\}$ be a set in which “ $-$ ” is defined by

$-$	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

Define a function $s : X \rightarrow [0, 1]$ by

$$s(0) = 0, s(x) = 0.3, s(y) = 0.5, s(1) = 0.8.$$

Then s is a Bosbach state on X .

Proposition 3.4. If $s : X \rightarrow [0, 1]$ is a Bosbach state on X , $s_a : X \rightarrow [0, 1]$, $s_a(x) = s(x - a)$ for every $x \in X$ is also a Bosbach state on X .

Proof. Let X be a subtraction algebra and $a \in X$. Then we have

$$s_a(0) = s(0 - a) = s(0) = 0.$$

Also, we have for all $x, y \in X$,

$$\begin{aligned} s_a(x) + s_a(y - x) &= s(x - a) + s((y - x) - a) \\ &= s(x - a) + s((y - a) + (x - a)) \\ &= s(y - a) + s((x - a) + (y - a)) \\ &= s(y - a) + s((x - y) - a) \\ &= s_a(y) + s_a(x - y). \end{aligned}$$

This completes the proof.

Proposition 3.5. Let $s : X \rightarrow [0, 1]$ be a Bosbach state on X . Then If $x \leq y$, then $s(x) \leq s(y)$ and $s(x) + s(y - x) = s(y)$.

Proof. Let s be a Bosbach state on X and $x \leq y$. Then $x - y = 0$. Since $s(x) + s(y - x) = s(y) + s(x - y)$, we have $0 \leq s(y - x) = s(x - y) + s(y) - s(x) = 0 + s(y) - s(x) = s(y) - s(x)$, and so $s(y) \geq s(x)$.

Proposition 3.6. Let $s : X \rightarrow [0, 1]$ be a Bosbach state on X . Then $\text{Ker}(s)$ is a subalgebra of X .

Proof. Let $x, y \in \text{Ker}(s)$. Then $s(x) = s(y) = 0$. Since $x - y \leq x$, we have $s(x - y) \leq s(x)$ by Proposition 3.5, and so $0 \leq s(x - y) \leq s(x) = 0$. This implies $s(x - y) = 0$, i.e., $x - y \in \text{Ker}(s)$.

Proposition 3.7. Let $s : X \rightarrow [0, 1]$ be a Bosbach state on X . Then $\text{Ker}(s)$ is an ideal of X .

Proof. Since $s(0) = 0$, we have $0 \in \text{Ker}(s)$. Let $x - y \in \text{Ker}(s)$ and $y \in \text{Ker}(s)$. So, $s(x - y) = s(y) = 0$. Since $y - x \leq y$, we have $s(y - x) \leq s(y)$ by Proposition 3.5, which implies $0 \leq s(x - y) \leq s(y) = 0$. Hence $s(y - x) = 0$. Now, since $s(x) + s(y - x) = s(y) + s(x - y)$, we obtain $s(x) + 0 = 0 + 0$. This implies $s(x) = 0$, i.e., $x \in \text{Ker}(s)$.

Theorem 3.8. Let $f : X \rightarrow Y$ be a homomorphism and $s : Y \rightarrow [0, 1]$ be a Bosbach state on Y . Then there exists an unique Bosbach state $t : X \rightarrow [0, 1]$ such that the following diagram is commutative (i.e., $t = s \circ f$).

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \exists t & \downarrow s \\ & & [0, 1] \end{array}$$

Proof. Let $t := s \circ f$. It is clear that t is well-defined. Now we show that t is a Bosbach state. We know that $t(0) = (s \circ f)(0) = s(f(0)) = s(0) = 0$. Now,

let $x, y \in X$. Then

$$\begin{aligned}
 t(x) + t(y - x) &= (s \circ f)(x) + (s \circ f)(y - x) \\
 &= s(f(x)) + s(f(y) - f(x)) \\
 &= s(f(y)) + s(f(x) - f(y)) \\
 &= s(f(y)) + s(f(x - y)) \\
 &= (s \circ f)(y) + (s \circ f)(x - y) \\
 &= t(y) + t(x - y).
 \end{aligned}$$

Hence t is a Bosbach state. Finally, we prove that t is unique. Suppose that there exists an another Bosbach state $r : X \rightarrow [0, 1]$ such that $r = s \circ f$. Then $r(x) = (s \circ f)(x) = t(x)$ for all $x \in X$. Hence we have $r = t$.

Theorem 3.9. Let $f : X \rightarrow Y$ be a bijection homomorphism and $s : X \rightarrow [0, 1]$ be a Bosbach state on X . Then there exists an unique Bosbach state $t : Y \rightarrow [0, 1]$ such that $s = t \circ f$.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow s & \swarrow \exists t & \\
 [0, 1] & &
 \end{array}$$

Proof. Let $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Put $t(y) = s(x)$. Then $t(y) = t(f(x)) = (t \circ f)(x) = s(x)$, and so $t \circ f = s$. Since f is one to one, then $\text{Ker}(f) = 0$, and so $t(0) = t(f(0)) = (t \circ f)(0) = s(0) = 0$. Let $y_1, y_2 \in Y$. Then there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then we have

$$\begin{aligned}
 t(y_1) + t(y_2 - y_1) &= t(f(x_1)) + t(f(x_2) - f(x_1)) \\
 &= t(f(x_1)) + t(f(x_2 - x_1)) \\
 &= (t \circ f)(x_1) + (t \circ f)(x_2 - x_1) \\
 &= s(x_1) + s(x_2 - x_1) \\
 &= s(x_2) + s(x_1 - x_2) \\
 &= (t \circ f)(x_2) + (t \circ f)(x_1 - x_2) \\
 &= t(f(x_2)) + t(f(x_1 - x_2)) \\
 &= t(f(x_2)) + t(f(x_1) - f(x_2)) \\
 &= t(y_2) + t(y_1 - y_2).
 \end{aligned}$$

Hence t is a Bosbach state. Suppose that there exists an another Bosbach state $r : X \rightarrow [0, 1]$ such that $s = r \circ f$. Let $y \in Y$. Then there exists $x \in X$

such that $f(x) = y$. Now, $r(y) = r(f(x)) = (r \circ f)(x) = s(x)$, but by definition of t , we get $r(y) = s(x) = t(y)$. Hence we have $r = t$.

Let I be an ideal of X . Consider relation θ on X by

$$(x, y) \in \theta \Leftrightarrow x - y \in I \quad \text{and} \quad y - x \in I.$$

Then θ is a congruence relation on X . For $x \in X$, we denote by $[x]$ the congruence class of x and let

$$X/\theta = \{[x] \mid x \in X\}.$$

Then X/θ is a subtraction algebra where $[x] - [y] = [x - y]$ for all $x, y \in X$. Also, we denote $p_\theta : X \rightarrow X/\theta$ by $p_\theta(x) = [x]$. If $x \in I$, we have $[x] = [0] = 0$. We introduce a relation " $\leq$ " on X/θ by $[x] \leq [y]$ if and only if $[x] - [y] = [0] = 0$.

Proposition 3.10. Let $s : X \rightarrow [0, 1]$ be a Bosbach state on X . Then

- (i) $[x] \leq [y] \Leftrightarrow s(x - y) = 0$.
- (ii) $[x] = [y] \Leftrightarrow s(x - y) = s(y - x) = 0$.

Proof. It is easy to prove from Proposition 3.5.

Theorem 3.11. Let I be an ideal of X and $s : X \rightarrow [0, 1]$ be a Bosbach state on X . Then there exists a unique Bosbach state $t : X/\theta \rightarrow [0, 1]$ such that the following diagram is commutative (i.e., $s = t \circ p_\theta$), in fact, θ is a congruence relation induced by ideal I .

$$\begin{array}{ccc} X & \xrightarrow{p_\theta} & X/\theta \\ & \searrow s & \downarrow \exists t \\ & & [0, 1] \end{array}$$

Corollary 3.12. Let X be a subtraction algebra and $s : X \rightarrow [0, 1]$ be a Bosbach state on X . Then there exists a unique Bosbach state $t : X/\theta \rightarrow [0, 1]$ such that the following diagram is commutative (i.e., $s = t \circ p_\theta$), in fact, θ is a congruence relation induced by ideal $\text{Ker}(s)$.

$$\begin{array}{ccc} X & \xrightarrow{p_\theta} & X/\theta \\ & \searrow s & \downarrow \exists t \\ & & [0, 1] \end{array}$$

References

- [1] J. C. Abbott, *Sets, Lattices and Boolean Algebras*, Allyn and Bacon, Boston 1969.
- [2] J. C. Abbott, *Algebras of implication and semi-lattices*, Séminaire Dubreil (Algèbre et théorie des nombres) **20(2)** (1966-1967), exp. n^o 20, 1–8.
- [3] B. M. Schein, *Difference Semigroups*, Comm. in Algebra **20** (1992), 2153–2169.
- [4] B. Zelinka, *Subtraction Semigroups*, Math. Bohemica, **120** (1995), 445–447.

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