Asymptotic Formulae for the Square Root of the $n$-th Perfect Power

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Abstract

In this note we obtain asymptotic formulae for the square root of the $n$-th perfect power.

Mathematics Subject Classification: 11A99, 11B99

Keywords: Asymptotic formulae, $n$-th perfect power, square root

1 Introduction and Lemmas

A natural number of the form $m^n$ where $m$ is a positive integer and $n \geq 2$ is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128 \ldots$$

and they are sequence A001597 in Sloane’s Encyclopedia. In this note $P_n$ denotes the $n$-th perfect power.

A quadratfrei number is a number without square factors, a product of different primes. The first few terms of the integer sequence of quadratfrei numbers are

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30 \ldots$$

On the other hand, the Mōbius function $\mu(n)$ is defined as follows: $\mu(1) = 1$, if $n$ is the product of $r$ different primes, then $\mu(n) = (-1)^r$, if the square of a
prime divides \( n \), then \( \mu(n) = 0 \). In this note \( q \) denotes a quadratfrei number.

On the other hand (as usual) \( p_n \) denotes the \( n \)-th prime number.

We shall need the following theorem.

**Theorem 1.1** Let \( p_h \) be the \( h \)-th prime with \( h \geq 3 \), where \( h \) is an arbitrary but fixed positive integer. We have the following asymptotic formula

\[
P_n = n^2 + \frac{13}{3} n^{8/6} + \frac{32}{15} n^{32/30} + \left( \sum_{\substack{2 \leq q \leq p_h \\text{that are prime} \qquad q \not= 2, 6, 30}} 2 \mu(q) n^{1 + \frac{2}{q}} \right) + o\left( n^{1 + \frac{2}{p_h}} \right) \tag{1}
\]

Note that \( 2 = 1 + \frac{2}{2}, \frac{8}{6} = 1 + \frac{2}{6} \) and \( \frac{32}{30} = 1 + \frac{2}{30} \).

Proof. See [1].

If \( h = 3 \) then Theorem 1.1 becomes

\[
P_n = n^2 - 2n^{\phi} - 2n^{\varphi} + o\left( n^{\phi} \right). \tag{2}
\]

If \( h = 4 \) then Theorem 1.1 becomes

\[
P_n = n^2 - 2n^{\phi} - 2n^{\varphi} + \frac{13}{3} n^{\phi} - 2n^{\psi} + o\left( n^{\psi} \right). \tag{3}
\]

If \( h = 5 \) then Theorem 1.1 becomes

\[
P_n = n^2 - 2n^{\phi} - 2n^{\varphi} + \frac{13}{3} n^{\phi} - 2n^{\psi} + 2n^{\varphi} - 2n^{\varphi} + o\left( n^{\varphi} \right). \tag{6}
\]

The following lemma is a immediate consequence of the binomial theorem.

**Lemma 1.2** We have the following formulae

\[
(1 + x)^{1/2} = 1 + \frac{1}{2} x + O(x^2) \quad (x \to 0) \tag{4}
\]

\[
(1 + x)^{1/2} = 1 + \frac{1}{2} x + \frac{1}{4} \left( \frac{1}{2} - 1 \right) x^2 + O(x^3) = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + O(x^3) \quad (x \to 0) \tag{5}
\]

In this note we obtain asymptotic formulae for \( \sqrt{P_n} \).

## 2 Main Results

**Theorem 2.1** Let \( p_h \) be the \( h \)-th prime with \( h \geq 3 \), where \( h \) is an arbitrary but fixed positive integer. We have the following asymptotic formula

\[
\sqrt{P_n} = n + \frac{5}{3} n^{2/6} + \frac{1}{15} n^{2/30} + \sum_{\substack{2 \leq q \leq p_h \\text{that are prime} \qquad q \not= 2, 6, 30}} \mu(q) n^{2/q} + o\left( n^{2/p_h} \right) \tag{6}
\]
Proof. First, we shall prove the theorem if \( h = 3 \), that is \( p_h = 5 \). We have (see (2))

\[
P_n = n^2 - 2n^{1 + \frac{2}{3}} - 2n^{1 + \frac{2}{5}} + o\left(n^{1 + \frac{2}{5}}\right) = n^2 \left(1 - 2n^{-1 + \frac{2}{3}} - 2n^{-1 + \frac{2}{5}} + o\left(n^{-1 + \frac{2}{5}}\right)\right)
\]

Therefore

\[
\sqrt{P_n} = n \left(1 - 2n^{-1 + \frac{2}{3}} - 2n^{-1 + \frac{2}{5}} + o\left(n^{-1 + \frac{2}{5}}\right)\right)^{1/2} = n \left(1 + x\right)^{1/2} \quad (7)
\]

where

\[
x = -2n^{-1 + \frac{2}{3}} - 2n^{-1 + \frac{2}{5}} + o\left(n^{-1 + \frac{2}{5}}\right) \sim -2n^{-1/3} \quad (8)
\]

Equations (7), (8) and (4) give

\[
\sqrt{P_n} = n \left(1 + \frac{1}{2} \left(-2n^{-1 + \frac{2}{3}} - 2n^{-1 + \frac{2}{5}} + o\left(n^{-1 + \frac{2}{5}}\right)\right) + O\left(n^{-2/3}\right)\right)
\]

\[
= n - n^{2/3} - n^{2/5} + o\left(n^{2/5}\right) + O\left(n^{2/6}\right) = n - n^{2/3} - n^{2/5} + o\left(n^{2/5}\right)
\]

That is, equation (6) if \( h = 3 \).

If \( h \geq 4 \), that is \( p_h \geq 7 \), equation (1) can be written in the form (see (3))

\[
P_n = n^2 - 2n^{1 + \frac{2}{3}} - 2n^{1 + \frac{2}{5}} + \frac{13}{3} n^{1 + \frac{2}{5}} + \frac{32}{15} n^{1 + \frac{2}{5}} + \sum_{\tau \leq q \leq p_h, q \neq 30} 2\mu(q)n^{1 + \frac{2}{q}} + o\left(n^{1 + \frac{2}{p_h}}\right) \quad (9)
\]

Note that if \( 7 \leq p_h \leq 29 \) equation (9) becomes

\[
P_n = n^2 - 2n^{1 + \frac{2}{5}} - 2n^{1 + \frac{2}{5}} + \frac{13}{3} n^{1 + \frac{2}{5}} + \frac{32}{15} n^{1 + \frac{2}{5}} + \sum_{\tau \leq q \leq p_h, q \neq 30} 2\mu(q)n^{1 + \frac{2}{q}} + o\left(n^{1 + \frac{2}{p_h}}\right)
\]

Since \( n^{1 + \frac{2}{30}} = o\left(n^{1 + \frac{2}{p_h}}\right) \).

Equation (9) gives

\[
P_n = n^2 \left(1 + x\right) \quad (10)
\]

where

\[
x = -2n^{-1 + \frac{2}{3}} - 2n^{-1 + \frac{2}{5}} + \frac{13}{3} n^{-1 + \frac{2}{5}} + \frac{32}{15} n^{-1 + \frac{2}{5}} + \sum_{\tau \leq q \leq p_h, q \neq 30} 2\mu(q)n^{-1 + \frac{2}{q}}
\]

\[
+ o\left(n^{-1 + \frac{2}{p_h}}\right) \sim -2n^{-1/3} \quad (11)
\]

Equations (10) and (5) give

\[
\sqrt{P_n} = n \left(1 + x\right)^{1/2} = n \left(1 + \frac{1}{2} x - \frac{1}{8} x^2 + O(x^3)\right)
\]

\[
= n + \frac{1}{2} nx - \frac{1}{8} nx^2 + nO(x^3) \quad (12)
\]
Equation (11) gives

\[ nO(x^3) = nO(n^{-1}) = O(1) = o \left( n^{2/p_h} \right) \]  
(13)

\[ \frac{1}{2} n x = -n^{2/3} - n^{2/5} + \frac{13}{6} n^{2/6} + \frac{16}{15} n^{2/30} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} \mu(q) n^{2/q} + o \left( n^{2/p_h} \right) \]  
(14)

\[ -\frac{1}{8} n x^2 \]

\[ = -\frac{1}{8} n \left( -2n^{-1+\frac{2}{5}} - 2n^{-1+\frac{2}{3}} + \frac{13}{3} n^{-1+\frac{2}{6}} + \frac{32}{15} n^{-1+\frac{2}{30}} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} 2\mu(q) n^{-1+\frac{2}{q}} \right)^2 \]

\[ + o(1) = -\frac{1}{8} n \left( -2n^{-1+\frac{2}{5}} \right)^2 - \frac{1}{8} n^2 \left( -2n^{-1+\frac{2}{3}} \right) \left( -2n^{-1+\frac{2}{5}} \right) + O(1) + o(1) \]

\[ = \frac{1}{2} n^{2/6} - n^{2/30} + o \left( n^{2/p_h} \right) \]  
(15)

Substituting (13), (14) and (15) into (12) we find that

\[ \sqrt{P_n} = n - n^{2/3} - n^{2/5} + \frac{5}{3} n^{2/6} + \frac{1}{15} n^{2/30} + \sum_{\substack{7 \leq q \leq p_h \\ q \neq 30}} \mu(q) n^{2/q} + o \left( n^{2/p_h} \right) \]

\[ = n + \frac{5}{3} n^{2/6} + \frac{1}{15} n^{2/30} + \sum_{\substack{2 \leq q \leq p_h \\ q \neq 2, 6, 30}} \mu(q) n^{2/q} + o \left( n^{2/p_h} \right) \]

That is, equation (6). The theorem is proved.

**ACKNOWLEDGEMENTS.** The author is very grateful to Universidad Nacional de Luján.

**References**


Received: March 29, 2013