Fredholm Type Integral Equations with Aleph-Function

and General Polynomials

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Abstract

This paper is devoted to the useful method of solving the one-dimensional integral equation of Fredholm type. The Mellin transform technique for solving a general fredholm type integral equation with the \( \aleph \)-function and a generalized polynomial in the kernel is considered. By specializing the coefficients and various parameters in the generalized polynomials and \( \aleph \)-function, our main theorem would readily yield several results involving simpler kernels.

Keywords: Fredholm type integral equation, Mellin transform technique & \( \aleph \)-function

1. Introduction

In the last several years a large number of Fredholm type integral equations involving various polynomials or special functions as their kernels have been studied by many authors notably
Buchman [10], Higgins [11], Love ([3] and [2]), Prabhakar and Kashyap [12], Srivastava and Raina [6] and others. In the present paper, we obtain the following one-dimensional Fredholm integral equation (1.1) involving the \( \aleph \) -function and a generalised polynomials in the kernel can be solved by resorting to the application of Mellin transforms.

\[
\int_0^\infty \alpha S_{n_1, \ldots, n_R}^{m_1 \ldots m_R} \left[ E\left( \frac{x}{y} \right)^p, \ldots, E\left( \frac{x}{y} \right)^p \right] \aleph_{p_i, q_i, \tau_i}^{m, n} \left[ z^q \right] \left[ (a_j, A_j)^{1, n}, \ldots, (a_j, A_j)^{1, n+1, pi} \right] \left[ (b_j, B_j)^{1, m}, \ldots, (b_j, B_j)^{1, m+1, qi} \right] f(y) dy = g(x)
\]

(1.1)

The \( \aleph \) -function occurring in (1.1) introduced by Sudland et al. [9] which is defined as a contour integral of Mellin-Barnes Type:

\[
\aleph[Z] = \aleph_{p_i, q_i, \tau_i}^{m, n} \left[ \frac{x}{y} \right] \left[ (a_j, A_j)^{1, n}, \ldots, (a_j, A_j)^{1, n+1, pi} \right] \left[ (b_j, B_j)^{1, m}, \ldots, (b_j, B_j)^{1, m+1, qi} \right]
\]

\[
\Omega = \frac{1}{2\pi i} \int_\gamma \left[ \frac{x}{y} \right] \left[ (a_j, A_j)^{1, n}, \ldots, (a_j, A_j)^{1, n+1, pi} \right] \left[ (b_j, B_j)^{1, m}, \ldots, (b_j, B_j)^{1, m+1, qi} \right] z^{-s} ds
\]

(1.2)

for all \( z \neq 0, \; \omega = \sqrt{-1} \) and

\[
\Omega_{p_i, q_i, \tau_i}^{m, n} = \prod_{j=1}^m \Gamma\left( b_j + B_j, s \right) \prod_{j=1}^n \Gamma\left( 1 - a_j, A_j, s \right)
\]

\[
\prod_{j=n+1}^m \Gamma\left( a_j + A_j, s \right) \prod_{j=m+1}^q \Gamma\left( 1 - b_j, B_j, s \right)
\]

(1.3)

The integration path \( \gamma = \gamma_{\psi, \omega} \) extends from \( \gamma - i\infty \) to \( \gamma + i\infty \), and is such that the poles, assumed to be simple, of \( \Gamma(1 - a_j - A_j, s) \), \( j = 1, n \) do not coincide with the pole \( \Gamma(1 - b_j - B_j, s), j = 1, m \). The parameters \( p_i, q_i \) are non-negative integers satisfying \( 0 \leq n \leq p_i, 1 \leq m \leq q_i \), \( \tau_i > 0 \) for \( i = 1, r \). The parameters \( A_j, B_j, A_{ji}, B_{ji}, a_j, b_j, a_{ji}, b_{ji} \in \mathbb{C} \). The empty product in (1.3) is interpreted as unity. The existence conditions for the defining integral (1.1) are given below:

\[
\psi_l > 0, \quad \left| \text{arg}(z) \right| < \frac{\pi}{2} \psi_l, \quad l = 1, r;
\]

(1.4)

\[
\psi_l \geq 0, \quad \left| \text{arg}(z) \right| < \frac{\pi}{2} \psi_l \text{ and } \Re\{\zeta_l\} + 1 > 0,
\]

(1.5)

Where
The general polynomials of $R$ variables given by Srivastava [4] defined and represented as:

$$s_{n_1, \ldots, n_R}^{m_1, \ldots, m_R} [x_1, \ldots, x_R] = \sum_{s_1 = 0}^{n_1} \cdots \sum_{s_R = 0}^{n_R} \left(\frac{(-1)^{m_1}s_1}{\Gamma(s_1)} \right) \cdots \left(\frac{(-1)^{m_R}s_R}{\Gamma(s_R)} \right)$$

(1.8)

Let $\int$ denote the space of all functions $f$ which are defined on $R^+ = [0, \infty)$ and satisfy

1. $f \in b^\infty (R^+)$,
2. $\lim_{x \to \infty} x^\gamma f^r (x) = 0$ for all non negative integers $\gamma$ and $r$.
3. $f(x) = O(1)$ as $x \to 0$.

For correspondence to the space of good functions defined on the whole real line $(-\infty, \infty)$ see (Lighthill) [8].

The Riemann – Liouville fractional integral (of order $\mu$) is defined by

$$D^{-\mu} \{f(x)\} = G_{\mu} \{f(x)\} = \frac{1}{\Gamma(\mu)} \int_0^x (x-w)^{\mu-1} f(w)dw, \quad (\text{Re}(\mu) > 0)$$

(1.9)

where $D^{-\mu} \{f(x)\} = \hat{\Phi}(x)$ is understood to mean that $\Phi$ is a locally integrable solution of $f(x) = D^{-\mu} \{\hat{\Phi}(x)\}$, implying that $D\mu$ is the inverse of the fractional integral operator $D^{-\mu}$ (whenever necessary, we shall simply write $D^{-\mu}$ for $D^{-\mu}$ for the Riemann-Liouville fractional integral operator defined by eq. (1.8) above).

$$W^{-h} \{f(x)\} = \Gamma(h) \int_x^\infty (\zeta - x)^{h-1} f(\zeta) d\zeta, (\text{Re}(h) > 0)$$

(1.10)

2. PRELIMINARY RESULTS

We first prove the following result which will be required in proving theorem 1 below.
Lemma 1. With the set of sufficient conditions (1.4), (1.5), (1.6) and (1.7) and let us suppose 
\(\text{Re } (\alpha) > \text{Re } (\beta); \text{ Re } [\beta + q (b/B_j)] > 0, (j = 1, m), q > 0.\)

Let

\[\binom{m, n + 1}{p_i + 1, q_i + 1, \tau; r} \text{ Re } \sum_{s_1 = 0}^{n_1/m_1} \sum_{s_R = 0}^{n_R/m_R} \left(\frac{x}{y}\right)^{s_1} \cdots \left(\frac{x}{y}\right)^{s_R} \binom{(-n_1)m_{1}s_1}{\sum_{s_1 = 0}^{n_1/m_1} \sum_{s_R = 0}^{n_R/m_R} A[n_1, s_1; \cdots; n_R s_R]} E^{s_1 + \cdots + s_R} \left(\frac{x}{y}\right)^{p_1 + \cdots + p_R} \right] \]

(2.1)

Proof: To prove Lemma 1, we first use the definition of Weyl fractional integral given in (1.10) express the \(\mathcal{R}\)-function and a generalized polynomial, then we change the order of summations and integration (which is justified under the stated conditions), evaluate the t-integral and interpreting the resulting Mellin-Barnes contour integral in terms of the \(\mathcal{R}\)-function, we easily arrive at the desired result.

Theorem 1 - Under the sufficient conditions (i), (ii), (iii) and (iv) of Lemma

\[\int_0^\infty \binom{m, n + 1}{p_i + 1, q_i + 1, \tau; r} \text{ Re } \sum_{s_1 = 0}^{n_1/m_1} \sum_{s_R = 0}^{n_R/m_R} \left(\frac{x}{y}\right)^{s_1} \cdots \left(\frac{x}{y}\right)^{s_R} \binom{(-n_1)m_{1}s_1}{\sum_{s_1 = 0}^{n_1/m_1} \sum_{s_R = 0}^{n_R/m_R} A[n_1, s_1; \cdots; n_R s_R]} E^{s_1 + \cdots + s_R} \left(\frac{x}{y}\right)^{p_1 + \cdots + p_R} \right] f(y) dy \]

(2.2)

Provided further \( f \in \mathcal{F}, \text{ and } x > 0. \)
Proof: Let $\eta$ denote the first member of the assertion (2.2). Then using Lemma 1 and applying (1.10), we have

$$
\eta (y) = \int_0^\infty \frac{f(y)}{\Gamma(\alpha - \beta)} \left( \int_y^\infty (\xi - y)^{a-\beta-1} \xi^{-a} S_{m_1 \ldots m_R}^{n_1 \ldots n_R} \left[ E \left( \frac{x}{y} \right)^p, \ldots, E \left( \frac{x}{y} \right)^p \right] \xi^{-p_i, q_i, \tau_i; r} \left[ \left( \frac{x}{y} \right)^q \right] \right) dy.
$$

(2.3)

$$
= \int_0^\infty \xi^{-a} S_{m_1 \ldots m_R}^{n_1 \ldots n_R} \left[ E \left( \frac{x}{y} \right)^p, \ldots, E \left( \frac{x}{y} \right)^p \right] \xi^{-p_i, q_i, \tau_i; r} \left[ \left( \frac{x}{y} \right)^q \right] \left( \int_y^\infty (\xi - y)^{a-\beta-1} f(y) \right) dy.
$$

(2.4)

Assuming the inversion of the order of integration to be permissible just as in the proof of Lemma 1.

Now, by definition (1.9), (2.4) gives

$$
\eta = \int_0^\infty \xi^{-a} S_{m_1 \ldots m_R}^{n_1 \ldots n_R} \left[ E \left( \frac{x}{y} \right)^p, \ldots, E \left( \frac{x}{y} \right)^p \right] \xi^{-p_i, q_i, \tau_i; r} \left[ \left( \frac{x}{y} \right)^q \right] D^{\beta-a} \{ f(\xi) \} d\xi.
$$

Which is the second member of (2.2).

3. SOLUTION OF A FREDHOLM TYPE INTEGRAL EQUATION

To obtain the solution of a fredholm type integral equation (1.1), we use the Mellin Transform technique.

Theorem 2. If

$$
\int f(\xi) D^{\alpha-\beta} \{ f(\xi) \} \text{ exists, } q > 0, x > 0, \psi_l > 0, \quad \text{s.t. } \arg(z) < \frac{\pi}{2} \psi_l, \quad l = 1, r;
$$

\( \psi_l \geq 0, \quad \arg(z) < \frac{\pi}{2} \psi_l \quad \text{and } \Re(\xi_1) + 1 < 0, \ (\psi_i \text{ given in (1.6) and } \xi_i \text{ is given in (1.7)})
$$

Re $(\alpha) > \Re (\beta) > 0$, and

$m_i$ is an arbitrary positive integer and coefficients $A_{n_1, s_1; \ldots; n_R, s_R}$ are arbitrary constants, real or complex, then the solution of the integral equation

$$
\int_0^\infty \xi^{-a} S_{m_1 \ldots m_R}^{n_1 \ldots n_R} \left[ E \left( \frac{x}{y} \right)^p, \ldots, E \left( \frac{x}{y} \right)^p \right] \xi^{-p_i, q_i, \tau_i; r} \left[ \left( \frac{x}{y} \right)^q \right] \left( \frac{(a_j, A_j)_{n+1, p_i}}{(b_j, B_j)_{m+1, q_i}} \right) \left[ \tau_j (a_j, A_j) \right] d\xi
$$

$$
f(y)dy = g(x)
$$

(3.1)

is given by
\[
f(y) = \frac{q}{2\pi w} \lim_{\rho \to \infty} \left( \sigma + \frac{\rho w}{\sigma} \right)^{\beta - 1} \left[ \sum_{s_1 = 0}^{n_1} \frac{m_1}{s_1} \ldots \sum_{s_R = 0}^{n_R} \frac{m_R}{s_R} \left( -n_1 \right)^{m_1 s_1} \ldots \left( -n_R \right)^{m_R s_R} A[s_1, \ldots, s_R] \right]^{-1} \left( -p(s_1 + \ldots + s_R) \right)^{-k_1} \frac{1}{q} \Omega \left( \frac{-p(s_1 + \ldots + s_R) - k_1}{q} \right) \phi(k_1) dk_1. \tag{3.2}
\]

provided further that \( \max \{ \Re [(a_{l-1}/\Phi)] \} < \Re [(p(s_1 + \ldots + s_R) + k_1)/q] < \min \{ \Re (b_j/B_j) \}, \)

\((j = 1, \ldots m) \) and \((l = 1, \ldots n) \)

Proof: On replacing \( f \) by \( D^{\alpha-\beta}[f] \) in (3.1) and applying (2.1), we have

\[
g(x) = \int_0^\infty \left[ \frac{x}{y} \right]^{\beta} \frac{1}{y} \sum_{s_1 = 0}^{n_1} \frac{m_1}{s_1} \ldots \sum_{s_R = 0}^{n_R} \frac{m_R}{s_R} \left( -n_1 \right)^{m_1 s_1} \ldots \left( -n_R \right)^{m_R s_R} A[s_1, \ldots, s_R] \left( x/y \right)^{p[s_1, \ldots, s_R]} \] 

\[
\Re_{p_i + 1, q_i + 1, \tau_i : r} \left[ \left( \frac{x}{y} \right)^q \left[ 1 - \beta - p(s_1 + \ldots + s_R); q, (a_j, A_j)_1, n, \ldots, [\tau_j (a_j, A_j)_n + 1, p_i] \right] \right] D^{\alpha-\beta}[f(y)] dy \tag{3.3}
\]

Multiplying both the sided of (3.3) by \( x^{n-1} \) and integrating with respect to \( x \) from 0 to \( \infty \), we have
Fredholm type integral equations

\[ \phi(s_i) = \int_0^{x_i} g(x) \, dx = \int_0^{\infty} y^{-\beta} D^{a-\beta} \{ f(y) \} \left( \int_0^{x_i} \sum_{s_1 = 0}^{n_1/m_1} ... \sum_{s_R = 0}^{n_R/m_R} \left( \frac{-n_1}{m_1} \right)^{s_1} \ldots \left( \frac{-n_R}{m_R} \right)^{s_R} \right) \, dx \]

\[ A[n_1, s_1; \ldots; n_R s_R] \sum_{p_1+1, q_1+1, \tau_1: r}^m \left( \frac{x}{y} \right)^q \left[ (1 - \beta - p(s_1 + \ldots + s_R); q), (a_j, A_j), n_1, \ldots, [\tau_j (a_j, A_j)]_{n+1, p_1} \right] \left( b_j, B_j \right)_{m+1, q_1} \left[ 1 - \alpha - p(s_1 + \ldots + s_R); q \right] \, dx \, dy \quad (3.4) \]

where we have assumed the absolute (and uniform) convergence of the integrals involved, with a view to justifying the inversion of the order of integration.

Now evaluate the inner integral in (3.4) by a simple change of variables in the familiar results (c.f., for example, [5] and [7]), eq. (3.4) reduces to

\[ \phi(s_1) = \int_0^{\infty} y^{-\beta} D^{a-\beta} \{ f(y) \} \left( \int_0^{x_i} \sum_{s_1 = 0}^{n_1/m_1} ... \sum_{s_R = 0}^{n_R/m_R} \left( \frac{-n_1}{m_1} \right)^{s_1} \ldots \left( \frac{-n_R}{m_R} \right)^{s_R} A[n_1, s_1; \ldots; n_R s_R] \right) \, dx \]

\[ E^{s_1 + \ldots + s_R} \left( \frac{-p(s_1 + \ldots + s_R) - k}{q} \right) \left( \frac{-p(s_1 + \ldots + s_R) - k}{q} \right) \left( \frac{-p(s_1 + \ldots + s_R) - k}{q} \right) \]
\[ \frac{\Gamma(\beta - k_i)}{\Gamma(\alpha - k_i)} p^{(s_1 + \ldots + s_R)} dy, \]  

(3.5)

where \( \Phi(s_i) \) is given by (3.4).

\[
\phi(s_1) = \frac{1}{q} \sum_{s_1 = 0}^{n_1/m_1} \ldots \sum_{s_R = 0}^{n_R/m_R} (-1)^{m_1 s_1} \cdots (-1)^{m_R s_R} \frac{A[s_1; \ldots; s_R]}{\sum_{s_1} s_1} \frac{(-p(s_1 + \ldots + s_R) - k_1)}{q} \left( \frac{\Gamma(\beta - k_i)}{\Gamma(\alpha - k_i)} \right) \]

(3.6)

Inverting (3.6) by applying the Mellin Inversion theorem \[1\], we get

\[
D^{\alpha - \beta} \{ f(y) \} = \frac{q}{2\pi i} \lim_{\rho \to \sigma + i\rho} j \left[ \sum_{s_1 = 0}^{n_1/m_1} \ldots \sum_{s_R = 0}^{n_R/m_R} (-1)^{m_1 s_1} \cdots (-1)^{m_R s_R} \frac{A[s_1; \ldots; s_R]}{\sum_{s_1} s_1} \frac{(-p(s_1 + \ldots + s_R) - k_1)}{q} \left( \frac{\Gamma(\beta - k_i)}{\Gamma(\alpha - k_i)} \right) \right]^{-1} \]

(3.7)

Operating upon both sides by \( D^{\beta - \alpha} \), (3.7) gives us
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\[ f(y) = \frac{q}{2\pi i} \int_{\rho} \frac{d\beta - \alpha}{\sigma + \rho \gamma} \lim_{\rho \to \infty} \left[ \sum_{s_1 = 0}^{n_1} \ldots \sum_{s_R = 0}^{m_R} \left( \begin{array}{c} n_1 \\ m_1 \\ \ldots \\ n_R \\ m_R \\ \ldots \\ (n_1 m_1) s_1 \\ \ldots \\ (n_R m_R) s_R \\ \ldots \\ A[n_1, s_1; \ldots; n_R s_R] \right) \right]^{-1} \]

\[ = \frac{-p(s_1 + \ldots + s_R) - k_1}{q} \frac{1}{\Omega} \frac{1}{\Gamma(\beta - k_1)} \left( \frac{-p(s_1 + \ldots + s_R) - k_1}{q} \right)^{-1} \phi(k_1) dk_1. \quad (3.8) \]

Which finally yields

\[ f(y) = \frac{q}{2\pi i} \lim_{\rho \to \infty} \int_{\rho} \frac{d\beta - \alpha}{\sigma - \rho \gamma} \left[ \sum_{s_1 = 0}^{n_1} \ldots \sum_{s_R = 0}^{m_R} \left( \begin{array}{c} n_1 \\ m_1 \\ \ldots \\ n_R \\ m_R \\ \ldots \\ (n_1 m_1) s_1 \\ \ldots \\ (n_R m_R) s_R \\ \ldots \\ A[n_1, s_1; \ldots; n_R s_R] \right) \right]^{-1} \]

\[ = \frac{-p(s_1 + \ldots + s_R) - k_1}{q} \frac{1}{\Omega} \left( \frac{-p(s_1 + \ldots + s_R) - k_1}{q} \right)^{-1} \phi(k_1) dk_1. \quad (3.8) \]

As the solution of the integral equation (3.1)

4. Special Case

1. In theorem 1, if we take \( R = 1 \), \( m_1 = 2 \) and \( A[n_1, s_1] = (-1)^{s_1} \), then we have the very interesting theorem, i.e.

Theorem 3- Suppose that the conditions corresponding to Theorem 2 are satisfied. Then

\[ \int_{0}^{\infty} y - \alpha E_1^{n_1 / 2} \left( \frac{x}{y} \right)^{2} H_{n_1} \left[ \frac{1}{2 \sqrt{E_1 \left( \frac{x}{y} \right)^s}} \right] \]
\[ f(y) \, dy = g(x) \]

is given by

\[
\begin{align*}
  f(y) &= \frac{q}{2\pi\rho} \lim_{\rho \to \infty} \int_{-\infty}^{\infty} \beta - k_1 - 1 \left\{ \sum_{s_1 = 0}^{n_1/2} \frac{(-n_1/2)_1}{\Omega(s_1)} (-1)^{s_1} \right\}^{-1} \\
  &\quad \cdot \frac{\mu + \rho \omega}{\rho} \left( - p(s_1 + \ldots + s_R) - k_1 \right) \\
  &\quad \cdot \frac{\phi(k_1) \, dk_1}{q} \\
  &\quad \cdot E^{s_1} Z^{s_1} \\
\end{align*}
\]

Provided integral exists.

2. If we take \( n_1 = \ldots = n_R \to 0 \), \( \tau_1 = \ldots = \tau_{r-1} = 1 \) and \( r = 1 \), Theorem 2 is seem to correspond to a result given by Srivastava and Raina [6]

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