Sums of Greatest Prime Factors

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Abstract

Suppose that \( k \geq 2 \) and \( m \geq 1 \) are fixed positive integers. Let \( B(n, p) \) be the number of positive integers not exceeding \( n \) such that the prime \( p \) is their greatest prime factor. In this article we obtain asymptotic formulae for

\[
C_{k,m}(n) = \sum_{\frac{n}{k} \leq p \leq n} p^m B(n, p)
\]

and

\[
D_{k,m}(n) = \sum_{2 \leq p \leq \frac{n}{k}} p^m B(n, p)
\]

Let \( b_m(n) \) be the \( m \)-th power of the greatest prime factor in the prime factorization of \( n \). In this article we proved the following asymptotic formula

\[
\sum_{i=2}^{n} b_m(i) \sim \frac{\zeta(m+1) n^{m+1}}{m+1 \log n},
\]

where \( \zeta(s) \) is the Riemann’s Zeta Function.

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1 Introduction and Preliminary Results

Let \( k \geq 2 \) a fixed positive integer. Let \( \beta_k(n) \) be the set of positive integers not exceeding \( n \) such that in their prime factorization appear some prime \( p \)
pertaining to the interval \( \left( \frac{n}{k}, n \right] \). That is, \( \beta_k(n) \) is the set of positive integers not exceeding \( n \) such that the greatest prime factor of these positive integers pertains to the interval \( \left( \frac{n}{k}, n \right] \). Let \( B_k(n) \) be the number of elements in the set \( \beta_k(n) \). In a previous article [3] we obtained the asymptotic formula

\[
B_k(n) \sim \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \right) \frac{n}{\log n} = o(n) \tag{1}
\]

Let \( m \) be a fixed positive integer and let \( C_{k,m}(n) \) be the sum of the \( m \)-th powers of the greatest prime factors of the positive integers pertaining to the set \( \beta_k(n) \). In this note we obtain asymptotic formulae for \( C_{k,m}(n) \).

Let \( k \geq 2 \) be a fixed positive integer. Let \( \alpha_k(n) \) be the set of positive integers not exceeding \( n \) such that in their prime factorization only appear primes \( p \) pertaining to the interval \( \left[ 0, \frac{n}{k} \right] \). That is, \( \alpha_k(n) \) is the set of positive integers not exceeding \( n \) such that the greatest prime factor of these positive integers pertains to the interval \( \left[ 0, \frac{n}{k} \right] \). We assume that 1 pertains to the set \( \alpha_k(n) \). These numbers are called smooth numbers.

Note that the sets \( \alpha_k(n) \) and \( \beta_k(n) \) are disjoints and \( \alpha_k(n) \cup \beta_k(n) = A \), where \( A \) is the set of positive integers \( k \) such that \( 1 \leq k \leq n \).

Let \( A_k(n) \) be the number of elements in the set \( \alpha_k(n) \). Consequently (see (1))

\[
A_k(n) \sim n. \tag{2}
\]

Let \( m \) be a fixed positive integer and let \( D_{k,m}(n) \) be the sum of the \( m \)-th powers of the greatest prime factors of the positive integers pertaining to the set \( \alpha_k(n) \). In this note we obtain asymptotic formulae for \( D_{k,m}(n) \).

Let \( B(n, p) \) be the number of positive integers not exceeding \( n \) such that the prime \( p \) is their greatest prime factor. Then

\[
\sum_{2 \leq p \leq n} B(n, p) = n - 1
\]

\[
C_{k,m}(n) = \sum_{\frac{n}{k} < p \leq n} p^m B(n, p) \tag{3}
\]

\[
D_{k,m}(n) = \sum_{2 \leq p \leq \frac{n}{k}} p^m B(n, p) \tag{4}
\]

Let \( b_m(n) \) be the \( m \)-th power of the greatest prime factor in the prime factorization of \( n \). In a previous article [2], we proved the following asymptotic formula

\[
\sum_{i=2}^{n} b_m(i) \sim \frac{\zeta(m+1) n^{m+1}}{(m+1) \log n},
\]

where \( \zeta(s) \) is the Riemann’s Zeta Function. In this article we give other proof more short and more direct of this formula.

The following lemma is a consequence of the prime number theorem (see either [1] or [4]).
Lemma 1.1 Let \( m \) be a nonnegative integer and let \( S_m(x) \) be the sum of the \( m \)-th powers of the primes not exceeding \( x \). The following asymptotic formula holds

\[
S_m(x) = \sum_{p \leq x} p^m \sim \frac{x^{m+1}}{(m + 1) \log x},
\]

where \( p \) denotes a positive prime.

Note that a consequence of equation (5) is the following inequality

\[
S_m(x) = \sum_{p \leq x} p^m < h \frac{x^{m+1}}{(m + 1) \log x},
\]

where \( h > 1 \). This inequality holds for \( x \geq x_0 \), where \( x_0 \) depend of \( m \).

2 Main Results

Theorem 2.1 The following formula holds

\[
C_{k,m}(n) \sim C_{k,m} \frac{n^{m+1}}{(m + 1) \log n},
\]

where

\[
C_{k,m} = 1 + \frac{1}{2m+1} + \frac{1}{3m+1} + \cdots + \frac{1}{km+1} - \frac{1}{km}
\]

Proof. Let \( k \geq 2 \) be a fixed positive integer. Consider the inequality

\[
\frac{n}{k} < p \leq n,
\]

where \( p \) denotes a positive prime number.

If \( n \geq k^2 \) equation (8) gives \( p > k \).

Consider the inequality

\[
\frac{n}{2} < p \leq n.
\]

The number of multiples of \( p \) not exceeding \( n \) is 1, namely \( p \), since \( p \leq n \) and \( 2p > n \). Therefore \( p \) is the greatest prime factor in these multiples of \( p \).

Consequently the sum of the \( m \)-th powers of the greatest prime factor in these multiples of \( p \) not exceeding \( n \) will be (see (5))

\[
S_m(n) - S_m(n/2).
\]

Consider the inequality

\[
\frac{n}{3} < p \leq \frac{n}{2}.
\]
The number of multiples of \( p \) not exceeding \( n \) is 2, namely \( p \) and \( 2p \), since \( 2p \leq n \) and \( 3p > n \). Therefore \( p \) is the greatest prime factor in these multiples of \( p \).

Consequently the sum of the \( m \)-th powers of the greatest prime factor in these multiples of \( p \) not exceeding \( n \) will be

\[
2 \left( S_m(n/2) - S_m(n/3) \right).
\]  

(10)

Consider the inequality

\[
\frac{n}{k} < p \leq \frac{n}{k-1}.
\]

The number of multiples of \( p \) not exceeding \( n \) is \( k-1 \), namely \( p, 2p, \ldots, (k-1)p \), since \( (k-1)p \leq n \) and \( kp > n \). Therefore \( p \) is the greatest prime factor in these multiples of \( p \). Since \( p > k \) (see above).

Consequently the sum of the \( m \)-th powers of the greatest prime factor in these multiples of \( p \) not exceeding \( n \) will be

\[
(k-1) \left( S_m(n/(k-1)) - S_m(n/k) \right).
\]  

(11)

Therefore, see (9), (10), \ldots, (11), we have

\[
C_{k,m}(n) = (S_m(n) - S_m(n/2)) + 2(S_m(n/2) - S_m(n/3)) + 3(S_m(n/3) - S_m(n/4)) + \cdots + (k-1)(S_m(n/(k-1)) - S_m(n/k))
\]

\[
= S_m(n) + S_m(n/2) + S_m(n/3) + \cdots + S_m(n/(k-1)) - (k-1)S_m(n/k).
\]  

(12)

Equation (5) implies

\[
\lim_{n \to \infty} \frac{S_m(n/j)}{S_m(n)} = \frac{1}{j^{m+1}}.
\]  

(13)

Equations (12) and (13) give

\[
\lim_{n \to \infty} \frac{C_{k,m}(n)}{S_m(n)} = 1 + \frac{1}{2m+1} + \frac{1}{3m+1} + \cdots + \frac{1}{(k-1)m+1} - (k-1)\frac{1}{k^{m+1}}
\]

\[
= 1 + \frac{1}{2m+1} + \frac{1}{3m+1} + \cdots + \frac{1}{k^{m+1}} - \frac{1}{k^m} = C_{k,m}
\]  

(14)

Finally, equations (14) and (5) give equation (7). The theorem is proved.

**Theorem 2.2** The following asymptotic formula holds

\[
\sum_{i=2}^{n} b_m(i) \sim \frac{\zeta(m+1) n^{m+1}}{m+1 \log n}
\]  

(15)
Proof. We have (see (3) and (4))

$$\sum_{i=2}^{n} b_m(i) = D_{k,m}(n) + C_{k,m}(n) = \sum_{2 \leq p \leq \frac{n}{k}} p^m B(n, p) + C_{k,m}(n) \quad (16)$$

where (see (7))

$$C_{k,m}(n) = C_{k,m}(n) + 1 \left( \frac{m}{m+1} \log n \right) + h_k(n) \left( \frac{m}{m+1} \log n \right) \quad (h_k(n) \to 0) \quad (17)$$

Consider the first sum in (16). Namely

$$D_{k,m}(n) = \sum_{2 \leq p \leq \frac{n}{k}} p^m B(n, p).$$

We have the following trivial inequality

$$B(n, p) \leq \left\lfloor \frac{n}{p} \right\rfloor \leq \frac{n}{p}.$$  

Since the multiples of $p$ not exceeding $n$ are $p.1, p.2, \ldots, \left\lfloor \frac{n}{p} \right\rfloor$.

Therefore (see (6))

$$\sum_{2 \leq p \leq \frac{n}{k}} p^m B(n, p) \leq \sum_{2 \leq p \leq \frac{n}{k}} \frac{p^n}{p} = n \sum_{2 \leq p \leq \frac{n}{k}} p^{m-1} \leq nh \frac{(\frac{n}{k})^m}{m \log \left( \frac{n}{k} \right)} \quad (\lambda > 0).$$  

That is

$$\sum_{2 \leq p \leq \frac{n}{k}} p^m B(n, p) = g_k(n) \frac{n^{m+1}}{(m+1) \log n}, \quad (18)$$

where

$$0 < g_k(n) < \frac{h(m+1)}{mk^m} + \lambda \quad (\lambda > 0). \quad (19)$$

We have

$$\sum_{i=2}^{n} b_m(i) = \frac{\zeta(m+1)}{m+1} n^{m+1} \log n + f(n) \frac{n^{m+1}}{(m+1) \log n}. \quad (20)$$
Now (see (16), (17) and (18))

\[
\sum_{i=2}^{n} b_{m}(i) = g_{k}(n) \frac{n^{m+1}}{(m+1) \log n} + \left(1 + \frac{1}{2^{m+1}} + \cdots + \frac{1}{k^{m+1}} - \frac{1}{k^{m}}\right) \frac{n^{m+1}}{(m+1) \log n} + h_{k}(n) \frac{n^{m+1}}{(m+1) \log n} = \zeta(m+1) \frac{n^{m+1}}{m+1 \log n} + \left( g_{k}(n) + h_{k}(n) - \frac{1}{k^{m}} - \sum_{j=k+1}^{\infty} \frac{1}{j^{m+1}}\right) \frac{n^{m+1}}{(m+1) \log n}
\]

Hence (see (20) and (21))

\[
f(n) = g_{k}(n) + h_{k}(n) - \frac{1}{k^{m}} - \sum_{j=k+1}^{\infty} \frac{1}{j^{m+1}}
\]

Let \(\epsilon > 0\). If we choose \(k\) sufficiently large then (see (19))

\[
0 < g_{k}(n) < \frac{\epsilon}{4}, \quad 0 < \frac{1}{k^{m}} < \frac{\epsilon}{4}, \quad 0 < \sum_{j=k+1}^{\infty} \frac{1}{j^{m+1}} < \frac{\epsilon}{4}
\]

On the other hand, we have (see (17)) \(h_{k}(n) \to 0\). Therefore if \(n\) is sufficiently large then

\[
|h_{k}(n)| < \frac{\epsilon}{4}
\]

Consequently we have (see (22), (23) and (24)) \(|f(n)| < \epsilon\). Now, \(\epsilon\) is arbitrarily small. Hence

\[
\lim_{n \to \infty} f(n) = 0.
\]

Equations (20) and (25) give (15). The theorem is proved.

**Corollary 2.3** The following asymptotic formula holds

\[
D_{k,m}(n) \sim D_{k,m} \frac{n^{m+1}}{(m+1) \log n}
\]

where

\[
D_{k,m} = \frac{1}{k^{m}} + \sum_{j=k+1}^{\infty} \frac{1}{j^{m+1}}
\]
Proof. We have (see (16), (17) and (15))

\[
D_{k,m}(n) = \sum_{i=2}^{n} b_m(i) - C_{k,m}(n) = \frac{\zeta(m+1) n^{m+1}}{m+1 \log n}
\]

\[
- \left(1 + \frac{1}{2^{m+1}} + \cdots + \frac{1}{k^{m+1}} - \frac{1}{k^m}\right) \frac{n^{m+1}}{(m+1) \log n}
\]

\[
+ f_k(n) \frac{n^{m+1}}{\log n} = \left(\frac{1}{k^m} + \sum_{j=k+1}^{\infty} \frac{1}{j^{m+1}}\right) \frac{n^{m+1}}{(m+1) \log n}
\]

\[
+ f_k(n) \frac{n^{m+1}}{\log n} \quad (f_k(n) \to 0)
\]

The corollary is proved.

**Remark 2.4** We have, (see equations (16), (7), (15) and (26))

\[
\sum_{i=2}^{n} b_m(i) = D_{k,m}(n) + C_{k,m}(n)
\]

Now, if \(k\) is large then \(D_{k,m}\) is very small (see (27)) in comparation with \(C_{k,m}\). Therefore the contribution to \(\sum_{i=2}^{n} b_m(i)\) of the smooth numbers whose density is 1 (see (2)) is insignificant in comparation with the contribution to \(\sum_{i=2}^{n} b_m(i)\) of the rest of numbers whose density is zero (see (1)).

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**References**


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