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A Note on Two-Sided Ideals in Locally C^*-Algebras

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Abstract. In the present note we show that if A is a locally C^*-algebra, and I and J are closed two-sided ideals in A, then $(I+J)^+ = I^+ + J^+$.

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1. Introduction

Let A be a C^*-algebra, and I and J be closed *-ideals in A. Let I^+ (resp. J^+) denotes the set of positive elements in I (resp. J). In 1964 in the first French edition of [3] Dixmier has formulated a problem whether or not

$$(I+J)^+ = I^+ + J^+?$$

Using the results of Effros [4] and Kadison [7],[8], Størmer was able in 1967 to settle this problem in affirmative in his paper [14]. In 1968 Pedersen has noticed that it was Combes who obtained a different proof of the aforementioned result of Størmer [14] as a corrolary of Pedersen's Decomposition Theorem for C^*-algebra (see [11] for details). In 1971 Bunce has given in [2] yet another a very short and elegant proof of the same result of Størmer from [14].

The Hausdorff projective limits of projective families of Banach algebras as natural locally-convex generalizations of Banach algebras have been studied sporadically by many authors since 1952, when they were first introduced by Arens [1] and Michael [10]. The Hausdorff projective limits of projective families of C^*-algebras were first mentioned by Arens [1]. They have since been studied under various names by many authors. Development of the subject is reflected in the monograph of Fragoulopoulou [5]. We will follow Inoue [6] in the usage of the name **locally C^*-algebras** for these algebras.

The purpose of the present notes is to extend the aforementioned result of Størmer from [14] to locally C^*-algebras.

2. Preliminaries

First, we recall some basic notions on topological *-algebras. A *-algebra (or involutive algebra) is an algebra A over $\mathbb{C}$ with an involution

$$^* : A \to A,$$

such that

$$(a + \lambda b)^* = a^* + \overline{\lambda} b^*,$$

and

$$(ab)^* = b^* a^*,$$

for every $a, b \in A$ and $\lambda \in \mathbb{C}$.

A seminorm $\|.\|$ on a *-algebra A is a C^*-seminorm if it is submultiplicative, i.e.

$$\|ab\| \leq \|a\| \|b\|,$$

and satisfies the C^*-condition, i.e.

$$\|a^* a\| = \|a\|^2,$$

for every $a, b \in A$. Note that the C^*-condition alone implies that $\|.\|$ is submultiplicative, and in particular

$$\|a^*\| = \|a\|,$$

for every $a \in A$ (cf. for example [5]).

When a seminorm $\|.\|$ on a *-algebra A is a C^*-norm, and A is complete in in the topology generated by this norm, A is called a **C^*-algebra**.

A topological *-algebra is a *-algebra A equipped with a topology making the operations (addition, multiplication, additive inverse, involution) jointly continuous. For a topological *-algebra A, one puts $N(A)$ for the set of continuous C^*-seminorms on A. One can see that $N(A)$ is a directed set with respect to pointwise ordering, because

$$\max\{\|.\|_\alpha, \|.\|_\beta\} \in N(A)$$

for every $\|.\|_\alpha, \|.\|_\beta \in N(A)$, where $\alpha, \beta \in \Lambda$, with Λ being a certain directed set.

For a topological *-algebra A, and $\|.\|_\alpha \in N(A)$, $\alpha \in \Lambda$,

$$\ker \|.\|_\alpha = \{a \in A : \|a\|_\alpha = 0\}$$

is a *-ideal in A, and $\|.\|_\alpha$ induces a C^*-norm (we as well denote it by $\|.\|_\alpha$) on the quotient $A_\alpha = A/\ker \|.\|_\alpha$, and A_α is automatically complete in the topology generated by the norm $\|.\|_\alpha$, thus is a C^*-algebra (see [5] for details). Each pair $\|.\|_\alpha, \|.\|_\beta \in N(A)$, such that

$$\beta \succeq \alpha,$$

$\alpha, \beta \in \Lambda$, induces a natural (continuous) surjective *-homomorphism

$$g_\alpha^\beta : A_\beta \to A_\alpha.$$

Let, again, Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) " $\preceq$ ". Let

$$\{A_\alpha, \alpha \in \Lambda\}$$

be a family of C^*-algebras, and g_α^β be, for

$$\alpha \preceq \beta,$$

the continuous linear *-mappings

$$g_\alpha^\beta : A_\beta \longrightarrow A_\alpha,$$

so that

$$g_\alpha^\alpha(x_\alpha) = x_\alpha,$$

for all $\alpha \in \Lambda$, and

$$g_\alpha^\beta \circ g_\beta^\gamma = g_\alpha^\gamma,$$

whenever

$$\alpha \preceq \beta \preceq \gamma.$$

Let Γ be the collections $\{g_\alpha^\beta\}$ of all such transformations. Let A be a *-subalgebra of the direct product algebra

$$\prod_{\alpha \in \Lambda} A_\alpha,$$

so that for its elements

$$x_\alpha = g_\alpha^\beta(x_\beta),$$

for all

$$\alpha \preceq \beta,$$

where

$$x_\alpha \in A_\alpha,$$

and

$$x_\beta \in A_\beta.$$

Definition 1. *The *-algebra A constructed above is called a* ***Hausdorff projective limit*** *of the projective family*

$$\{A_\alpha, \alpha \in \Lambda\},$$

relatively to the collection

$$\Gamma = \{g_\alpha^\beta : \alpha, \beta \in \Lambda : \alpha \preceq \beta\},$$

and is denoted by

$$\varprojlim A_\alpha,$$

$\alpha \in \Lambda$, *and is called the Arens-Michael decomposition of A.*

It is well known (see, for example [15]) that for each $x \in A$, and each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$, there is a natural projection

$$\pi_\beta : A \longrightarrow A_\beta,$$

defined by

$$\pi_\alpha(x) = g_\alpha^\beta(\pi_\beta(x)),$$

and each projection π_α for all $\alpha \in \Lambda$ is continuous.

Definition 2. *A topological *-algebra (A, τ) over $\mathbb{C}$ is called a* ***locally** C^***-algebra*** *if there exists a projective family of C^*-algebras*

$$\{A_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\},$$

so that

$$A \cong \varprojlim A_\alpha,$$

$\alpha \in \Lambda$, *i.e. A is topologically *-isomorphic to a projective limit of a projective family of C^*-algebras, i.e. there exits its Arens-Michael decomposition of A composed entirely of C^*-algebras.*

A topological *-algebra (A, τ) over $\mathbb{C}$ is a locally C^*-algebra iff A is a complete Hausdorff topological *-algebra in which the topology τ is generated by a saturated separating family $\mathcal{F}$ of C^*-seminorms (see [5] for details).

Example 1. *Every C^*-algebra is a locally C^*-algebra.*

Example 2. *A closed *-subalgebra of a locally C^*-algebra is a locally C^*-algebra.*

Example 3. *The product $\prod_{\alpha \in \Lambda} A_\alpha$ of C^*-algebras A_α, with the product topology, is a locally C^*-algebra.*

Example 4. *Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$). Then the algebra $C(X)$ of all continuous, not necessarily bounded complex-valued functions on X, with the topology of uniform convergence on compact subsets, is a locally C^*-algebra. It is well known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated (see* [9] *for details).*

Let A be a locally C^*-algebra. Then an element $a \in A$ is called **bounded**, if

$$\|a\|_{\infty} = \{\sup \|a\|_{\alpha}, \alpha \in \Lambda : \|.\|_{\alpha} \in N(A)\} < \infty.$$

The set of all bounded elements of A is denoted by $b(A)$.

It is well-known that for each locally C^*-algebra A, its set $b(A)$ of bounded elements of A is a locally C^*-subalgebra, which is a C^*-algebra in the norm $\|.\|_{\infty}$, such that it is dense in A in its topology (see for example [5]).

2.1. **The cone of positive elements in a locally C*-algebra.** If (A, τ) is a unital topological *-algebra, then the **spectrum** $sp_A(a)$ of an element $a \in A$ is the set

$$sp_A(a) = \{z \in \mathbb{C} : z\mathbf{e}_A - a \notin G_A\},$$

where $\mathbf{e}_A$ is a unital element in A, and G_A is the group of invertable elements in A.

An element $a \in A$ in a unital topological *-algebra (A, τ) is called positive, and we write

$$a \geq \mathbf{0}_A,$$

if $a \in A_{sa}$, and

$$sp_A(a) \subseteq [0, \infty).$$

We denote the set of positive elements is (A, τ) by A^+.

Let A,τ be a locally C^*-algebra, and

$$A \cong \varprojlim A_{\alpha},$$

$\alpha \in \Lambda$, be its Arens-Michael decomposition. Then

$$a \geq \mathbf{0}_A \longleftrightarrow a_{\alpha} \geq \mathbf{0}_{A_{\alpha}}, \text{ for all } \alpha \in \Lambda,$$

where $a_{\alpha} = \pi_{\alpha}(a)$.

A **positive part** A^+ of any locally C^*-algebra is not empty, and, together with each $a \in A_{sa}$, it contains positive elements a^+, a^- and $|a|$, such that

$$a = a^+ - a^-,$$

$$a^+ a^- = \mathbf{0}_A = a^- a^+,$$

$$|a| = a^+ + a^-.$$

For any positive $a \in A^+$ in any locally C^*-algebra A there exists a unique **positive squire root** $b \in A^+$, such that

$$a = b^2.$$

(see [5] for details).

The following theorem is valid:

Theorem 1 (Inoue-Schmüdgen)**.** *Let (A, τ) be a locally C^*-algebra. The following are equivalent:*

1. $a \geq \mathbf{0}_A$;
2. $a = b^*b$, *for some* $b \in A$;
3. $a = c^2$, *for some* $c \in A_{sa}$.

Proof. See [6] and [13] for details. □

As a corollary from this theorem one can see that $A^+ = \{a^*a : a \in A\}$ in each locally C^*-algebra A.

The following theorem is valid:

Theorem 2 (Inoue-Schmüdgen)**.** *Let (A, τ) be a locally C^*-algebra. Then A^+ is a closed convex cone such that $A^+ \cap (-A^+) = \{\mathbf{0}_A\}$.*

Proof. See [6] and [13] for details. □

3. Positive parts of two-sided ideals in locally C*-algebras

We start by recalling the following well-known result:

Proposition 1. *Let (A, τ_A) and (B, τ_B) be two locally C^*-algebras, and*

$$\varphi : A \to B,$$

be a *-homomorphism. Then

$$\varphi(A^+) = B^+ \cap \varphi(A).$$

Proof. See for example [5] for details. □

As a corrolary from that theorem one gets:

Corollary 1. *Let (B, τ_B) be a locally C^*-algebra, and (A, τ_A) be its closed *-subalgebra (with $\tau_A = \tau_B|_A$). Then*

$$A^+ = B^+ \cap A.$$

Proof. See for example [5] for details. □

Now, let (A, τ_A) be an unital locally C^*-algebra, I and J be closed *-ideals of (A, τ_A), B be a C^*-algebra, and

$$\varphi : A \to B,$$

be a surjective continuous *-homomorphism from A onto B. In the following series of lemmata we analyse what $\varphi(I), \varphi(J), \varphi(A^+), \varphi(I^+), \varphi(J^+), \varphi(I^+ + J^+)$ and $\varphi((I + J)^+)$ are in B.

Lemma 1. *Let $A, B, I\ (J), \varphi$ be as above. Then $\varphi(I)$ (resp. $\varphi(J)$) is a closed *-ideal in B.*

Proof. Because φ is continuous surjection from A onto B, and I is a *-subalgebra (as a *-ideal), it follows that $\varphi(I)$ is a closed *-subalgebra of B. It remained to show that if $x \in B$, then

$$x \cdot \varphi(I) \subset \varphi(I).$$

Let a be a fixed arbitrary element

$$a \in \varphi^{-1}(x) \subset A$$

(it means that $\varphi(a) = x$), and $b \in I$. Because I is an *-ideal in A, it follows that

$$ab = c \in I.$$

Therefore, from

$$\varphi(a) \cdot \varphi(b) = \varphi(ab) = \varphi(c) \in \varphi(I)$$

it follows that

$$x \cdot \varphi(b) \in \varphi(I),$$

for all $b \in I$, Q.E.D. □

Lemma 2. *Let A, B, A^+, B^+ and φ be as above. Then*

$$\varphi(A^+) = B^+.$$

Proof. From Proposition 1 it follows that

$$\varphi(A^+) = B^+ \cap \varphi(A) \subset B^+.$$

Therefore, it is enough to show that

$$B^+ \subset \varphi(A^+).$$

Let x be an arbitrary element in B^+. Then (see for example [12]) there exists a positive $y \in B^+$, such that

$$x = y^2.$$

Let $b \in A$ be an arbitrary element in

$$\varphi^{-1}(y) \subset A$$

(it means that $\varphi(b) = y$). From Inoue-Schmüdgen Theorem 1 above it follows that

$$b^2 \in A^+,$$

and

$$\varphi(b^2) = (\varphi(b))^2 = y^2 = x,$$

Q.E.D. □

Lemma 3. *Let A, B, I (resp. J), I^+ (resp. J^+) and φ be as above. Then*

$$\varphi(I^+) = (\varphi(I))^+$$

(resp. $\varphi(J^+) = (\varphi(J))^+$).

Proof. Because I is a locally C^*-algebra, and $\varphi(I)$ is a C^*-algebra, the statement is immediately follows from Lemma 2 above. □

Lemma 4. *Let A, B, I, J, I^+, J^+ and φ be as above. Then*

$$\varphi(I^+ + J^+) = (\varphi(I))^+ + ((\varphi(J))^+$$

Proof. Because φ is a *-homomorphism, from Lemma 2 above it follows that

$$\varphi(I^+ + J^+) = \varphi(I^+) + \varphi(J^+) = (\varphi(I))^+ + ((\varphi(J))^+,$$

Q.E.D. □

Lemma 5. *Let A, B, I, J, I^+, J^+ and φ be as above. Then*

$$\varphi((I + J)^+) = (\varphi(I) + \varphi(J))^+$$

Proof. Let c be an arbitrary element in $(I + J)^+$. From Inoue-Schmüdgen Theorem 1 above it follows that there exist $a \in I$ and $b \in J$, such that

$$c = (a + b)^*(a + b).$$

Let us denote

$$x = \varphi(a), \text{ and } y = \varphi(b).$$

Then

$$\varphi(c) = \varphi((a + b)^*(a + b)) = \varphi((a + b)^*)\varphi(a + b) =$$

$$= (\varphi(a) + \varphi(b))^*(\varphi(a) + \varphi(b)) \in (\varphi(I) + \varphi(J))^+,$$

thus

$$\varphi((I + J)^+) \subset (\varphi(I) + \varphi(J))^+.$$

Inversly, let t be an arbitrary element

$$t \in (\varphi(I) + \varphi(J))^+.$$

From the basic properties of the cone of positive elements in a C^*-algera (see for example [12]) if follows that there exist

$$x \in \varphi(I), \text{ and } y \in \varphi(J),$$

such that

$$t = (x + y)^*(x + y).$$

Let a be an arbitrary element in

$$\varphi^{-1}(x) \subset I,$$

and b be an arbitrary element in

$$\varphi^{-1}(y) \subset J.$$

It means that

$$x = \varphi(a), \text{ and } y = \varphi(b).$$

Then from Inoue-Schmüdgen Theorem 1 above it follows that the element

$$(a+b)^*(a+b) \in (I+J)^+,$$

and

$$\varphi((a+b)^*(a+b)) = \varphi((a+b)^*)\varphi(a+b) = (\varphi(a)+\varphi(b))^*(\varphi(a)+\varphi(b)) =$$

$$= (x+y)^*(x+y) = t,$$

thus

$$\varphi((I+J)^+) \supset (\varphi(I)+\varphi(J))^+,$$

Q.E.D. □

Now we are ready to present the main theorem of the current notes.

Theorem 3. *Let (A, τ_A) be an unital locally C^*-algebra, and (I, τ_I) and (J, τ_J) be two *-ideals in A, such that*

$$\tau_I = \tau_A|_I \text{ and } \tau_J = \tau_A|_J.$$

Then

$$(I+J)^+ = I^+ + J^+.$$

Proof. Let now (A, τ) be a locally C^*-algebra, and let

$$A = \varprojlim A_\alpha,$$

$\alpha \in \Lambda$, be its Arens-Michael decomposition, built using the family of seminorms $\|.\|_\alpha, \alpha \in \Lambda$, that defines the topology τ. Let

$$\pi_\alpha : A \to A_\alpha,$$

$\alpha \in \Lambda$, be a projection from A onto A_α, for each $\alpha \in \Lambda$. Each π_α is an injective *-homomorphism from A onto $A_\alpha, \alpha \in \Lambda$, thus, we can apply to A, A_α, and π_α lemmata 3-5 above for all $\alpha \in \Lambda$. Let

$$I_\alpha = \pi_\alpha(I)$$

(resp. $J_\alpha = \pi_\alpha(J)$). Therefore, for all $\alpha \in \Lambda$, Størmer's result from [14] for C^*-algebras implies that

$$(I_\alpha + J_\alpha)^+ = I_\alpha^+ + J_\alpha^+.$$

Thus, because

$$(I+J)^+ = \varprojlim (I+J)_\alpha^+,$$

$$I^+ = \lim_{\longleftarrow} I^+_\alpha,$$

and

$$J^+ = \lim_{\longleftarrow} J^+_\alpha,$$

$\alpha \in \Lambda$, we get

$$(I + J)^+ = I^+ + J^+,$$

Q.E.D. □

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