Asymptotic Formulae for the Square Root of the n -th Perfect Power

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Abstract

In this note we obtain asymptotic formulae for the square root of the n -th perfect power.

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1 Introduction and Lemmas

A natural number of the form m^n where m is a positive integer and $n \geq 2$ is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \dots$$

and they are sequence A001597 in Sloane's Encyclopedia. In this note P_n denotes the n -th perfect power.

A quadratfrei number is a number without square factors, a product of different primes. The first few terms of the integer sequence of quadratfrei numbers are

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, \dots$$

On the other hand, the Möbius function $\mu(n)$ is defined as follows: $\mu(1) = 1$, if n is the product of r different primes, then $\mu(n) = (-1)^r$, if the square of a

prime divides n , then $\mu(n) = 0$. In this note q denotes a quadratfrei number. On the other hand (as usual) p_n denotes the n -th prime number.

We shall need the following theorem.

Theorem 1.1 *Let p_h be the h -th prime with $h \geq 3$, where h is an arbitrary but fixed positive integer. We have the following asymptotic formula*

$$P_n = n^2 + \frac{13}{3}n^{8/6} + \frac{32}{15}n^{32/30} + \left(\sum_{\substack{2 \leq q \leq p_h \\ q \neq 2, 6, 30}} 2\mu(q)n^{1+\frac{2}{q}} \right) + o\left(n^{1+\frac{2}{p_h}}\right) \quad (1)$$

Note that $2 = 1 + \frac{2}{2}$, $\frac{8}{6} = 1 + \frac{2}{6}$ and $\frac{32}{30} = 1 + \frac{2}{30}$.

Proof. See [1].

If $h = 3$ then Theorem 1.1 becomes

$$P_n = n^2 - 2n^{\frac{5}{3}} - 2n^{\frac{7}{5}} + o\left(n^{\frac{7}{5}}\right). \quad (2)$$

If $h = 4$ then Theorem 1.1 becomes

$$P_n = n^2 - 2n^{\frac{5}{3}} - 2n^{\frac{7}{5}} + \frac{13}{3}n^{\frac{4}{3}} - 2n^{\frac{9}{7}} + o\left(n^{\frac{9}{7}}\right). \quad (3)$$

If $h = 5$ then Theorem 1.1 becomes

$$P_n = n^2 - 2n^{\frac{5}{3}} - 2n^{\frac{7}{5}} + \frac{13}{3}n^{\frac{4}{3}} - 2n^{\frac{9}{7}} + 2n^{\frac{6}{5}} - 2n^{\frac{13}{11}} + o\left(n^{\frac{13}{11}}\right).$$

The following lemma is a immediate consequence of the binomial theorem.

Lemma 1.2 *We have the following formulae*

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + O(x^2) \quad (x \rightarrow 0) \quad (4)$$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2}x^2 + O(x^3) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3) \quad (x \rightarrow 0) \quad (5)$$

In this note we obtain asymptotic formulae for $\sqrt{P_n}$.

2 Main Results

Theorem 2.1 *Let p_h be the h -th prime with $h \geq 3$, where h is an arbitrary but fixed positive integer. We have the following asymptotic formula*

$$\sqrt{P_n} = n + \frac{5}{3}n^{2/6} + \frac{1}{15}n^{2/30} + \sum_{\substack{2 \leq q \leq p_h \\ q \neq 2, 6, 30}} \mu(q)n^{2/q} + o\left(n^{2/p_h}\right) \quad (6)$$

Proof. First, we shall prove the theorem if $h = 3$, that is $p_h = 5$. We have (see (2))

$$P_n = n^2 - 2n^{1+\frac{2}{3}} - 2n^{1+\frac{2}{5}} + o\left(n^{1+\frac{2}{5}}\right) = n^2 \left(1 - 2n^{-1+\frac{2}{3}} - 2n^{-1+\frac{2}{5}} + o\left(n^{-1+\frac{2}{5}}\right)\right)$$

Therefore

$$\sqrt{P_n} = n \left(1 - 2n^{-1+\frac{2}{3}} - 2n^{-1+\frac{2}{5}} + o\left(n^{-1+\frac{2}{5}}\right)\right)^{1/2} = n(1+x)^{1/2} \quad (7)$$

where

$$x = -2n^{-1+\frac{2}{3}} - 2n^{-1+\frac{2}{5}} + o\left(n^{-1+\frac{2}{5}}\right) \sim -2n^{-1/3} \quad (8)$$

Equations (7), (8) and (4) give

$$\begin{aligned} \sqrt{P_n} &= n \left(1 + \frac{1}{2} \left(-2n^{-1+\frac{2}{3}} - 2n^{-1+\frac{2}{5}} + o\left(n^{-1+\frac{2}{5}}\right)\right) + O\left(n^{-2/3}\right)\right) \\ &= n - n^{2/3} - n^{2/5} + o\left(n^{2/5}\right) + O\left(n^{2/6}\right) = n - n^{2/3} - n^{2/5} + o\left(n^{2/5}\right) \end{aligned}$$

That is, equation (6) if $h = 3$.

If $h \geq 4$, that is $p_h \geq 7$, equation (1) can be written in the form (see (3))

$$P_n = n^2 - 2n^{1+\frac{2}{3}} - 2n^{1+\frac{2}{5}} + \frac{13}{3}n^{1+\frac{2}{6}} + \frac{32}{15}n^{1+\frac{2}{30}} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} 2\mu(q)n^{1+\frac{2}{q}} + o\left(n^{1+\frac{2}{p_h}}\right) \quad (9)$$

Note that if $7 \leq p_h \leq 29$ equation (9) becomes

$$P_n = n^2 - 2n^{1+\frac{2}{3}} - 2n^{1+\frac{2}{5}} + \frac{13}{3}n^{1+\frac{2}{6}} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} 2\mu(q)n^{1+\frac{2}{q}} + o\left(n^{1+\frac{2}{p_h}}\right)$$

Since $n^{1+\frac{2}{30}} = o\left(n^{1+\frac{2}{p_h}}\right)$.

Equation (9) gives

$$P_n = n^2(1+x) \quad (10)$$

where

$$\begin{aligned} x &= -2n^{-1+\frac{2}{3}} - 2n^{-1+\frac{2}{5}} + \frac{13}{3}n^{-1+\frac{2}{6}} + \frac{32}{15}n^{-1+\frac{2}{30}} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} 2\mu(q)n^{-1+\frac{2}{q}} \\ &+ o\left(n^{-1+\frac{2}{p_h}}\right) \sim -2n^{-1/3} \end{aligned} \quad (11)$$

Equations (10) and (5) give

$$\begin{aligned} \sqrt{P_n} &= n(1+x)^{1/2} = n \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)\right) \\ &= n + \frac{1}{2}nx - \frac{1}{8}nx^2 + nO(x^3) \end{aligned} \quad (12)$$

Equation (11) gives

$$nO(x^3) = nO(n^{-1}) = O(1) = o(n^{2/p_h}) \quad (13)$$

$$\frac{1}{2}nx = -n^{2/3} - n^{2/5} + \frac{13}{6}n^{2/6} + \frac{16}{15}n^{2/30} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} \mu(q)n^{2/q} + o(n^{2/p_h}) \quad (14)$$

$$\begin{aligned} & -\frac{1}{8}nx^2 \\ = & -\frac{1}{8}n \left(-2n^{-1+\frac{2}{3}} - 2n^{-1+\frac{2}{5}} + \frac{13}{3}n^{-1+\frac{2}{6}} + \frac{32}{15}n^{-1+\frac{2}{30}} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} 2\mu(q)n^{-1+\frac{2}{q}} \right)^2 \\ + & o(1) = -\frac{1}{8}n \left(-2n^{-1+\frac{2}{3}} \right)^2 - \frac{1}{8}n2 \left(-2n^{-1+\frac{2}{3}} \right) \left(-2n^{-1+\frac{2}{5}} \right) + O(1) + o(1) \\ = & -\frac{1}{2}n^{2/6} - n^{2/30} + o(n^{2/p_h}) \end{aligned} \quad (15)$$

Substituting (13), (14) and (15) into (12) we find that

$$\begin{aligned} \sqrt{P_n} &= n - n^{2/3} - n^{2/5} + \frac{5}{3}n^{2/6} + \frac{1}{15}n^{2/30} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} \mu(q)n^{2/q} + o(n^{2/p_h}) \\ &= n + \frac{5}{3}n^{2/6} + \frac{1}{15}n^{2/30} + \sum_{\substack{2 \leq q \leq p_h \\ q \neq 2, 6, 30}} \mu(q)n^{2/q} + o(n^{2/p_h}) \end{aligned}$$

That is, equation (6). The theorem is proved.

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References

- [1] R. Jakimczuk, Asymptotic formulae for the n -th perfect power, *J. Integer Seq.* **15** (2012), Article 12.5.5.

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