A Note on Two-Sided Ideals in Locally $C^*$-Algebras

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Abstract. In the present note we show that if $A$ is a locally $C^*$-algebra, and $I$ and $J$ are closed two-sided ideals in $A$, then $(I + J)^+ = I^+ + J^+$.

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1. Introduction


$$(I + J)^+ = I^+ + J^+?$$

Using the results of Effros [4] and Kadison [7],[8], Stormer was able in 1967 to settle this problem in affirmative in his paper [14]. In 1968 Pedersen has noticed that it was Combes who obtained a different proof of the aforementioned result of Stormer [14] as a corollary of Pedersen’s Decomposition Theorem for $C^*$-algebra (see [11] for details). In 1971 Bunce has given in [2] yet another a very short and elegant proof of the same result of Stormer from [14].

The Hausdorff projective limits of projective families of Banach algebras as natural locally-convex generalizations of Banach algebras have been studied
sporadically by many authors since 1952, when they were first introduced by Arens [1] and Michael [10]. The Hausdorff projective limits of projective families of $C^*$-algebras were first mentioned by Arens [1]. They have since been studied under various names by many authors. Development of the subject is reflected in the monograph of Fragoulopoulou [5]. We will follow Inoue [6] in the usage of the name **locally $C^*$-algebras** for these algebras.

The purpose of the present notes is to extend the aforementioned result of Størmer from [14] to locally $C^*$-algebras.

2. Preliminaries

First, we recall some basic notions on topological *-algebras. A *-algebra (or involutive algebra) is an algebra $A$ over $\mathbb{C}$ with an involution

$$* : A \to A,$$

such that

$$(a + \lambda b)^* = a^* + \bar{\lambda} b^*,$$

and

$$(ab)^* = b^* a^*,$$

for every $a, b \in A$ and $\lambda \in \mathbb{C}$.

A seminorm $\| \cdot \|$ on a *-algebra $A$ is a $C^*$-seminorm if it is submultiplicative, i.e.

$$\|ab\| \leq \|a\| \|b\|,$$

and satisfies the $C^*$-condition, i.e.

$$\|a^* a\| = \|a\|^2,$$

for every $a, b \in A$. Note that the $C^*$-condition alone implies that $\| \cdot \|$ is submultiplicative, and in particular

$$\|a^*\| = \|a\|,$$

for every $a \in A$ (cf. for example [5]).

When a seminorm $\| \cdot \|$ on a *-algebra $A$ is a $C^*$-norm, and $A$ is complete in the topology generated by this norm, $A$ is called a $C^*$-algebra.

A topological *-algebra is a *-algebra $A$ equipped with a topology making the operations (addition, multiplication, additive inverse, involution) jointly continuous. For a topological *-algebra $A$, one puts $N(A)$ for the set of continuous $C^*$-seminorms on $A$. One can see that $N(A)$ is a directed set with respect to pointwise ordering, because

$$\max\{\| \cdot \|_\alpha, \| \cdot \|_\beta\} \in N(A)$$

for every $\| \cdot \|_\alpha, \| \cdot \|_\beta \in N(A)$, where $\alpha, \beta \in \Lambda$, with $\Lambda$ being a certain directed set.
For a topological *-algebra $A$, and $\|\|_\alpha \in N(A), \alpha \in \Lambda$,
\[
\ker \|\|_\alpha = \{a \in A : \|a\|_\alpha = 0\}
\]
is a *-ideal in $A$, and $\|\|_\alpha$ induces a $C^*$-norm (we as well denote it by $\|\|_\alpha$) on the quotient $A_\alpha = A/\ker \|\|_\alpha$, and $A_\alpha$ is automatically complete in the topology generated by the norm $\|\|_\alpha$, thus is a $C^*$-algebra (see [5] for details). Each pair $\|\|_\alpha, \|\|_\beta \in N(A)$, such that
\[
\beta \succeq \alpha,
\]
$\alpha, \beta \in \Lambda$, induces a natural (continuous) surjective *-homomorphism
\[
g^\beta_\alpha : A_\beta \rightarrow A_\alpha.
\]
Let, again, $\Lambda$ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) $\succeq$. Let
\[
\{A_\alpha, \alpha \in \Lambda\}
\]
be a family of $C^*$-algebras, and $g^\beta_\alpha$ be, for
\[
\alpha \preceq \beta,
\]
the continuous linear *-mappings
\[
g^\beta_\alpha : A_\beta \longrightarrow A_\alpha,
\]
so that
\[
g^\alpha_\alpha(x_\alpha) = x_\alpha,
\]
for all $\alpha \in \Lambda$, and
\[
g^\beta_\alpha \circ g^\gamma_\beta = g^\gamma_\alpha,
\]
whenever
\[
\alpha \preceq \beta \preceq \gamma.
\]
Let $\Gamma$ be the collections $\{g^\beta_\alpha\}$ of all such transformations. Let $A$ be a *-subalgebra of the direct product algebra
\[
\prod_{\alpha \in \Lambda} A_\alpha,
\]
so that for its elements
\[
x_\alpha = g^\beta_\alpha(x_\beta),
\]
for all
\[
\alpha \preceq \beta,
\]
where
\[
x_\alpha \in A_\alpha,
\]
and
\[ x_\beta \in A_\beta. \]

**Definition 1.** The \(*\)-algebra \(A\) constructed above is called a **Hausdorff projective limit** of the projective family
\[ \{A_\alpha, \alpha \in \Lambda\}, \]
relatively to the collection
\[ \Gamma = \{g^\beta_\alpha : \alpha, \beta \in \Lambda : \alpha \preceq \beta\}, \]
and is denoted by
\[ \lim\leftarrow A_\alpha, \alpha \in \Lambda, \]
and is called the Arens-Michael decomposition of \(A\).

It is well known (see, for example [15]) that for each \(x \in A\), and each pair \(\alpha, \beta \in \Lambda\), such that \(\alpha \preceq \beta\), there is a natural projection
\[ \pi_\beta : A \longrightarrow A_\beta, \]
defined by
\[ \pi_\alpha(x) = g^\beta_\alpha(\pi_\beta(x)), \]
and each projection \(\pi_\alpha\) for all \(\alpha \in \Lambda\) is continuous.

**Definition 2.** A topological \(*\)-algebra \((A, \tau)\) over \(\mathbb{C}\) is called a **locally \(C^*\)-algebra** if there exists a projective family of \(C^*\)-algebras
\[ \{A_\alpha; g^\beta_\alpha; \alpha, \beta \in \Lambda\}, \]
so that
\[ A \cong \lim\leftarrow A_\alpha, \alpha \in \Lambda, \]
\(\alpha \in \Lambda\), i.e. \(A\) is topologically \(*\)-isomorphic to a projective limit of a projective family of \(C^*\)-algebras, i.e. there exits its Arens-Michael decomposition of \(A\) composed entirely of \(C^*\)-algebras.

A topological \(*\)-algebra \((A, \tau)\) over \(\mathbb{C}\) is a locally \(C^*\)-algebra iff \(A\) is a complete Hausdorff topological \(*\)-algebra in which the topology \(\tau\) is generated by a saturated separating family \(\mathcal{F}\) of \(C^*\)-seminorms (see [5] for details).

**Example 1.** Every \(C^*\)-algebra is a locally \(C^*\)-algebra.

**Example 2.** A closed \(*\)-subalgebra of a locally \(C^*\)-algebra is a locally \(C^*\)-algebra.

**Example 3.** The product \(\prod_{\alpha \in \Lambda} A_\alpha\) of \(C^*\)-algebras \(A_\alpha\), with the product topology, is a locally \(C^*\)-algebra.
Example 4. Let $X$ be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$). Then the algebra $C(X)$ of all continuous, not necessarily bounded complex-valued functions on $X$, with the topology of uniform convergence on compact subsets, is a locally $C^*$-algebra. It is well known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated (see [9] for details).

Let $A$ be a locally $C^*$-algebra. Then an element $a \in A$ is called bounded, if

$$\|a\|_\infty = \{\sup \|a\|_\alpha, \alpha \in \Lambda : \|\cdot\|_\alpha \in N(A)\} < \infty.$$  

The set of all bounded elements of $A$ is denoted by $b(A)$.

It is well-known that for each locally $C^*$-algebra $A$, its set $b(A)$ of bounded elements of $A$ is a locally $C^*$-subalgebra, which is a $C^*$-algebra in the norm $\|\cdot\|_\infty$, such that it is dense in $A$ in its topology (see for example [5]).

2.1. The cone of positive elements in a locally $C^*$-algebra. If $(A, \tau)$ is a unital topological $*$-algebra, then the spectrum $sp_A(a)$ of an element $a \in A$ is the set

$$sp_A(a) = \{z \in \mathbb{C} : ze_A - a \notin G_A\},$$

where $e_A$ is a unital element in $A$, and $G_A$ is the group of invertible elements in $A$.

An element $a \in A$ in a unital topological $*$-algebra $(A, \tau)$ is called positive, and we write

$$a \geq 0_A,$$

if $a \in A_{sa}$, and

$$sp_A(a) \subseteq [0, \infty).$$

We denote the set of positive elements is $(A, \tau)$ by $A^+$. Let $A, \tau$ be a locally $C^*$-algebra, and

$$A \cong \lim_{\alpha} A_\alpha,$$

$\alpha \in \Lambda$, be its Arens-Michael decomposition. Then

$$a \geq 0_A \iff a_\alpha \geq 0_{A_\alpha}, \text{ for all } \alpha \in \Lambda,$$

where $a_\alpha = \pi_\alpha(a)$.

A positive part $A^+$ of any locally $C^*$-algebra is not empty, and, together with each $a \in A_{sa}$, it contains positive elements $a^+, a^-$ and $|a|$, such that

$$a = a^+ - a^-,$$

$$a^+a^- = 0_A = a^-a^+,$$

$$|a| = a^+ + a^-.$$
For any positive $a \in A^+$ in any locally $C^*$-algebra $A$ there exists a unique positive square root $b \in A^+$, such that $a = b^2$.

(see [5] for details).

The following theorem is valid:

**Theorem 1** (Inoue-Schmüdgen). Let $(A, \tau)$ be a locally $C^*$-algebra. The following are equivalent:

1. $a \geq 0_A$;
2. $a = b^*b$, for some $b \in A$;
3. $a = c^2$, for some $c \in A_{sa}$.


As a corollary from this theorem one can see that $A^+ = \{a^*a : a \in A\}$ in each locally $C^*$-algebra $A$.

The following theorem is valid:

**Theorem 2** (Inoue-Schmüdgen). Let $(A, \tau)$ be a locally $C^*$-algebra. Then $A^+$ is a closed convex cone such that $A^+ \cap (-A^+) = \{0_A\}$.


3. **Positive parts of two-sided ideals in locally $C^*$-algebras**

We start by recalling the following well-known result:

**Proposition 1.** Let $(A, \tau_A)$ and $(B, \tau_B)$ be two locally $C^*$-algebras, and $\varphi : A \to B$, be a $*$-homomorphism. Then

$$\varphi(A^+) = B^+ \cap \varphi(A).$$


As a corollary from that theorem one gets:

**Corollary 1.** Let $(B, \tau_B)$ be a locally $C^*$-algebra, and $(A, \tau_A)$ be its closed $*$-subalgebra (with $\tau_A = \tau_B|_A$). Then

$$A^+ = B^+ \cap A.$$ 


Now, let $(A, \tau_A)$ be an unital locally $C^*$-algebra, $I$ and $J$ be closed $*$-ideals of $(A, \tau_A)$, $B$ be a $C^*$-algebra, and $\varphi : A \to B$, be a surjective continuous $*$-homomorphism from $A$ onto $B$. In the following series of lemmata we analyse what $\varphi(I), \varphi(J), \varphi(A^+), \varphi(I^+), \varphi(J^+), \varphi(I^+ + J^+)$ and $\varphi((I + J)^+)$ are in $B$. 
Lemma 1. Let $A, B, I (J), \varphi$ be as above. Then $\varphi(I)$ (resp. $\varphi(J)$) is a closed $^*$-ideal in $B$.

Proof. Because $\varphi$ is continuous surjection from $A$ onto $B$, and $I$ is a $^*$-subalgebra (as a $^*$-ideal), it follows that $\varphi(I)$ is a closed $^*$-subalgebra of $B$. It remained to show that if $x \in B$, then

$$x \cdot \varphi(I) \subset \varphi(I).$$

Let $a$ be a fixed arbitrary element

$$a \in \varphi^{-1}(x) \subset A$$

(it means that $\varphi(a) = x$), and $b \in I$. Because $I$ is an $^*$-ideal in $A$, it follows that

$$ab = c \in I.$$

Therefore, from

$$\varphi(a) \cdot \varphi(b) = \varphi(ab) = \varphi(c) \in \varphi(I)$$

it follows that

$$x \cdot \varphi(b) \in \varphi(I),$$

for all $b \in I$, Q.E.D.

Lemma 2. Let $A, B, A^+, B^+$ and $\varphi$ be as above. Then

$$\varphi(A^+) = B^+.$$ 

Proof. From Proposition 1 it follows that

$$\varphi(A^+) = B^+ \cap \varphi(A) \subset B^+.$$ 

Therefore, it is enough to show that

$$B^+ \subset \varphi(A^+).$$ 

Let $x$ be an arbitrary element in $B^+$. Then (see for example [12]) there exists a positive $y \in B^+$, such that

$$x = y^2.$$ 

Let $b \in A$ be an arbitrary element in

$$\varphi^{-1}(y) \subset A$$

(it means that $\varphi(b) = y$). From Inoue-Schmüdgen Theorem 1 above it follows that

$$b^2 \in A^+,$$

and

$$\varphi(b^2) = (\varphi(b))^2 = y^2 = x,$$

Q.E.D.
Lemma 3. Let $A, B, I$ (resp. $J$), $I^+$ (resp. $J^+$) and $\varphi$ be as above. Then
\[ \varphi(I^+) = (\varphi(I))^+ \]
(resp. $\varphi(J^+) = (\varphi(J))^+$).

Proof. Because $I$ is a locally $C^*$-algebra, and $\varphi(I)$ is a $C^*$-algebra, the statement is immediately follows from Lemma 2 above.

Lemma 4. Let $A, B, I, J, I^+, J^+$ and $\varphi$ be as above. Then
\[ \varphi(I^+ + J^+) = (\varphi(I))^+ + (\varphi(J))^+ \]

Proof. Because $\varphi$ is a $^*$-homomorphism, from Lemma 2 above it follows that
\[ \varphi(I^+ + J^+) = \varphi(I^+) + \varphi(J^+) = (\varphi(I))^+ + (\varphi(J))^+, \]
Q.E.D.

Lemma 5. Let $A, B, I, J, I^+, J^+$ and $\varphi$ be as above. Then
\[ \varphi((I + J)^+) = (\varphi(I) + \varphi(J))^+ \]

Proof. Let $c$ be an arbitrary element in $(I + J)^+$. From Inoue-Schm"udgen Theorem 1 above it follows that there exist $a \in I$ and $b \in J$, such that
\[ c = (a + b)^*(a + b). \]
Let us denote
\[ x = \varphi(a), \text{ and } y = \varphi(b). \]
Then
\[ \varphi(c) = \varphi((a + b)^*(a + b)) = \varphi((a + b)^*)\varphi(a + b) = \]
\[ = (\varphi(a) + \varphi(b))^*(\varphi(a) + \varphi(b)) \in (\varphi(I) + \varphi(J))^+, \]
thus
\[ \varphi((I + J)^+) \subset (\varphi(I) + \varphi(J))^+. \]
Inversly, let $t$ be an arbitrary element
\[ t \in (\varphi(I) + \varphi(J))^+. \]
From the basic properties of the cone of positive elements in a $C^*$-algera (see for example [12]) if follows that there exist
\[ x \in \varphi(I), \text{ and } y \in \varphi(J), \]
such that
\[ t = (x + y)^*(x + y). \]
Let $a$ be an arbitrary element in
\[ \varphi^{-1}(x) \subset I, \]
and \( b \) be an arbitrary element in
\[
\varphi^{-1}(y) \subset J.
\]
It means that
\[
x = \varphi(a), \text{ and } y = \varphi(b).
\]
Then from Inoue-Schmüdgen Theorem 1 above it follows that the element
\[
(a + b)^*(a + b) \in (I + J)^+,
\]
and
\[
\varphi((a + b)^*(a + b)) = \varphi((a + b)^*)\varphi(a + b) = (\varphi(a) + \varphi(b))^*(\varphi(a) + \varphi(b)) =
\]
\[
= (x + y)^*(x + y) = t,
\]
thus
\[
\varphi((I + J)^+) \supset (\varphi(I) + \varphi(J))^+,
\]
Q.E.D.

Now we are ready to present the main theorem of the current notes.

**Theorem 3.** Let \((A, \tau_A)\) be an unital locally \(C^*\)-algebra, and \((I, \tau_I)\) and \((J, \tau_J)\) be two \(*\)-ideals in \(A\), such that
\[
\tau_I = \tau_A|_I \text{ and } \tau_J = \tau_A|_J.
\]
Then
\[
(I + J)^+ = I^+ + J^+.
\]

**Proof.** Let now \((A, \tau)\) be a locally \(C^*\)-algebra, and let
\[
A = \varprojlim A_\alpha,
\]
\(\alpha \in \Lambda\), be its Arens-Michael decomposition, built using the family of seminorms \(\| . \|_\alpha, \alpha \in \Lambda\), that defines the topology \(\tau\). Let
\[
\pi_\alpha : A \to A_\alpha,
\]
\(\alpha \in \Lambda\), be a projection from \(A\) onto \(A_\alpha\), for each \(\alpha \in \Lambda\). Each \(\pi_\alpha\) is an injective \(*\)-homomorphism from \(A\) onto \(A_\alpha\), \(\alpha \in \Lambda\), thus, we can apply to \(A, A_\alpha\), and \(\pi_\alpha\) lemmata 3-5 above for all \(\alpha \in \Lambda\). Let
\[
I_\alpha = \pi_\alpha(I)
\]
(resp. \(J_\alpha = \pi_\alpha(J)\)). Therefore, for all \(\alpha \in \Lambda\), Størmer’s result from [14] for \(C^*\)-algebras implies that
\[
(I_\alpha + J_\alpha)^+ = I_\alpha^+ + J_\alpha^+.
\]
Thus, because
\[
(I + J)^+ = \varprojlim (I + J)_\alpha^+,
\]
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\[ I^+ = \lim_{\alpha} I^+_\alpha, \]

and

\[ J^+ = \lim_{\alpha} J^+_\alpha, \]

\( \alpha \in \Lambda, \) we get

\[ (I + J)^+ = I^+ + J^+, \]

Q.E.D.

REFERENCES


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