Some Identities on the Generalized q -Euler Polynomials with Weak Weight

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

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Abstract

In this paper, we study the symmetry for generalized q -Euler numbers $E_{n,\chi,q}^{(\alpha)}$ and polynomials $E_{n,\chi,q}^{(\alpha)}(x)$ with weak weight α . We obtain some interesting identities of the power sums and generalized q -Euler polynomials $E_{n,\chi,q}^{(\alpha)}(x)$ using the symmetric properties for the p -adic invariant q -integral on $\mathbb{Z}_p$.

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1 Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p -adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p -adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q -extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$ (see [1-9]). Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$

Hence, $\lim_{q \rightarrow 1} [x] = x$ for any x with $|x|_p \leq 1$ in the present p -adic case.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the fermionic p -adic invariant q -integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [1, 2] .} \quad (1.1)$$

If we take $g_n(x) = g(x+n)$ in (1.1), then we see that

$$q^n I_q(g_n) + (-1)^{n-1} I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \quad (1.2)$$

Note that

$$\lim_{q \rightarrow 1} I_{-q}(g) = I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x).$$

Let a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N \mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$ (cf. [1, 2, 3, 4, 5, 6]). In [8], we introduced generalized q -Euler numbers $E_{n,\chi,q}^{(\alpha)}$ and polynomials $E_{n,\chi,q}^{(\alpha)}(x)$ attached to χ . Let χ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. We assume that $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$. Let $g(y) = \chi(y)e^{(y+x)t}$. By (1.1), we derive

$$\begin{aligned} \int_X \chi(y) e^{(y+x)t} d\mu_{-q^\alpha}(y) &= \frac{[2]_{q^\alpha} \sum_{a=0}^{d-1} \chi(a) (-1)^a q^{\alpha a} e^{at}}{q^{\alpha d} e^{dt} + 1} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(\alpha)}(x) \frac{t^n}{n!}. \end{aligned} \quad (1.3)$$

By using Taylor series of $e^{(y+x)t}$ in the above equation (1.3), we obtain

$$\sum_{n=0}^{\infty} \left(\int_X \chi(y) (y+x)^n d\mu_{-q^\alpha}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(\alpha)}(x) \frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the generalized q -Euler polynomials attached to χ as follows:

Theorem 1.1 *For positive integers n , we have*

$$E_{n,\chi,q}^{(\alpha)}(x) = \int_X \chi(y)(y+x)^n d\mu_{-q^\alpha}(y). \quad (1.4)$$

Observe that for $x = 0$, the equation (1.4) reduces to (1.5).

Corollary 1.2 *For positive integers n , we have*

$$E_{n,\chi,q}^{(\alpha)} = \int_X \chi(y)y^n d\mu_{-q^\alpha}(y). \quad (1.5)$$

By (1.4) and (1.5), we have the following theorem.

Theorem 1.3 *For positive integers n , we have*

$$E_{n,\chi,q}^{(\alpha)}(x) = \sum_{l=0}^n \binom{n}{l} E_{l,\chi,q}^{(\alpha)} x^{n-l}.$$

2 Some identities for generalized q -Euler polynomials

In this section, we assume that $q \in \mathbb{C}_p$. We obtain some interesting identities of the power sums and generalized q -Euler polynomials $E_{n,\chi,q}^{(\alpha)}(x)$ using the symmetric properties for the p -adic invariant q -integral on $\mathbb{Z}_p$. If n is odd from (1.2), we obtain

$$q^n I_q(g_n) + I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l g(l). \quad (2.1)$$

Substituting $g(x) = \chi(x)e^{xt}$ into the above, we obtain

$$\begin{aligned} & q^{\alpha nd} \int_X \chi(x+nd) e^{(x+nd)t} d\mu_{-q^\alpha}(x) + \int_X \chi(x) e^{xt} d\mu_{-q^\alpha}(x) \\ &= [2]_{q^\alpha} \sum_{j=0}^{nd-1} (-1)^j \chi(j) q^{\alpha j} e^{jt}. \end{aligned} \quad (2.2)$$

For $k \in \mathbb{Z}_+$, let us define the p -adic functional $T_{k,\chi,q}^{(\alpha)}(n)$ as follows:

$$T_{k,\chi,q}^{(\alpha)}(n) = \sum_{l=0}^n (-1)^l \chi(l) q^{\alpha l} l^k. \quad (2.3)$$

After some elementary calculations, we have

$$\begin{aligned} & q^{\alpha nd} \int_X \chi(x) e^{(x+nd)t} d\mu_{-q^\alpha}(x) + \int_X \chi(x) e^{xt} d\mu_{-q^\alpha}(x) \\ &= (1 + q^{\alpha nd} e^{ndt}) \frac{[2]_{q^\alpha} \sum_{a=0}^{d-1} \chi(a) (-1)^a q^{\alpha a} e^{at}}{q^{\alpha d} e^{dt} + 1}. \end{aligned}$$

From the above, we get

$$\begin{aligned} & q^{\alpha nd} \int_X \chi(x) e^{(x+nd)t} d\mu_{-q^\alpha}(x) + \int_X \chi(x) e^{xt} d\mu_{-q^\alpha}(x) \\ &= \frac{2 \int_X \chi(x) e^{xt} d\mu_{-q^\alpha}(x)}{\int_{\mathbb{Z}_p} q^{\alpha ndx} e^{ndtx} d\mu_{-1}(x)}. \end{aligned} \quad (2.4)$$

By (2.2), (2.3), and (2.4), we arrive at the following theorem:

Theorem 2.1 *Let n be odd positive integer. Then we obtain*

$$\frac{\int_X \chi(x) e^{xt} d\mu_{-q^\alpha}(x)}{\int_{\mathbb{Z}_p} q^{\alpha ndx} e^{ndtx} d\mu_{-1}(x)} = \sum_{m=0}^{\infty} \left(\frac{[2]_{q^\alpha}}{2} T_{m,\chi,q}^{(\alpha)}(nd-1) \right) \frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. By Theorem 2.1, and after some elementary calculations, we have the following theorem.

Theorem 2.2 *Let w_1 and w_2 be odd positive integers. Then we have*

$$\frac{\int_X \chi(x) e^{w_2 xt} d\mu_{-q^{w_2}}(x)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 dx} e^{w_1 w_2 tdx} d\mu_{-1}(x)} = \frac{[2]_{q^{w_2}}}{2} \sum_{m=0}^{\infty} (T_{m,\chi,q}^{(w_2)}(w_1 d - 1) w_2^m) \frac{t^m}{m!}.$$

Then we set

$$S(w_1, w_2) = \frac{\int_X \int_X \chi(x_1) \chi(x_2) e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x) t} d\mu_{-q^{w_1}}(x_1) d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 dx} e^{w_1 w_2 tdx} d\mu_{-1}(x)}.$$

By $S(w_1, w_2)$ and Theorem 2.2, after elementary calculations, we obtain

$$\begin{aligned} & S(w_1, w_2) \\ &= \left(\int_X \chi(x_1) e^{(w_1 x_1 + w_1 w_2 x) t} d\mu_{-q^{w_1}}(x_1) \right) \left(\frac{\int_X \chi(x_2) e^{x_2 w_2 t} d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 dx} e^{w_1 w_2 tdx} d\mu_{-1}(x)} \right) \\ &= \left(\sum_{m=0}^{\infty} E_{m,\chi,q}^{(w_1)}(w_2 x) w_1^m \frac{t^m}{m!} \right) \left(\frac{[2]_{q^{w_2}}}{2} \sum_{m=0}^{\infty} T_{m,\chi,q}^{(w_2)}(w_1 d - 1) w_2^m \frac{t^m}{m!} \right). \end{aligned}$$

By using Cauchy product in the above, we obtain

$$S(w_1, w_2) = \sum_{m=0}^{\infty} \left(\frac{[2]_{q^{w_2}}}{2} \sum_{j=0}^m \binom{m}{j} E_{j,\chi,q}^{(w_1)}(w_2 x) w_1^j T_{m-j,\chi,q}^{(w_2)}(w_1 d - 1) w_2^{m-j} \right) \frac{t^m}{m!}.$$

From the symmetry of $S(w_1, w_2)$ in w_1 and w_2 , we also see that

$$\begin{aligned} S(w_1, w_2) &= \left(\int_X \chi(x_2) e^{(w_2 x_2 + w_1 w_2 x) t} d\mu_{-q^{w_2}}(x_2) \right) \left(\frac{\int_X \chi(x_1) e^{x_1 w_1 t} d\mu_{-q^{w_1}}(x_1)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 dx} e^{w_1 w_2 dx t} d\mu_{-1}(x)} \right) \\ &= \left(\sum_{m=0}^{\infty} E_{m, \chi, q}^{(w_2)}(w_1 x) w_2^m \frac{t^m}{m!} \right) \left(\frac{[2]_{q^{w_1}}}{2} \sum_{m=0}^{\infty} T_{m, \chi, q}^{(w_1)}(w_2 d - 1) w_1^m \frac{t^m}{m!} \right). \end{aligned}$$

Thus we obtain

$$S(w_1, w_2) = \sum_{m=0}^{\infty} \left(\frac{[2]_{q^{w_1}}}{2} \sum_{j=0}^m \binom{m}{j} E_{j, \chi, q}^{(w_2)}(w_1 x) w_2^j T_{m-j, \chi, q}^{(w_1)}(w_2 d - 1) w_1^{m-j} \right) \frac{t^m}{m!}.$$

Thus we arrive at the following theorem:

Theorem 2.3 *Let w_1 and w_2 be odd positive integers. Then we have*

$$\begin{aligned} [2]_{q^{w_1}} \sum_{j=0}^m \binom{m}{j} w_1^{m-j} w_2^j E_{j, \chi, q}^{(w_2)}(w_1 x) T_{m-j, \chi, q}^{(w_1)}(w_2 d - 1) \\ = [2]_{q^{w_2}} \sum_{j=0}^m \binom{m}{j} w_1^j w_2^{m-j} E_{j, \chi, q}^{(w_1)}(w_2 x) T_{m-j, \chi, q}^{(w_2)}(w_1 d - 1), \end{aligned}$$

where $E_{k, \chi, q}^{(w_1)}(x)$ and $T_{m, \chi, q}^{(w_1)}(k)$ denote generalized q -Euler polynomials with weak weight w_1 and p -adic functional, respectively.

By Theorem 1.3 and Theorem 2.3, we have the following corollary.

Corollary 2.4 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned} [2]_{q^{w_1}} \sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j E_{k, \chi, q}^{(w_2)} T_{m-j, \chi, q}^{(w_1)}(w_2 d - 1) x^{j-k} \\ = [2]_{q^{w_2}} \sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} \binom{j}{k} w_1^j w_2^{m-k} E_{k, \chi, q}^{(w_1)} T_{m-j, \chi, q}^{(w_2)}(w_1 d - 1) x^{j-k}. \end{aligned}$$

If we take $x = 0$ in Theorem 2.3, we also derive the interesting identity for generalized q -Euler numbers with weak weight as follows:

Corollary 2.5 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned} [2]_{q^{w_1}} \sum_{j=0}^m \binom{m}{j} w_1^{m-j} w_2^j E_{j, \chi, q}^{(w_2)} T_{m-j, \chi, q}^{(w_1)}(w_2 d - 1) \\ = [2]_{q^{w_2}} \sum_{j=0}^m \binom{m}{j} w_1^j w_2^{m-j} E_{j, \chi, q}^{(w_1)} T_{m-j, \chi, q}^{(w_2)}(w_1 d - 1). \end{aligned}$$

By substituting Taylor series of e^{xt} into (2.2), we obtain

$$\begin{aligned} & \sum_{m=0}^{\infty} \left(q^{\alpha nd} \int_X \chi(x+nd)(x+nd)^m d\mu_{-q^\alpha}(x) + \int_X \chi(x)x^m d\mu_{-q^\alpha}(x) \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left([2]_{q^\alpha} \sum_{j=0}^{nd-1} (-1)^j \chi(j) q^{\alpha j} j^m \right) \frac{t^m}{m!}. \end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation and (2.3), we have the following theorem.

Theorem 2.6 *Let χ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then we have*

$$T_{m,\chi,q}^{(\alpha)}(nd-1) = \frac{q^{\alpha nd} E_{m,\chi,q}^{(\alpha)}(nd) + E_{m,\chi,q}^{(\alpha)}}{[2]_{q^\alpha}}.$$

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