On Toader-Sángor Mean

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Abstract

In this paper, we establish some inequalities for the Toader-Sángor mean by use of Jensen inequality and convexity.

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1. Introduction

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Recently, the means $M(a, b)$ of two positive numbers $a$ and $b$ have been the subject of intensive research. In particular, many remarkable inequalities between different means can be found in the literature [1-26].

Let $\mathbb{R}$ be the real numbers set and $\mathbb{R}_+$ the positive real numbers set, and $J \subseteq \mathbb{R}_+$ be an interval. If $f : J \rightarrow \mathbb{R}$ be a strictly monotone function and $p : J \rightarrow \mathbb{R}_+$ be a positive function, then the weighted quasi-arithmetic integral mean $M(f, p)$ [27] on $J$ is defined by

$$M(f, p) = M(f, p)(a, b) = f^{-1}\left(\frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx}\right), \quad \forall a, b \in J. \quad (1.1)$$

The weighted quasi-arithmetic integral mean $M(f, p)$ was considered in [28] for arbitrary weight function $p$ and special $f$ defined by

$$f(x) = \begin{cases} x^n, & n \neq 0, \\ \log x, & n = 0. \end{cases}$$

More means of type $M(f, p)$ are given in [29, 30], but only for special cases of functions $f$ or $p$.

In [31], Sándor and Toader proved that inequality $M(f, p) < M(g, p)$ holds for any $p$ if $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly monotone, $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing, and the composed function $g \circ f^{-1}$ is convex.

Making use of two functions but only one integral, Toader and Sándor [32] defined another integral mean $N(f, p)$ as follows.

Let $f : J \rightarrow \mathbb{R}$ and $p : J \rightarrow \mathbb{R}$ be two strictly monotone functions on $J$. Then the Toader-Sándor mean $N(f, p)$ [8] on $J$ is defined by

$$N(f, p) = N(f, p)(a, b) = f^{-1}\left(\int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1-t) \cdot p(b)]dt\right). \quad (1.2)$$

It is not difficult to verify that if the function $p$ is differentiable with $p' > 0$, then

$$N(f, p) = M(f, p'). \quad (1.3)$$

In [33], the Schur convexity and concavity for $M(f, p')$ were discussed.

The following Theorems 1.1 and 1.2 can be found in [32].

**Theorem 1.1.** If the function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly monotone, the function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing, and the composed function $g \circ f^{-1}$ is convex, then the inequality

$$N(f, p) < N(g, p)$$

holds for every monotone function $p$. 
**Theorem 1.2.** If $p$ is a strictly monotone real function on $J$ and $q$ is a strictly increasing real function on $J$, such that $q \circ p^{-1}$ is strictly convex, then

$$N(f, p) < N(f, q)$$

for each strictly monotone function $f$.

The following Lemmas 1.3 and 1.4 were established by Hutník in [34].

**Lemma 1.3.** Let $f : [a, b] \to [c, d]$ and let $f^{-1}$ be the inverse function to the function $f$.

1. If $f$ is strictly increasing and convex, or a strictly decreasing and concave function on $[a, b]$, then $f^{-1}$ is a concave function on $[c, d]$.
2. If $f$ is strictly decreasing and convex, or a strictly increasing and concave function on $[a, b]$, then $f^{-1}$ is a convex function on $[c, d]$.

**Lemma 1.4.** Let $\varphi : [a, b] \to [c, d]$ and $h : [c, d] \to \mathbb{R}$.

1. If $\varphi$ is convex on $[a, b]$ and $h$ is convex increasing on $[c, d]$, or $\varphi$ is concave on $[a, b]$ and $h$ is convex decreasing on $[c, d]$, then $h \circ \varphi$ is convex on $[c, d]$.
2. If $\varphi$ is convex on $[a, b]$ and $h$ is concave decreasing on $[c, d]$, or $\varphi$ is concave on $[a, b]$ and $h$ is concave increasing on $[c, d]$, then $h \circ \varphi$ is concave on $[c, d]$.

In general measure theoretical notation the Jensen inequality theorem sounds as follows: let $(\Omega, A, \mu)$ be a measurable space, such that $\mu(\Omega) = 1$. If $f$ is a real $\mu$-integrable function and $\phi$ is a convex (concave) function on the range of $f$, then

$$\phi \left( \int_{\Omega} f \, d\mu \right) \leq \left( \geq \right) \int_{\Omega} \phi \circ f \, d\mu.$$

The purpose of this paper is to establish some new and interesting inequalities for the Toader-Sándor mean $N(f, p)$.

**2. Main Result**

**Theorem 2.1.** Let $f$ and $p$ be two strictly monotone functions, and $A(a, b) = (a + b)/2$ the arithmetic mean. Then the following statements are true.

1. If $f^{-1}$ and $p^{-1}$ are convex, then

$$N(f, p)(a, b) \leq A(a, b). \quad (2.1)$$

2. If $f^{-1}$ and $p^{-1}$ are concave, then inequality (2.1) is reversed.

**Proof.** We only give the proof of inequality (2.1) in detail, proof of the remaining part is similar.

If $f^{-1}$ is convex, then Jensen inequality implies that

$$N(f, p) = f^{-1} \left( \int_0^1 (f \circ p^{-1})(t \cdot p(a) + (1 - t) \cdot p(b)) \, dt \right)$$

$$\leq \int_0^1 p^{-1}(t \cdot p(a) + (1 - t) \cdot p(b)) \, dt = N(I, p), \quad (2.2)$$
where $I = I(t) = t$ denotes the identity transformation.

If $p^{-1}$ is convex, then

$$N(I, p) = \int_0^1 p^{-1}[t \cdot p(a) + (1 - t) \cdot p(b)]dt$$

$$\leq \int_0^1 [ta + (1 - t)b]dt = \frac{a + b}{2} = A(a, b). \quad (2.3)$$

Therefore, inequality (2.1) follows from inequalities (2.2) and (2.3). \(\square\)

**Theorem 2.2.** Let $p$ be a strictly monotone function, and $f$ a strictly increasing function.

(1) If $f$ is concave, then

$$N(f, p) \leq N(f^{-1}, p). \quad (2.4)$$

(2) If $f$ is convex, then inequality (2.4) is reversed.

**Proof.** If $f$ is strictly increasing and concave, then by Lemma 1.3(2) we know that $f^{-1}$ is convex. Therefore, inequality (2.2) holds again and it follows from Jensen inequality that

$$f^{-1}\left(\int_0^1 p^{-1}[t \cdot p(a) + (1 - t) \cdot p(b)]dt\right)$$

$$\leq \int_0^1 (f^{-1} \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)]dt.$$

Moreover, if $f$ is strictly increasing, then the above inequality leads to

$$N(I, p) = \int_0^1 p^{-1}[t \cdot p(a) + (1 - t) \cdot p(b)]dt$$

$$\leq f\left(\int_0^1 (f^{-1} \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)]dt\right) = N(f^{-1}, p). \quad (2.5)$$

Therefore, inequality (2.4) follows from inequalities (2.2) and (2.5).

The proof of part (2) is similar. \(\square\)

**Theorem 2.3.** Let $f$ and $p$ be two strictly monotone functions and $g$ defined on the range of $f$. Then the following statements are true.

(1) If $g$ is convex and $g \circ f$ is strictly increasing, or $g$ is concave and $g \circ f$ is strictly decreasing, then

$$N(f, p) \leq N(g \circ f, p). \quad (2.6)$$

(2) If $g$ is concave and $g \circ f$ is strictly increasing, or $g$ is convex and $g \circ f$ is strictly decreasing, then inequality (2.6) is reversed.
Proof. If $g$ is convex, then the Jensen inequality implies that

$$g \left( \int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right)$$

$$\leq \int_0^1 (g \circ f \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt.$$ 

If $g \circ f$ is strictly increasing, then $(g \circ f)^{-1}$ exists and is strictly increasing. By composition with $(g \circ f)^{-1}$ on the both sides of the above inequality, one has

$$f^{-1} \left( \int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right)$$

$$\leq (g \circ f)^{-1} \left( \int_0^1 (g \circ f \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right),$$

i.e. inequality (2.6) holds.

The proofs of remaining parts are similar. □

Theorem 2.4. Let $f$ and $p$ be two strictly monotone functions. Then we have

1. If $f \circ g \circ f^{-1}$ is convex and $f \circ g$ is strictly increasing, or $f \circ g \circ f^{-1}$ is concave and $f \circ g$ is strictly decreasing, then

$$N(f, p) \leq N(f \circ g, p). \quad (2.7)$$

2. If $f \circ g \circ f^{-1}$ is convex and $f \circ g$ is strictly decreasing, or $f \circ g \circ f^{-1}$ is concave and $f \circ g$ is strictly increasing, then inequality (2.7) is reversed.

Proof. If $f \circ g \circ f^{-1}$ is convex, then the Jensen inequality implies that

$$(f \circ g \circ f^{-1}) \left( \int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right)$$

$$\leq \int_0^1 (f \circ g \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt.$$ 

If $f \circ g$ is strictly increasing, then $(f \circ g)^{-1}$ exists and is strictly increasing. By composition with $(f \circ g)^{-1}$ on the both sides of the above inequality, we have

$$f^{-1} \left( \int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right)$$

$$\leq (f \circ g)^{-1} \left( \int_0^1 (f \circ g \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right),$$

i.e., inequality (2.7) holds.

The proofs of the remaining parts are similar. □

Theorem 2.5. Let $f$ and $p$ be two strictly monotone functions, then the following statements are true:
(1) If function $g$ is strictly decreasing and $g \circ f^{-1}$ is concave, then
\[ N(f, p) \leq N(g, p). \]  
\hspace{1cm} (2.8)

(2) If function $g$ is strictly decreasing and $g \circ f^{-1}$ is convex, or $g$ is strictly increasing and $g \circ f^{-1}$ is concave, then inequality (2.8) is reversed.

**Proof.** The proof is similar to the proof of Theorem 1.1. For the reader’s convenience, we only give the proof of inequality (2.8) in detail as follows.

If $g \circ f^{-1}$ is concave, then by the Jensen inequality, one has
\[
(g \circ f^{-1}) \left( \int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right) 
\geq \int_0^1 (g \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt.
\]

If $g$ is strictly decreasing, then $g^{-1}$ exists and is strictly decreasing. By composition with $g^{-1}$ on both sides of the above inequality, we get
\[
f^{-1} \left( \int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right) 
\leq g^{-1} \left( \int_0^1 (g \circ p^{-1})[t \cdot p(a) + (1 - t) \cdot p(b)] dt \right).
\]

Therefore, inequality (2.8) holds. 

\[ \Box \]

**Theorem 2.6.** Let $f$ and $p$ be two strictly monotone functions.

(1) If function $q$ is strictly decreasing and $q \circ p^{-1}$ is concave, then
\[ N(f, p) \leq N(f, q). \]  
\hspace{1cm} (2.9)

(2) If function $q$ is strictly increasing and $q \circ p^{-1}$ is concave, or $q$ is strictly decreasing and $q \circ p^{-1}$ is convex, then inequality (2.9) is reversed.

**Proof.** The proof is similar to that of Theorem 1.2. For convenience of the reader, we only give the proof of (2.9) in detail.

Let $p(a) = c$, $p(b) = d$. If $q \circ p^{-1}$ is concave, then
\[
(q \circ p^{-1})[tc + (1 - t)d] \geq t(q \circ p^{-1})(c) + (1 - t)(q \circ p^{-1})(d).
\]

If $q$ is strictly decreasing, then $q^{-1}$ exists and is strictly decreasing. By composition with $q^{-1}$ on both sides of the above inequality, one has
\[
p^{-1}[tp(a) + (1 - t)p(b)] \leq q^{-1}[tq(a) + (1 - t)q(b)]. \hspace{1cm} (2.10)
\]

We divide the discussion into two cases.

**Case 1** $f$ is strictly increasing. Then it follows from inequality (2.10) that
\[
(f \circ p^{-1})[tp(a) + (1 - t)p(b)] \leq (f \circ q^{-1})[tq(a) + (1 - t)q(b)]. \hspace{1cm} (2.11)
\]
Inequality (2.11) leads to

\[ \int_0^1 (f \circ p^{-1})(tp(a) + (1 - t)p(b))dt \leq \int_0^1 (f \circ q^{-1})(tq(a) + (1 - t)q(b))dt. \ (2.12) \]

Composing with \( f^{-1} \) on both sides of inequality (2.12), we obtain the desired inequality (2.9).

**Case 2** If \( f \) is strictly decreasing. Both inequalities (2.11) and (2.12) are reversed. But the inequality (2.9) holds again. \( \Box \)

**Theorem 2.7.** Let \( f \) be a strictly monotone function and \( p \) a strictly increasing function, and \( p : [a, b] \to [c, d] \) with \([c, d] \subseteq [a, b] \).

1. If \( p \) is convex, then

\[ N(f, p^{-1})(c, d) \leq N(f, p)(c, d). \ (2.13) \]

2. If \( p \) is concave, then inequality (2.13) is reversed.

**Proof.** We only give the proof of inequality (2.13) in detail.

If \( p \) is convex, then

\[ p(t \cdot p^{-1}(c) + (1 - t)p^{-1}(d)) \leq tc + (1 - t)d. \ (2.14) \]

If \( p \) is strictly increasing and convex, then Lemma 1.3(1) implies that \( p^{-1} \) exists and is concave. Therefore,

\[ tc + (1 - t)d = t(p^{-1} \circ p)(c) + (1 - t)(p^{-1} \circ p)(d) \]
\[ \leq p^{-1}(t \cdot p(c) + (1 - t) \cdot p(d)). \ (2.15) \]

Inequalities (2.14) and (2.15) implies that

\[ p(t \cdot p^{-1}(c) + (1 - t)p^{-1}(d)) \leq p^{-1}(t \cdot p(c) + (1 - t) \cdot p(d)) \ (2.16) \]

if \( p \) is strictly increasing and convex.

We divide the discuss into two cases.

**Case 1** \( f \) is strictly increasing. Then it follows from (2.16) that

\[ (f \circ p)(t \cdot p^{-1}(c) + (1 - t)p^{-1}(d)) \leq (f \circ p^{-1})(t \cdot p(c) + (1 - t) \cdot p(d)). \ (2.17) \]

It follows from (2.17) that

\[ \int_0^1 (f \circ p)(t \cdot p^{-1}(c) + (1 - t)p^{-1}(d))dt \leq \int_0^1 (f \circ p^{-1})(t \cdot p(c) + (1 - t) \cdot p(d))dt. \ (2.18) \]
Therefore, inequality (2.13) follows from (2.18).

Case 2. If \( f \) is decreasing. Then inequalities (2.17) and (2.18) are reversed. But inequality (2.13) holds again. \( \square \)

**Theorem 2.8.** Let \( f \) and \( p \) be two strictly monotone functions and \( q \) a strictly decreasing function. Then the following statements are true.

1. If \( q \) is concave and \( p \) is convex and strictly decreasing, or \( q \) is concave and \( p \) is concave and strictly increasing, then
   \[ N(f, p) \leq N(f, p \circ q). \] (2.19)

2. If \( q \) is convex and \( p \) is concave and strictly decreasing, or \( q \) is convex and \( p \) is convex and strictly increasing, then inequality (2.19) is reversed.

**Proof.** If \( p \) is convex and strictly decreasing, or concave and strictly increasing, then Lemma 1.3(2) implies that \( p^{-1} \) is convex. Therefore,

\[
p^{-1}[tp(a) + (1 - t)p(b)] \leq ta + (1 - t)b. \] (2.20)

If \( q \) is concave and strictly decreasing, then it follows from Lemma 1.3(1) that \( q^{-1} \) is concave and decreasing. Because \( p^{-1} \) is convex, Lemma 1.4(2) leads to that \( q^{-1} \circ p^{-1} = (p \circ q)^{-1} \) is concave. Therefore,

\[
ta + (1 - t)b \leq (p \circ q)^{-1}[t(p \circ q)(a) + (1 - t)(p \circ q)(b)]. \] (2.21)

From inequalities (2.20) and (2.21) we get

\[
p^{-1}[tp(a) + (1 - t)p(b)] \leq (p \circ q)^{-1}[t(p \circ q)(a) + (1 - t)(p \circ q)(b)]. \] (2.22)

Therefore, inequality (2.19) follows easily from inequality (2.22).

The proof of part (2) is similar. \( \square \)

**Theorem 2.9.** Let \( f, p \) and \( q \) be three strictly monotone functions.

1. If \( p^{-1} \) is strictly increasing and \( q^{-1} \) is concave, or \( p^{-1} \) is strictly decreasing and \( q^{-1} \) is convex, then
   \[ N(f, p) \leq N(f, q \circ p). \] (2.23)

2. If \( p^{-1} \) is strictly decreasing and \( q^{-1} \) is concave, or \( p^{-1} \) is strictly increasing and \( q^{-1} \) is convex, then inequality (2.23) is reversed.

**Proof.** If \( q^{-1} \) is concave, then

\[
tp(a) + (1 - t)p(b) \leq q^{-1}[t(q \circ p)(a) + (1 - t)(q \circ p)(b)]. \]

It follows from the strict monotonicity of \( p^{-1} \) that

\[
p^{-1}[tp(a) + (1 - t)p(b)] \leq (q \circ p)^{-1}[t(q \circ p)(a) + (1 - t)(q \circ p)(b)]. \] (2.24)

Therefore, inequality (2.23) follows easily from (2.24).

The proofs of the remaining parts are similar. \( \square \)
References


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