On Locally Reduced and Locally Multiplication Modules

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Abstract

This paper presents and defines two types of modules which are called locally reduced and locally multiplication modules and several properties of these types of modules are studied and proved.

Keywords: reduced modules, multiplication modules, locally reduced modules, locally multiplication modules

1 Introduction

Let $R$ be a commutative ring with identity, $M$ be an $R$-module and $N$ be a submodule of $M$. The Jacobson radical of $R$, denoted by $J(R)$, is the intersection of all maximal ideals of $R$. $M$ is called a multiplication module if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$ [2]. $N$ is called a prime submodule of $M$ if $N \neq M$ and for $r \in R, m \in M$, the condition $rm \in N$ implies that $m \in N$ or $rM \subseteq N$ [8]. The spectrum of $M$, denoted by $\text{Spec}(M)$, is the set of all prime submodules of $M$, that is, $\text{Spec}(M) = \{ P : P$ is a prime submodule of $M \}$ and $M$ is called a reduced module if $\cap \text{Spec}(M) = 0$ [7]. An element $r \in R$ is called prime to $N$ if $rm \in N$, for $m \in M$, implies that $m \in N$ [1], equivalently, $r \in R$ is not prime to $N$ if $rm \in N$ for some $m \in M - N$. If we denote the set of all elements of $R$ that are not prime to $N$ by $S_M(N)$, then we have $S_M(N) = \{ r \in R : rm \in N, \text{ for some } m \in M - N \}$, specially, if $N = 0$, then $S_M(0) = \{ r \in R : rm = 0 \}$.
If $x/ab \in R$ of $Ann$ $N \cap$ commutative ring with identity and make $\text{duced} R$ which we get.

So that when we say $\text{minimal ideal}$ $P$ multiplication.

Then:

$x$ module operations $\text{called an essential submodule of}$ $N$ $\text{is a strongly reduced}$ $R$ $\text{is a multiplicative set in}$ $S$ $\text{Spec}$ $N = 0$. The annihilator of $K$ is a multiplicative closed set if $0 \in S$ and $a, b \in S$ implies that $ab \in S$ [5]. If $S$ is a multiplicative set in $R$, then one can easily make $R_S$ as a commutative ring with identity [5] and make $M_S$ as an $R_S$–module under the module operations $\frac{x}{s} + \frac{y}{t} = \frac{tx + sy}{st}$ and $\frac{x}{s} \cdot \frac{y}{t} = \frac{yx}{us}$, for $\frac{x}{s}, \frac{y}{t} \in M_S$ [6], so that when we say $M_S$ is a module we mean $M_S$ is an $R_S$–module.

Throughout this paper $R$ is a commutative ring with identity and $M$ is a non zero left unitary $R$–module unless otherwise stated.

## 2 The Results

First, we introduce the following definitions.

**Definition 2.1.** An $R$–module $M$ is called locally reduced if $M_P$ is a reduced $R_P$–module for each maximal ideal $P$ of $R$ and it is called a locally multiplication $R$–module if $M_P$ is a multiplication $R_P$–module for each maximal ideal $P$ of $R$.

**Definition 2.2.** Let $M$ be an $R$–module and $P$ a maximal ideal of $R$, we define $\text{Spec}_P(M) = \{N : N \in \text{Spec}(M) \text{ and } S_M(N) \subseteq P\}$ and we say that $M$ is a strongly reduced $R$–module if $\cap \text{Spec}_P(M) = 0$, for each maximal ideal $P$ of $R$ and we denote the set of all maximal submodules of $M$ by $\text{Spec}_\text{max}(M)$.

**Proposition 2.3.** Every maximal submodule $N$ of an $R$–module $M$ is prime.

Proof. $N$ is a proper submodule of $M$. Let $rx \in N$, for $r \in R$ and $x \in M$. If $x \notin N$, then we have $N \subset< x > + N$ and as $N$ is maximal in $M$, we get $< x > + N = M$. Now, if $m \in M$ then $m = sx + b$, for $s \in R$ and $b \in N$, from which we get $rm = srx + rb \in N$, so that $rM \subseteq N$ and thus $N$ is a prime submodule of $M$.

**Theorem 2.4.** Let $M$ be an $R$–module and let $P$ be a maximal ideal of $R$, then:

1. $(\cap \text{Spec}(M))_P \subseteq \cap \text{Spec}(M_P)$.
2. $\cap \text{Spec}(M) \subseteq \cap \text{Spec}_P(M)$.
3. $\cap \text{Spec}(M_P) \subseteq (\cap \text{Spec}_P(M))_P$.
4. $\cap \text{Spec}(M) \subseteq \cap \text{Spec}_\text{max}(M)$.

Proof. (1) Let $\frac{x}{p} \in (\cap \text{Spec}(M))_P$, for $x \in M$ and $p \notin P$. Then $qx \in
∩Spec(M), for some \( q \notin P \). Now, let \( \overline{N} \in \text{Spec}(M_P) \), that is \( \overline{N} \) is a prime submodule of \( M_P \), then by \([3, \text{Proposition 2.16}]\), \( \overline{N} = N_P \), for the prime submodule \( N = \{ x \in M : \frac{x}{1} \in \overline{N} \} \) of \( M \) and by \([4, \text{Lemma 2.27}]\), we get \( S_M(N) \subseteq P \), that means \( N \in \text{Spec}(M) \) and thus \( qx \in N \). If \( x \notin N \), then \( q \in S_M(N) \subseteq P \), which is a contradiction and thus we must have \( x \in N \), that gives \( \frac{x}{p} \in N_P = \overline{N} \), so that we get \( \frac{x}{p} \in \cap \text{Spec}(M,P) \). Hence \( (\cap \text{Spec}(M))_P \subseteq \cap \text{Spec}(M,P) \).

(2) Since \( \text{Spec}_P(M) \subseteq \text{Spec}(M) \), so we have \( \cap \text{Spec}(M) \subseteq \cap \text{Spec}_P(M) \).

(3) Let \( \frac{x}{p} \in \cap \text{Spec}(M,P) \), where \( x \in M \) and \( p \notin P \). Let \( N \in \cap \text{Spec}_P(M) \), then \( N \in \text{Spec}(M) \) and \( S_M(N) \subseteq P \), so by \([4, \text{Proposition 2.21}]\), we have \( N_P \) is a prime submodule of \( M_P \), that means \( N_P \in \text{Spec}(M_P) \), so that \( \frac{x}{p} \in N_P \) and then by \([4, \text{Lemma 2.1}]\), we have \( x \in N \), thus we get \( x \in \cap \text{Spec}_P(M) \), this gives \( \frac{x}{p} \in (\cap \text{Spec}_P(M))_P \). Hence \( \cap \text{Spec}(M,P) \subseteq (\cap \text{Spec}_P(M))_P \).

(4) By Proposition 2.3, we have \( \text{Spec}_m(M) \subseteq \text{Spec}(M) \), so that \( \cap \text{Spec}(M) \subseteq \text{Spec}_m(M) \).

As a corollary we give:

**Corollary 2.5.** Let \( M \) be an \( R \)-module, then:

(1) If \( M \) is locally reduced, then it is reduced.

(2) If \( M \) is a strongly reduced \( R \)-module, then it is locally reduced (and hence it is reduced).

(3) If \( \cap \text{Spec}_m(M) = 0 \), then \( M \) is reduced.

Proof. (1) Let \( P \) be any maximal ideal of \( R \) (such maximal ideals exist since \( R \) is a commutative ring with identity), then \( M_P \) is reduced, that is \( \cap \text{Spec}(M_P) = 0 \). Then by Theorem 2.4 (1), we get \( (\cap \text{Spec}(M))_P = 0 \) and by \([3, \text{Corollary 2.3}]\), we get \( \cap \text{Spec}(M) = 0 \), thus \( M \) is reduced.

(2) Let \( P \) be any maximal ideal of \( R \). As \( M \) is strongly reduced, we have \( \cap \text{Spec}_P(M) = 0 \). Then by Theorem 2.4 (3), we have \( \cap \text{Spec}(M,P) = 0 \), so that \( M_P \) is reduced and thus \( M \) is locally reduced.

(3) The proof follows directly from Theorem 2.4 (4).

**Proposition 2.6.** Let \( M \) be an \( R \)-module and \( P \) a maximal ideal of \( R \). If \( N \) is a maximal submodule of \( M \) such that \( S_M(N) \subseteq P \), then \( N_P \) is a maximal submodule of \( M_P \).

Proof. If \( N_P = M_P \), then for any \( m \in M \), we have \( \frac{m}{1} \in N_P \) and then by \([4, \text{Lemma 2.1}]\), we get \( m \in N \), which gives \( N = M \), that is a contradiction, so that \( N_P \) is a proper submodule of \( M_P \). Next, suppose that \( \overline{K} \) is a proper submodule of \( M_P \) such that \( N_P \subseteq \overline{K} \), then by \([3, \text{proposition 2.16}]\), we have \( \overline{K} = K_P \), for the submodule \( K = \{ x \in M : \frac{x}{1} \in \overline{K} \} \) of \( M \) and hence we get \( N_P \subseteq K_P \). If \( m \in N \), then \( \frac{m}{1} \in K_P = \overline{K} \), so that \( m \in K \) and thus we have \( N \subseteq K \subseteq M \). As \( N \) is a maximal submodule and \( K \neq M \) (otherwise \( \overline{K} = K_P = M_P \), that is a contradiction), so we must have \( N = K \) and thus \( N_P = K_P = \overline{K} \). Hence \( N_P \) is a maximal submodule of \( M_P \).

**Proposition 2.7.** Let \( M \) be an \( R \)-module and \( P \) a maximal ideal of \( R \). If
is a maximal submodule of \(M_P\), then there exists a submodule \(N\) of \(M\) with \(\overline{N} = N_P\) and such that \(N\) is maximal with respect to the relation \(S_M(N) \subseteq P\).

Proof. By [3, Proposition 2.16], we have \(\overline{N} = N_P\), for the submodule \(N = \{x \in M : \frac{x}{1} \in \overline{N}\}\) of \(M\). By [4, Lemma 2.27], we have \(S_M(N) \subseteq P\). Next, let \(K\) be any proper submodule of \(M\) such that \(N \subseteq K\) with \(S_M(K) \subseteq P\).

Then \(\overline{N} = N_P \subseteq K_P\). If \(K_P = M_P\), then for each \(m \in M\), we have \(\frac{m}{1} \in K_P\), so by [4, Lemma 2.1], we get \(m \in K\), that means \(K = M\) and that is a contradiction, so that \(K_P\) is a proper submodule of \(M_P\). As \(\overline{N}\) is maximal in \(M_P\), we get \(\overline{N} = N_P = K_P\), from which one can easily get that \(N = K\). Hence \(N\) is a maximal submodule of \(M\).

Combining Proposition 2.6 and Proposition 2.7, we give the following theorem.

**Theorem 2.8.** Let \(M\) be an \(R\)-module and \(P\) a maximal ideal of \(R\), then there is a one to one correspondence between the maximal submodules \(N\) of \(M\) for which \(S_M(N) \subseteq P\) and the maximal submodules of \(M_P\).

Proof. Suppose that \(F = \{N : N\) is a maximal submodule of \(M\) for which \(S_M(N) \subseteq P\}\) and \(H = \{\overline{N} : \overline{N}\) is a maximal submodule of \(M_P\}\). Define \(f : F \to H\) as follows: let \(N \in F\), then by Proposition 2.6, we have \(N_P \in H\), so we set \(f(N) = N_P\). One can easily show that this definition provides a one to one correspondence between \(F\) and \(H\).

It is necessary to mention that, if \(M\) is a non zero \(R\)-module and \(P\) is a maximal ideal of \(R\) with \(S_M(0) \subseteq P\), then we have \(M_P \neq 0\). To show this, suppose that \(M_P = 0\), then for any \(x \in M\), we have \(\frac{x}{1} = 0\) and as \(S_M(0) \subseteq P\), by [4, Lemma 2.1], we get \(x = 0\), so that \(M = 0\), that is a contradiction and hence \(M_P \neq 0\). As especial case, if \(S_M(0) \subseteq J(R)\), then for each maximal ideal \(P\) of \(R\) we have \(S_M(0) \subseteq P\) and thus \(M_P \neq 0\), for each maximal ideal \(P\) of \(R\).

**Proposition 2.9.** If \(M\) is a non zero locally reduced and a locally multiplication \(R\)-module with \(S_M(0) \subseteq J(R)\) and \(N\) is a submodule of \(M\), then:

(1) \(N \cap \text{Ann}(N)M = 0\).

(2) \(\text{Ann}(N + \text{Ann}(N)M) = \text{Ann}(M)\).

Proof. Let \(P\) be any maximal ideal of \(R\), then \(M_P\) is a reduced and a multiplication \(R\)-module and as \(M \neq 0\), so by what we have mentioned in the above we have \(M_P \neq 0\).

(1) By [7, Lemma 2.3], we have \(N_P \cap (\text{Ann}(N)M_P) = 0\) and by [4, Proposition 2.4], we have \((\text{Ann}(N))_P \subseteq \text{Ann}(N_P)\), this gives that \([N \cap \text{Ann}(N)M]_P = N_P \cap (\text{Ann}(N))_P = N_P \cap \text{Ann}(N)_P \subseteq N_P \cap \text{Ann}(N_P)M_P = 0\), thus by [3, Corollary 2.3], we get \(N \cap \text{Ann}(N)M = 0\).

(2) By [7, Lemma 2.3], we have \(\text{Ann}[N_P + \text{Ann}(N)M_P] = \text{Ann}(M_P)\) and as \(S_M(0) \subseteq J(R) \subseteq P\), by using [4, Proposition 2.5], we get \((\text{Ann}(N) + \text{Ann}(N)M)_P = (\text{Ann}(M))_P\). Hence by [3, Corollary 2.2] and [3, Corollary 2.3], we get \(\text{Ann}(N + \text{Ann}(N)M) = \text{Ann}(M)\).

**Proposition 2.10.** Let \(M\) be an \(R\)-module and \(N\) a submodule of \(M\). If
$P$ is a maximal ideal of $R$ with $S_M(0) \subseteq P$, then $\sqrt{Ann(NP)} = (\sqrt{Ann(N)})_P$.

Proof. Let $\frac{r}{p} \in \sqrt{Ann(NP)}$, where $r \in R, p \notin P$. Then $\frac{r^n}{p^n} = (\frac{r}{p})^n \in Ann(NP)$, for some positive integer $n$. So that $\frac{r^n}{p^n}NP = 0$, then by [3, Corollary 2.9], we get $(r^nN)_P = 0$. Let $m \in N$ be any element, then we have $\frac{r^n}{p^n}m = 0$, so by [4, Lemma 2.1], we get $r^nm = 0$, so that $r^nN = 0$ and thus $r^n \in Ann(N)$, that is $r \in \sqrt{Ann(N)}$ and thus $\frac{r}{p} \in (\sqrt{Ann(N)})_P$. Hence $\sqrt{Ann(NP)} \subseteq (\sqrt{Ann(N)})_P$. Conversely, let $\frac{r}{p} \in (\sqrt{Ann(N)})_P$, for $r \in R, p \notin P$. Then $q^nr^n = (qr)^n \in Ann(N)$, for some positive integer $n$ and some $q \notin P$. So that $q^nr^nN = 0$. If $\frac{m}{u} \in NP$ is any element, where $m \in M, u \notin P$, then $vm \in N$, for some $v \notin P$, so that $(\frac{r}{p})^n m = \frac{r^n}{p^n} m = \frac{q^n}{q^n} \frac{r^n}{p^n} \frac{u}{v} m = \frac{q^n}{q^n} \frac{r^n}{p^n} vu \frac{u}{v} m = 0$, that means $(\frac{r}{p})^n \in Ann(NP)$, which gives that $\frac{r}{p} \in \sqrt{Ann(NP)}$. Thus $(\sqrt{Ann(N)})_P \subseteq \sqrt{Ann(NP)}$. Hence we have $\sqrt{Ann(NP)} = (\sqrt{Ann(N)})_P$.

**Proposition 2.11.** Let $M$ be an $R$–module, $N$ a proper submodule of $M$ and $P$ is a maximal ideal of $R$. If $L$ is a prime submodule of $M$ for which $N \nsubseteq L$ and $S_M(L) \subseteq P$, then $L_P$ is a prime submodule of $MP$ such that $NP \nsubseteq L_P$.

Proof. By [3, Proposition 2.17], we have $L_P$ is a proper submodule of $MP$. Let $\frac{r}{p}m \frac{m}{q} \in L_P$, where $r \in R, m \in M$ and $p, q \notin P$. Then $urm \in L$, for some $u \notin P$ and so $u \notin S_M(L)$, from which we get $rm \in L$. As $L$ is prime we get $m \in L$ or $rM \subseteq L$. The first case gives $\frac{m}{q} \in L_P$ and by using [3, Corollary 2.9], the second case leads to $\frac{r}{p}M_P = rM_P \subseteq L_P$. Hence $L_P$ is a prime submodule of $MP$. If possible suppose that $NP \subseteq L_P$. Let $x \in N$, then $\frac{x}{1} \in L_P$, from which, by [4, Lemma 2.1], we get $x \in L$, so $N \subseteq L$, which is a contradiction and thus $NP \nsubseteq L_P$.

**Proposition 2.12.** Let $M$ be an $R$–module and $N$ a proper submodule of $M$ and $P$ is a maximal ideal of $R$. If $L$ is a prime submodule of $MP$ such that $NP \nsubseteq L$, then there exists a prime submodule $L$ of $M$ with $L = L_P$ for which $N \nsubseteq L$ and $S_M(L) \subseteq P$.

Proof. By [4, Lemma 2.27], we have $L_P = L_P$, for the prime submodule $L = \{x \in M : \frac{x}{1} \in L\}$ of $M$ and $S_M(L) \subseteq P$. So we have $NP \nsubseteq L_P$, which gives $N \nsubseteq L$.

Combining Proposition 2.11 and Proposition 2.12, we get the following theorem.

**Theorem 2.13.** Let $M$ be an $R$–module and $N$ a proper submodule of $M$. If $P$ is a maximal ideal of $R$, then there is a one to one correspondence between the prime submodules $L$ of $M$, for which $N \nsubseteq L$ and $S_M(L) \subseteq P$ and the prime submodules of $MP$ that do not contain $NP$.

Proof. Let $D_P(N) = \{L : L$ is a prime submodule of $M$ such that $N \nsubseteq L$ and $S_M(L) \subseteq P\}$ and $D(NP) = \{L : L$ is a prime submodule of $MP$ such that $NP \nsubseteq L\}$. Then $f : D_P(N) \to D(NP)$, defined by $f(L) = L_P$ is the required correspondence.
Proposition 2.14. Let $M$ be an $R$–module and $P$ a maximal ideal of $R$. If $N$ is a proper submodule of $M$, then $\cap D(N_P) = (\cap D_P(N))_P$.

Proof. Let $\frac{x}{p} \in \cap D(N_P)$, for $x \in M, p \notin P$. Let $L \in D_P(N)$, then $L$ is a prime submodule of $M$ with $N \notin L$ and $S_M(L) \subseteq P$. By Proposition 2.11, we have $L_P$ is a prime submodule of $M_P$ with $N_P \notin L_P$, so that $L_P \subseteq D(N_P)$ and thus $\frac{x}{p} \in L_P$ and then by [4, Lemma 2.1], we get $x \in L$, so that $x \in \cap D_P(N)$, which gives that $\frac{x}{p} \in (\cap D_P(N))_P$. Hence $\cap D(N_P) \subseteq (\cap D_P(N))_P$. Conversely, let $\frac{x}{p} \in (\cap D_P(N))_P$, for $x \in M, p \notin P$. Then $qx \in \cap D_P(N)$, for some $q \notin P$. Let $\bar{\mathcal{T}} \in D(N_P)$, that is, $\bar{\mathcal{T}}$ is a prime submodule of $M_P$ and $N_P \notin \bar{\mathcal{T}}$, so by Proposition 2.12, $\mathcal{T} = L_P$, where $L = \{x \in M : \frac{x}{p} \in \bar{\mathcal{T}}\}$ is a prime submodule of $M$ for which $N \notin L$ and $S_M(L) \subseteq P$, that means $L \subseteq D_P(N)$ and thus we have $qx \in L$. If $x \notin L$, then $q \notin S_M(L) \subseteq P$, which is a contradiction, so we must have $x \in L$, which gives that $\frac{x}{p} \in \bar{\mathcal{T}}$ and then $\frac{x}{p} = \frac{1}{p} \frac{x}{1} \in \bar{\mathcal{T}}$, thus $\frac{x}{p} \in \cap D(N_P)$, so that $(\cap D_P(N))_P \subseteq \cap D(N_P)$. Hence $\cap D(N_P) = (\cap D_P(N))_P$.

Lemma 2.15. Let $M$ be an $R$–module and $P$ a maximal ideal of $R$. If $N$ and $L$ are submodule of $M$ such that $S_M(N) \subseteq P$ and $S_M(L) \subseteq P$, then $N = L$ if and only if $N_P = L_P$.

Proof. Let $N = L$, then it is obvious that $N_P = L_P$. Conversely, suppose that $N_P = L_P$. Let $x \in N$, then $\frac{x}{p} \in L_P$ and by [4, Lemma 2.1], we get $x \in L$, so that $N \subseteq L$. In a similar argument we can prove that $L \subseteq N$ and thus we get $N = L$.

Proposition 2.16. Let $M$ be a non zero locally reduced and a locally multiplication $R$–module with $S_M(0) \subseteq J(R)$ and let $N$ be a proper submodule of $M$. If $S_M(Ann(N)M) \subseteq J(R), S_M(\cap D_P(N)) \subseteq J(R)$, then $\cap D_P(N) = Ann(M)_M$, for each maximal ideal $P$ of $R$.

Proof. Let $P$ be any maximal ideal of $R$. Then $M_P$ is a reduced and a multiplication $R_P$–module and as $M \neq 0$, we get $M_P \neq 0$, so by [7, Lemma 2.4], we have $Ann(N_P)M_P = \cap D(N_P)$. Then by using [4, Proposition 2.5] and Proposition 2.14, we get $(Ann(N)M)_P = (Ann(N))_P M_P = Ann(N_PM_P = \cap D(N_P) = (\cap D_P(N))_P$. As we have $S_M(Ann(N)M) \subseteq J(R) \subseteq P, S_M(\cap D_P(N)) \subseteq J(R) \subseteq P$, so by Lemma 2.15, we get $\cap D_P(N) = Ann(M)_M$.

Proposition 2.17. Let $M$ be a non zero locally reduced and a locally multiplication $R$–module and $N$ is a proper submodule of $M$. If $S_M(0) \subseteq J(R)$, then $\sqrt{Ann(N)} = Ann(N)$.

Proof. Let $P$ be any maximal ideal of $R$. Then as $S_M(0) \subseteq J(R)$, we have $S_M(0) \subseteq P$ and $M_P$ is a reduced and a multiplication $R$–module and since $M \neq 0$, so $M_P \neq 0$. Thus by [7, Lemma 2.4], we have $\sqrt{Ann(N_P)} = Ann(N_P)$, then by Proposition 2.10 and [4, Proposition 2.5], we have $(\sqrt{Ann(N)})_P = \sqrt{Ann(N_P)} = Ann(N_P) = (Ann(N))_P$. Hence by [3, Corollary 2.2], we get $\sqrt{Ann(N)} = Ann(N)$. 

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**Proposition 2.18.** Let $M$ be a non zero locally reduced and a locally multiplication $R$–module and $N, L$ are proper submodules of $M$ with $N \subseteq L, S_M(L) \subseteq J(R)$ and $S_M(0) \subseteq J(R)$, then $N$ is essential in $L$ if and only if $Ann(N) = Ann(L)$.

Proof. Let $P$ be any maximal ideal of $R$, so that $M$ is both a reduced and a multiplication $R$–module and as $S_M(0) \subseteq J(R)$, we have $S_M(0) \subseteq P$ and $S_M(L) \subseteq P$. Now, $N \subseteq L$ gives $N_P \subseteq L_P$ and as $M \neq 0$, $S_M(0) \subseteq P$, we have $M_P \neq 0$. Now, suppose that $N$ is essential in $L$. First, we claim that $N_P$ is essential in $L_P$. Let $\overline{K}$ be a submodule of $L_P$ such that $N_P \cap \overline{K} = 0$, then $\overline{K} = K_P$, for some submodule $K$ of $L$. Let $x \in N \cap K$, then $\overline{x} \in (N \cap K)_P = N_P \cap K_P = 0$, so that $\overline{x} = 0$ and thus by [4, Lemma 2.1], we get $x = 0$, so that $N \cap K = 0$. Then as $N$ is essential in $L$, we get $K = 0$ which gives $\overline{K} = K_P = 0$. Hence $N_P$ is essential in $L_P$. Then by [7, Theorem 2.5], we have $Ann(N_P) = Ann(L_P)$, so by [4, Proposition 2.5], we get $(Ann(N))_P = (Ann(L))_P$ and by [3, Corollary 2.2], we get $Ann(N) = Ann(L)$. Conversely, suppose that $Ann(N) = Ann(L)$. Then by [4, Proposition 2.5], we have $Ann(N_P) = (Ann(N))_P = (Ann(L))_P = Ann(L_P)$ and as $N_P \subseteq L_P$, by [7, Theorem 2.5], we have $N_P$ is essential in $L_P$. Let $K$ be any submodule of $L$ such that $N \cap K = 0$, then $K_P$ is a submodule of $L_P$ and then we have $N_P \cap K_P = (N \cap K)_P = 0$, so we get $K_P = 0$, and as $S_M(0) \subseteq P$, by [4, Lemma 2.1], we get $K = 0$, so $N$ is essential in $L$.

**Definition 2.19.** Let $M$ be an $R$–module and $P$ be a maximal ideal of $R$. For a submodule $N$ of $M$, we define $V_P(N) = \{ L \in Spec(M) : N \subseteq L \text{ and } S_M(L) \subseteq P \}$.

**Proposition 2.20.** Let $M$ be an $R$–module and $N$ a proper submodule of $M$. If $P$ is a maximal ideal of $R$, then $V_P(N) = \phi$ if and only if $V(N_P) = \phi$.

Proof. Let $V_P(N) = \phi$. If $V(N_P) \neq \phi$, then there exists $\overline{L} \in V(N_P)$, that is $\overline{L}$ is a prime submodule of $M_P$ such that $N_P \subseteq \overline{L}$. Then by [3, Proposition 2.16], $\overline{L} = L_P$, for the prime submodule $L = \{ x \in M : \overline{x} \in \overline{L} \}$ of $M$ and by [4, Lemma 2.27], we have $S_M(L) \subseteq P$. If $N \not\subseteq L$, then there exists $x \in N$ and $x \notin L$, then $\overline{x} \in N_P \not\subseteq \overline{L}$, from which we get $x \in L$, that is a contradiction, so that $N \subseteq L$, which implies that $L \in V_P(N)$ and thus $V_P(N) \neq \phi$, that is a contradiction. Hence $V(N_P) = \phi$. Conversely, suppose that $V(N_P) = \phi$. If $V_P(N) \neq \phi$, then there exists $L \in V_P(N)$, that is, $L$ is a prime submodule of $M$ with $N \subseteq L$ and $S_M(L) \subseteq P$, this gives $N_P \subseteq L_P$ and by [4, Proposition 2.21], we get $L_P$ is a prime submodule of $M_P$ with $N_P \subseteq L_P$, so that $L_P \in V(N_P)$ and thus $V(N_P) \neq \phi$, which is a contradiction, so that $V_P(N) = \phi$.

**Proposition 2.21.** Let $M$ be a $R$–module and $N$ a proper submodule of $M$. If $P$ is a maximal ideal of $R$ such that $S_M(0) \subseteq P$, then $N$ is essential in $M$ if and only if $N_P$ is essential in $M_P$.

Proof. Let $N$ be essential in $M$. To show $N_P$ is essential in $M_P$. Let $\overline{K}$ be a submodule of $M_P$ such that $N_P \cap \overline{K} = 0$, then $\overline{K} = K_P$, for some submodule
$K$ of $M$. Let $x \in N \cap K$, then $\bar{x} \in (N \cap K)_P = N_P \cap K_P = 0$, so that $\bar{x} = 0$ and thus by [4, Lemma 2.1], we get $x = 0$, so that $N \cap K = 0$. Then as $N$ is essential in $M$, we get $K = 0$ which gives $\overline{K} = K_P = 0$. Hence $N_P$ is essential in $M_P$. Conversely, suppose that $N_P$ is essential in $M_P$. Let $K$ be any submodule of $M$ such that $N \cap K = 0$, then we get $N_P \cap K_P = (N \cap K)_P = 0$, this gives $K_P = 0$ and then by [4, Lemma 2.1], we have $K = 0$. Hence $N$ is essential in $M$.

**Proposition 2.22.** Let $M$ be a non zero locally reduced and a locally multiplication $R$–module and $P$ is a maximal ideal of $R$. If $N$ a non zero proper submodule of $M$ such that $S_M(N) \subseteq P$ and $S_M(0) \subseteq P$, then $N$ is essential in $M$ if and only if $V_P(N) = \phi$.

Proof. As $M \neq 0$ and $S_M(0) \subseteq P$, we have $M_P \neq 0$. Let $N$ be essential in $M$, then by Proposition 2.21, $N_P$ is essential in $M_P$ and as $N \neq 0$, we have $N_P \neq 0$ (otherwise, as $S_M(N) \subseteq P$, by [4, Lemma 2.1], we get $N = 0$). Hence by [7, Corollary 2.6], we get $V(N_P) = \phi$ and so by Proposition 2.20, we get $V_P(N) = \phi$. Conversely, if $V_P(N) = \phi$, then by Proposition 2.20, we have $V(N_P) = \phi$ and then by [7, Corollary 2.6], we have $N_P$ is essential in $M_P$ and by Proposition 2.21, $N$ is essential in $M$.

**References**


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