Asymptotic Frequencies of Beams Left Fixed and Right Supported

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Abstract

The purpose of this paper is to construct the asymptotic for natural frequencies of the Euler-Bernoulli beam problem with non-uniform section using the method of Wentzel-Kramers-Brillouin (WKB) and find the secular equation for the eigenvalues a non-uniform beam embedded at one end and supported at the other end.

Mathematics Subject Classification: 35B40

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1 Introduction

The beam is one of the structural elements most used. Its importance lies in the strength of materials, the completeness of which is taught in schools stress analysis techniques in the beams, so as its wide use in all types of bridges, buildings, trucks, etc.

Therefore, the vibration of beams has been studied extensively and continues to receive attention in the current literature. Initially, Daniel Bernoulli in the eighteenth century became interested for determining the elasticity of the rods, the vibrations of the beams and developing the differential equation for the vibration a bar. Leonhard Euler was born in the eighteenth century in Basel (Switzerland), influenced by Daniel Bernoulli study problems elasticity of the beams and columns, use the method of minimum work and contributed his valuable discussion on “ buckling of the columns. ”

(Mabie, Rogers [6], 1974) studied various cases of vibrations cantilevered transverse beams with different conditions ends. (Naguleswaran [7], 1994) found a solution for transverse vibration of Euler Bernoulli beams and conical beams. In (Abrate [1], 1995), were obtained natural frequencies Euler Bernoulli beams (EB) when solution of the equation of EB was expressed by elementary functions. In (Hsu, Lai, Chen [5], 2008) was studied vibration of non-uniform beams using the Euler Bernoulli modified decomposition method of Adomian.

For the equation of the beam with a uniform small parameter, you can use the method (WKB) Wentzel - Kramers - Brillouin, also known in the literature as the approximation Liouville - Green. In (Akulenko, Nesterov [2], 2005) WKB method was used for finding asymptotic high frequency, this method is to obtain asymptotic series for solutions powers with a small parameter.

Although the WKB method is developed for differential equations of n -th order (Fedoryuk [4], 1993), this technique was applied in the equation EB fourth order in (Aya, Cano, Zhevandrov [3], 2011) to non-uniform embedded and cantilevered beams.

In the next section one takes the Bernoulli Euler equation, for non-uniform beams.

2 Mathematical formulation

In this section we find the approximate solution as a linear combination of four linearly independent solutions. After imposing the boundary conditions for the case of fixed at one end and supported at the other, which is reduced to a system of four equations with four unknowns which has nontrivial solutions if its determinant is zero. And finally we obtain the asymptotic determinant equation when $\epsilon \rightarrow 0$. Then we have eigenvalues.

2.1 Euler-Bernoulli equation

We take the Euler–Bernoulli equation for transverse vibrations in the nonuniform beam approximation have been described by (Tijonov–Samarsky [8], 1972)

$$\rho S(x) \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(EJ(x) \frac{\partial^2 y}{\partial x^2} \right) = 0, \quad (1)$$

where E is the Young's modulus of the beam, ρ is the density, $S(x)$ is the cross-sectional area, $J(x)$ is the moment of inertia of the cross section with respect to its horizontal axis and y is the transverse displacement. It is assumed that functions $S(x)$ and $J(x)$ are smooth and positive $0 \leq x \leq l$ where l is the length of the beam.

Looking for the solution of the equation of Bernoulli (1) in the form

$$y(x, t) = e^{i\omega t} v(x), \quad (2)$$

where ω is the frequency. Substituting (2) in the equation (1) we obtain fourth order ordinary differential equation for $v(x)$

$$-\omega^2 \rho S(x) v(x) + (EJ(x) v''(x))'' = 0. \quad (3)$$

Taking the boundary conditions for beams fixed in an end and supported at the other. Equation (3) is transformed into the following problem:

$$\begin{aligned} \frac{1}{\rho S(x)} (EJ(x) v''(x))'' &= \omega^2 v(x). \\ v(0) = v(l) = 0, v'(0) &= v''(l) = 0. \end{aligned} \quad (4)$$

The discrete spectrum of the problem (EB) (4) constitutes a sequence w_n of real numbers tending to infinity when $n \rightarrow \infty$. In view of the above can be consider $w^2 = \frac{1}{\epsilon^4}$, where $\epsilon \rightarrow 0$ (Akulenko, Nesterov [2], 2005). Therefore the problem (EB), (4) becomes

$$\frac{1}{\rho S(x)} (EJ(x) v''(x))'' = \frac{1}{\epsilon^4} v(x). \quad (5)$$

$$v(0) = v(l) = 0, v'(0) = v''(l) = 0. \quad (6)$$

In the next section we calculate the secular equation of the problem (5)–(6).

3 Main result

This section states and solves the problem of Euler Bernoulli beams at high frequency, which is applied the WKB method.

The mathematical formulation of the problem is the searching for nontrivial solutions of the problem (5)–(6).

The main result is as follows

Theorem 3.1. *The eigenvalues of the problem (5)–(6) are given by $w_n^2 = \frac{1}{\varepsilon_n^4}$, with*

$$\omega_n = \left(\frac{(4n+1)\pi}{4 \int_0^l a^{-\frac{1}{4}}(x) dx} \right)^2 \left(1 + O\left(\frac{1}{n}\right) \right), \quad n = 0, 1, \dots, \quad n \rightarrow \infty,$$

where ε_n is the solution of the secular equation (25). The function $a^{-\frac{1}{4}}(x)$ is defined by equation (23)

Proof. Given that $\omega^2 = \frac{1}{\varepsilon^4}$ then equation (5) can be transformed into

$$v(x) = \frac{\varepsilon^4}{\rho S(x)} (EJ(x)v''(x))''$$

$$v(x) = \frac{\varepsilon^4}{\rho S(x)} (EJ''(x)v''(x) + 2EJ'(x)v'''(x) + EJ(x)v''''(x)). \quad (7)$$

Following the traditional WKB method, the analytical solution approximates equation (5) can be replaced by a power series given by the following

$$v(x) = A(x, \varepsilon) e^{\frac{i\phi(x)}{\varepsilon}}, \quad \varepsilon \rightarrow 0, \quad (8)$$

where

$$A(x, \varepsilon) = A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \dots, \quad \varepsilon \rightarrow 0, \quad (9)$$

with $\phi(x)$ and $A_j(x)$, $j = 0, 1, 2, \dots$ are smooth functions and unknown.

Replacing (8) and each of the derivatives of $v(x)$ in (7), we have the following expression

$$A(x, \varepsilon) = \frac{E}{\rho S(x)} [J(x)\phi_x^4(x)A(x, \varepsilon) - 2i\varepsilon(2J(x)\phi_x^3(x)A_x(x, \varepsilon) \quad (10)$$

$$+ (J_x(x)\phi_x^3(x) + J(x)3\phi_x^2(x)\phi_{xx}(x))A(x, \varepsilon)] + O(\varepsilon^2)$$

$$= \frac{E}{\rho S(x)} [J(x)\phi_x^4(x)A(x, \varepsilon) - 2i\varepsilon(2J(x)\phi_x^3(x)A_x(x, \varepsilon) \quad (11)$$

$$+ (J(x)\phi_x^3(x))'A(x, \varepsilon)] + O(\varepsilon^2), \quad \varepsilon \rightarrow 0.$$

Replacing (9) on both sides of (10) we obtain

$$\begin{aligned}
A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \cdots &= \frac{E}{\rho S(x)} [J(x)\phi_x^4(x)(A_0(x) + \varepsilon A_1(x) \\
&+ \varepsilon^2 A_2(x) + \cdots) - 2i\varepsilon(2J(x)\phi_x^3(x)(A_{0x}(x) + \varepsilon A_{1x}(x) + \varepsilon^2 A_{2x}(x) + \cdots)) \\
&\quad + (J(x)\phi_x^3(x))'(A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \cdots)] + O(\varepsilon^2) \\
&= \frac{EJ(x)}{\rho S(x)}\phi_x^4(x)A_0(x) + \varepsilon \left[\frac{EJ(x)}{\rho S(x)}\phi_x^4(x)A_1(x) \right. \\
&\quad \left. - \frac{4iEJ(x)}{\rho S(x)}\phi_x^3(x)A_{0x}(x) - \frac{2iE}{\rho S(x)}(J(x)\phi_x^3(x))'A_0(x) \right] \\
&\quad + O(\varepsilon^2), \quad \varepsilon \rightarrow 0.
\end{aligned} \tag{12}$$

Equating the coefficients of the asymptotic series in ε and taking corresponding to ε^0 in (12) and using that $A_0 \neq 0$ as seen in the equation (16), it can be obtained

$$\varepsilon^0 : \frac{EJ(x)}{\rho S(x)}\phi_x^4(x) = 1. \tag{13}$$

From equation (13) and choosing the corresponding equality to ε^1 in (12), we obtain

$$\varepsilon^1 : 2J(x)\phi_x^3(x)A_{0x}(x) + (J(x)\phi_x^3(x))'A_0(x) = 0. \tag{14}$$

Equating the asymptotic series are obtained more equations but we will consider only the first two equations, other equations are of order $O(\varepsilon^2)$. Since equation (13) has two real roots and two pure imaginary roots with opposite signs, we obtain

$$\phi_k(x) = (i)^k \int \left(\frac{EJ(x)}{\rho S(x)} \right)^{-\frac{1}{4}} dx, \quad k = 1, 2, 3, 4. \tag{15}$$

From the equation (14) and separating the functions $A_0(x)$, $J(x)\phi_x(x)^3$ and integrating on both sides, it follows that

$$A_0(x) = C(J(x)\phi_x(x)^3)^{-\frac{1}{2}}, \tag{16}$$

where C is a non-zero arbitrary constant. Therefore, differentiating with respect to x in the equation (15) and substituting (16), function $A_0(x)$ can be expressed as follows

$$\begin{aligned} A_0(x) &= C \left(J(x) \left(\frac{J(x)}{S(x)} \right)^{-\frac{3}{4}} \right)^{-\frac{1}{2}} \\ &= C \left(J^{\frac{1}{4}}(x) S^{\frac{3}{4}}(x) \right)^{-\frac{1}{2}} \\ &= C J^{-\frac{1}{8}}(x) S^{-\frac{3}{8}}(x). \end{aligned} \quad (17)$$

Therefore, replacing (15) and (17) in (8) which is the solution $v(x)$ of (5), we have

$$v_1(x) = d_1 J^{-\frac{1}{8}}(x) S^{-\frac{3}{8}}(x) \sin \left(\frac{\psi_1(x)}{\varepsilon} \right) + O(\varepsilon), \quad \varepsilon \rightarrow 0, \quad (18)$$

$$v_2(x) = d_2 J^{-\frac{1}{8}}(x) S^{-\frac{3}{8}}(x) \cos \left(\frac{\psi_1(x)}{\varepsilon} \right) + O(\varepsilon), \quad \varepsilon \rightarrow 0, \quad (19)$$

$$v_3(x) = d_3 J^{-\frac{1}{8}}(x) S^{-\frac{3}{8}}(x) e^{-\frac{\psi_1(x)}{\varepsilon}} + O(\varepsilon), \quad \varepsilon \rightarrow 0, \quad (20)$$

$$v_4(x) = d_4 J^{-\frac{1}{8}}(x) S^{-\frac{3}{8}}(x) e^{-\frac{\psi_2(x)}{\varepsilon}} + O(\varepsilon), \quad \varepsilon \rightarrow 0, \quad (21)$$

and writing the solution (5) as the first term of the linear combination of $v_1(x), v_2(x), v_3(x), v_4(x)$

$$\begin{aligned} v(x) &= J^{-\frac{1}{8}}(x) S^{-\frac{3}{8}}(x) \\ &\left(c_1 \sin \left(\frac{\psi_1(x)}{\varepsilon} \right) + c_2 \cos \left(\frac{\psi_1(x)}{\varepsilon} \right) + c_3 e^{-\frac{\psi_1(x)}{\varepsilon}} + c_4 e^{-\frac{\psi_2(x)}{\varepsilon}} \right), \end{aligned} \quad (22)$$

$\varepsilon \rightarrow 0$, where

$$\psi_1(x) = \int_0^x a^{-\frac{1}{4}}(t) dt, \quad \psi_2(x) = \int_x^l a^{-\frac{1}{4}}(t) dt, \quad a(t) = \frac{EJ(t)}{\rho S(t)}, \quad (23)$$

and c_1, c_2, c_3, c_4 are constants. It is noted that $\sin \left(\frac{\psi_1}{\varepsilon} \right), \cos \left(\frac{\psi_1}{\varepsilon} \right), e^{-\frac{\psi_1}{\varepsilon}}, e^{-\frac{\psi_2}{\varepsilon}}$ are linearly independent of the fact that $\psi_2(x) = L - \psi_1(x)$, where L is defined in (26) and since $\sin \left(\frac{\psi_1}{\varepsilon} \right), \cos \left(\frac{\psi_1}{\varepsilon} \right), e^{-\frac{\psi_1}{\varepsilon}}, e^{\frac{\psi_1}{\varepsilon}}$ are linearly independent.

The boundary conditions for a beam fixed on the left and supported on the right side are given by

$$v(0) = v(l) = 0, \quad \varepsilon v'(0) = \varepsilon^2 v''(l) = 0. \quad (24)$$

From the solution (22) and boundary conditions (24), yields a homogeneous system of four equations for four constants $c_i, i = 1, 2, 3, 4$. This system has nontrivial solutions when

$$\begin{vmatrix} 0 & -1 & -1 & -e^{-\frac{L}{\varepsilon}} \\ -\operatorname{sen}\left(\frac{L}{\varepsilon}\right) & -\cos\left(\frac{L}{\varepsilon}\right) & -e^{-\frac{L}{\varepsilon}} & -1 \\ -\gamma & \frac{\alpha\varepsilon}{8} & \gamma + \frac{\alpha\varepsilon}{8} & \left(\frac{\alpha\varepsilon}{8} - \gamma\right)e^{-\frac{L}{\varepsilon}} \\ R_1 & R_2 & R_3 & R_4 \end{vmatrix} = 0, \quad (25)$$

where

$$\begin{aligned} L &= \int_0^l a^{-\frac{1}{4}}(t) dt, \quad \alpha = J^{-1}(0)J'(0) + 3S^{-1}(0)S'(0), \quad \theta = (\psi'_1(l))^2, \\ \beta &= \frac{3(S''(l))}{8S(l)} - \frac{33(S'(l))^2}{64S(l)^2} - \frac{3J'(l)S'(l)}{32J(l)S(l)} + \frac{J''(l)}{8J(l)} - \frac{9(J'(l))^2}{64J(l)^2}, \\ \mu &= -\psi''_1(l) + \frac{\psi'_1(l)J'(l)}{4J(l)} + \frac{3\psi'_1(l)S'(l)}{4S(l)}, \quad \gamma = a^{-\frac{1}{4}}(0), \end{aligned} \quad (26)$$

and

$$\begin{aligned} R_1 &= (\theta + \beta\varepsilon^2) \operatorname{sen}\left(\frac{L}{\varepsilon}\right) + \mu\varepsilon \cos\left(\frac{L}{\varepsilon}\right), \\ R_2 &= -\mu\varepsilon \operatorname{sen}\left(\frac{L}{\varepsilon}\right) + (\theta + \beta\varepsilon^2) \cos\left(\frac{L}{\varepsilon}\right), \\ R_3 &= (-\theta - \mu\varepsilon + \beta\varepsilon^2)e^{-\frac{L}{\varepsilon}}, \\ R_4 &= -\theta + \mu\varepsilon + \beta\varepsilon^2. \end{aligned}$$

The equation (25) is the secular equation for natural frequency $\omega_n = \varepsilon_n^{-2}$.

They get the asymptotic fundamental system of solutions with the precision $O(\varepsilon)$, when $\varepsilon \rightarrow 0$, we have to eliminate higher order terms (e.g, $e^{-\frac{L}{\varepsilon}} = O(\varepsilon^\infty)$, $\varepsilon \rightarrow 0$) en (25). It is understood that $O(\varepsilon^\infty)$ is the same as $O(\varepsilon^n)$ for all $n \in \mathbb{N}$. Therefore the secular equation (25) is

$$\begin{vmatrix} 0 & 1 & 1 & O(\varepsilon^\infty) \\ \operatorname{sen}\left(\frac{L}{\varepsilon}\right) & \cos\left(\frac{L}{\varepsilon}\right) & O(\varepsilon^\infty) & 1 \\ 1 & O(\varepsilon) & -1 + O(\varepsilon) & O(\varepsilon^\infty) \\ -\operatorname{sen}\left(\frac{L}{\varepsilon}\right) + O(\varepsilon) & -\cos\left(\frac{L}{\varepsilon}\right) + O(\varepsilon) & O(\varepsilon^\infty) & 1 + O(\varepsilon) \end{vmatrix} = 0.$$

Therefore

$$\tan\left(\frac{L}{\varepsilon}\right) = 1 + O(\varepsilon), \quad \varepsilon \rightarrow 0, \quad \varepsilon = \varepsilon_n = \frac{4L}{(4n+1)\pi} (1 + O(\varepsilon)) \quad (27)$$

then

$$\omega_n = \left(\frac{(4n+1)\pi}{4 \int_0^l a^{-\frac{1}{4}}(t) dt} \right)^2 \left(1 + O\left(\frac{1}{n}\right) \right), \quad n \rightarrow \infty, n = 0, 1, \dots$$

From (27) and initial conditions, the eigenfunction is

$$v_n(x) = c_3 J^{-\frac{1}{8}}(x) S^{-\frac{3}{8}}(x) \left(\operatorname{sen} \left(\frac{\psi_1(x)}{\varepsilon_n} \right) - \cos \left(\frac{\psi_1(x)}{\varepsilon_n} \right) + e^{-\frac{\psi_1(x)}{\varepsilon_n}} \right),$$

where c_3 is an arbitrary constant. $\square$

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