International Mathematical Forum, Vol. 8, 2013, no. 18, 851 - 858
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On Locally Reduced and Locally Multiplication Modules

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Abstract

This paper presents and defines two types of modules which are called locally reduced and locally multiplication modules and several properties of these types of modules are studied and proved.

Keywords: reduced modules, multiplication modules, locally reduced modules, locally multiplication modules

1 Introduction

Let R be a commutative ring with identity, M be an R-module and N be a submodule of M. The Jacobson radical of R, denoted by $J(R)$, is the intersection of all maximal ideals of R. M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N = IM$ [2]. N is called a prime submodule of M if $N \neq M$ and for $r \in R, m \in M$, the condition $rm \in N$ implies that $m \in N$ or $rM \subseteq N$ [8]. The spectrum of M, denoted by $Spec(M)$, is the set of all prime submodules of M, that is, $Spec(M) = \{P : P$ is a prime submodule of $M\}$ and M is called a reduced module if $\cap Spec(M) = 0$ [7]. An element $r \in R$ is called prime to N if $rm \in N$, for $m \in M$, implies that $m \in N$ [1], equivalently, $r \in R$ is not prime to N if $rm \in N$ for some $m \in M - N$. If we denote the set of all elements of R that are not prime to N by $S_M(N)$, then we have $S_M(N) = \{r \in R : rm \in N,$ for some $m \in M - N\}$, specially, if $N = 0$, then $S_M(0) = \{r \in R : rm =$

0, for some $m \neq 0\}$. N is called a maximal submodule of M if $N \neq M$ and N is not properly contained in any proper submodule of M [9] and it is called an essential submodule of M if K is any submodule of M such that $N \cap K = 0$, then $K = 0$. The annihilator of N, denoted by $Ann(N)$, is the set $Ann(N) = \{x \in M : xN = 0\}$ and the annihilator of M is defined as $Ann(M) = \{r \in R : rM = 0\}$. $V(N) = \{L : L$ is a prime submodule of M for which $N \subseteq L\}$ and $D(N) = Spec(M) - V(N)$ [7], or equivalently, we can say that $D(N) = \{L : L \in Spec(M)$ and $N \nsubseteq L\}$. A non empty subset S of R is called a multiplicative closed set if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$ [5]. If S is a multiplicative set in R, then one can easily make R_S as a commutative ring with identity [5] and make M_S as an R_S-module under the module operations $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$ and $\frac{r}{u}.\frac{x}{s} = \frac{rx}{us}$, for $\frac{r}{u} \in R_S$ and $\frac{x}{s}, \frac{y}{t} \in M_S$ [6], so that when we say M_S is a module we mean M_S is an R_S-module.

Throughout this paper R is a commutative ring with identity and M is a non zero left unitary $R-$module unless otherwise stated.

2 The Results

First, we introduce the following definitions.

Definition2.1. An $R-$module M is called locally reduced if M_P is a reduced R_P-module for each maximal ideal P of R and it is called a locally multiplication $R-$module if M_P is a multiplication R_P-module for each maximal ideal P of R.

Definition2.2. Let M be an $R-$module and P a maximal ideal of R, we define $Spec_P(M) = \{N : N \in Spec(M)$ and $S_M(N) \subseteq P\}$ and we say that M is a strongly reduced $R-$module if $\cap Spec_P(M) = 0$, for each maximal ideal P of R and we denote the set of all maximal submodules of M by $Spec_m(M)$.

Proposition2.3. Every maximal submodule N of an $R-$module M is prime.

Proof. N is a proper submodule of M. Let $rx \in N$, for $r \in R$ and $x \in M$. If $x \notin N$, then we have $N \subset < x > +N$ and as N is maximal in M, we get $< x > +N = M$. Now, if $m \in M$ then $m = sx + b$, for $s \in R$ and $b \in N$, from which we get $rm = srx + rb \in N$, so that $rM \subseteq N$ and thus N is a prime submodule of M.

Theorem2.4. Let M be an $R-$module and let P be a maximal ideal of R, then:

1. $(\cap Spec(M))_P \subseteq \cap Spec(M_P)$.
2. $\cap Spec(M) \subseteq \cap Spec_P(M)$.
3. $\cap Spec(M_P) \subseteq (\cap Spec_P(M))_P$.
4. $\cap Spec(M) \subseteq \cap Spec_m(M)$.

Proof. (1) Let $\frac{x}{p} \in (\cap Spec(M))_P$, for $x \in M$ and $p \notin P$. Then $qx \in$

$\cap Spec(M)$, for some $q \notin P$. Now, let $\overline{N} \in Spec(M_P)$, that is $\overline{N}$ is a prime submodule of M_P, then by [3, Proposition 2.16], $\overline{N} = N_P$, for the prime submodule $N = \{x \in M : \frac{x}{1} \in \overline{N}\}$ of M and by [4, Lemma 2.27], we get $S_M(N) \subseteq P$, that means $N \in Spec(M)$ and thus $qx \in N$. If $x \notin N$, then $q \in S_M(N) \subseteq P$, which is a contradiction and thus we must have $x \in N$, that gives $\frac{x}{p} \in N_P = \overline{N}$, so that we get $\frac{x}{p} \in \cap Spec(M_P)$. Hence $(\cap Spec(M))_P \subseteq \cap Spec(M_P)$.

(2) Since $Spec_P(M) \subseteq Spec(M)$, so we have $\cap Spec(M) \subseteq \cap Spec_P(M)$.

(3) Let $\frac{x}{p} \in \cap Spec(M_P)$, where $x \in M$ and $p \notin P$. Let $N \in \cap Spec_P(M)$, then $N \in Spec(M)$ and $S_M(N) \subseteq P$, so by [4, Proposition 2.21], we have N_P is a prime submodule of M_P, that means $N_P \in Spec(M_P)$, so that $\frac{x}{p} \in N_P$ and then by [4, Lemma 2.1], we have $x \in N$, thus we get $x \in \cap Spec_P(M)$, this gives $\frac{x}{p} \in (\cap Spec_P(M))_P$. Hence $\cap Spec(M_P) \subseteq (\cap Spec_P(M))_P$.

(4) By Proposition 2.3, we have $Spec_m(M) \subseteq Spec(M)$, so that $\cap Spec(M) \subseteq Spec_m(M)$.

As a corollary we give:

Corollary2.5. Let M be an $R-$module, then:

(1) If M is locally reduced, then it is reduced.

(2) If M is a strongly reduced $R-$module, then it is locally reduced (and hence it is reduced).

(3) If $\cap Spec_m(M) = 0$, then M is reduced.

Proof. (1) Let P be any maximal ideal of R (such maximal ideals exist since R is a commutative ring with identity), then M_P is reduced, that is $\cap Spec(M_P) = 0$. Then by Theorem 2.4 (1), we get $(\cap Spec(M))_P = 0$ and by [3, Corollary 2.3], we get $\cap Spec(M) = 0$, thus M is reduced.

(2) Let P be any maximal ideal of R. As M is strongly reduced, we have $\cap Spec_P(M) = 0$, Then by Theorem 2.4 (3), we have $\cap Spec(M_P) = 0$, so that M_P is reduced and thus M is locally reduced.

(3) The proof follows directly from Theorem 2.4 (4).

Proposition2.6. Let M be an $R-$module and P a maximal ideal of R. If N is a maximal submodule of M such that $S_M(N) \subseteq P$, then N_P is a maximal submodule of M_P.

Proof. If $N_P = M_P$, then for any $m \in M$, we have $\frac{m}{1} \in N_P$ and then by [4, Lemma 2.1], we get $m \in N$, which gives $N = M$, that is a contradiction, so that N_P is a proper submodule of M_P. Next, suppose that $\overline{K}$ is a proper submodule of M_P such that $N_P \subseteq \overline{K}$, then by [3, proposition 2.16], we have $\overline{K} = K_P$, for the submodule $K = \{x \in M : \frac{x}{1} \in \overline{K}\}$ of M and hence we get $N_P \subseteq K_P$. If $m \in N$, then $\frac{m}{1} \in K_P = \overline{K}$, so that $m \in K$ and thus we have $N \subseteq K \subseteq M$. As N is a maximal submodule and $K \neq M$ (otherwise $\overline{K} = K_P = M_P$, that is a contradiction), so we must have $N = K$ and thus $N_P = K_P = \overline{K}$. Hence N_P is a maximal submodule of M_P.

Proposition2.7. Let M be an $R-$module and P a maximal ideal of R. If

$\overline{N}$ is a maximal submodule of M_P, then there exists a submodule N of M with $\overline{N} = N_P$ and such that N is maximal with respect to the relation $S_M(N) \subseteq P$.

Proof. By [3, Proposition 2.16], we have $\overline{N} = N_P$, for the submodule $N = \{x \in M : \frac{x}{1} \in \overline{N}\}$ of M. By [4, Lemma 2.27], we have $S_M(N) \subseteq P$. Next, let K be any proper submodule of M such that $N \subseteq K$ with $S_M(K) \subseteq P$. Then $\overline{N} = N_P \subseteq K_P$. If $K_P = M_P$, then for each $m \in M$, we have $\frac{m}{1} \in K_P$, so by [4, Lemma 2.1], we get $m \in K$, that means $K = M$ and that is a contradiction, so that K_P is a proper submodule of M_P. As $\overline{N}$ is maximal in M_P, we get $\overline{N} = N_P = K_P$, from which one can easily get that $N = K$. Hence N is a maximal submodule of M.

Combining Proposition 2.6 and Proposition 2.7, we give the following theorem.

Theorem2.8. Let M be an $R-$module and P a maximal ideal of R, then there is a one to one correspondence between the maximal submodules N of M for which $S_M(N) \subseteq P$ and the maximal submodules of M_P.

Proof. Suppose that $F = \{N : N$ is a maximal submodule of M for which $S_M(N) \subseteq P\}$ and $H = \{\overline{N} : \overline{N}$ is a maximal submodule of $M_P\}$. Define $f : F \to H$ as follows: let $N \in F$, then by Proposition 2.6, we have $N_P \in H$, so we set $f(N) = N_P$. One can easily show that this definition provides a one to one correspondence between F and H.

It is necessary to mention that, if M is a non zero $R-$module and P is a maximal ideal of R with $S_M(0) \subseteq P$, then we have $M_P \neq 0$. To show this, suppose that $M_P = 0$, then for any $x \in M$, we have $\frac{x}{1} = 0$ and as $S_M(0) \subseteq P$, by [4, Lemma 2.1], we get $x = 0$, so that $M = 0$, that is a contradiction and hence $M_P \neq 0$. As especial case, if $S_M(0) \subseteq J(R)$, then for each maximal ideal P of R we have $S_M(0) \subseteq P$ and thus $M_P \neq 0$, for each maximal ideal P of R.

Proposition2.9. If M is a non zero locally reduced and a locally multiplication $R-$module with $S_M(0) \subseteq J(R)$ and N is a submodule of M, then:

(1) $N \cap Ann(N)M = 0$.

(2) $Ann(N + Ann(N)M) = Ann(M)$.

Proof. Let P be any maximal ideal of R, then M_P is a reduced and a multiplication $R-$module and as $M \neq 0$, so by what we have mentioned in the above we have $M_P \neq 0$.

(1) By [7, Lemma 2.3], we have $N_P \cap (Ann(N_P)M_P) = 0$ and by [4, Proposition 2.4], we have $(Ann(N))_P \subseteq Ann(N_P)$, this gives that $[N \cap Ann(N)M]_P = N_P \cap (Ann(N)M)_P = N_P \cap (Ann(N))_P M_P \subseteq N_P \cap Ann(N_P)M_P = 0$, thus by [3, Corollary 2.3], we get $N \cap Ann(N)M = 0$.

(2) By [7, Lemma 2.3], we have $Ann[N_P + Ann(N_P)M_P] = Ann(M_P)$ and as $S_M(0) \subseteq J(R) \subseteq P$, by using [4, Proposition 2.5], we get $(Ann(N) + Ann(N)M)_P = (Ann(M))_P$. Hence by [3, Corollary 2.2] and [3, Corollary 2.3], we get $Ann(N + Ann(N)M) = Ann(M)$.

Proposition2.10. Let M be an $R-$module and N a submodule of M. If

P is a maximal ideal of R with $S_M(0) \subseteq P$, then $\sqrt{Ann(N_P)} = (\sqrt{Ann(N)})_P$.

Proof. Let $\frac{r}{p} \in \sqrt{Ann(N_P)}$, where $r \in R, p \notin P$. Then $\frac{r^n}{p^n} = (\frac{r}{p})^n \in Ann(N_P)$, for some positive integer n. So that $\frac{r^n}{p^n}N_P = 0$, then by [3, Corollary 2.9], we get $(r^nN)_P = 0$. Let $m \in N$ be any element, then we have $\frac{r^n m}{1} = 0$, so by [4, Lemma 2.1], we get $r^n m = 0$, so that $r^n N = 0$ and thus $r^n \in Ann(N)$, that is $r \in \sqrt{Ann(N)}$ and thus $\frac{r}{p} \in (\sqrt{Ann(N)})_P$. Hence $\sqrt{Ann(N_P)} \subseteq (\sqrt{Ann(N)})_P$. Conversely, let $\frac{r}{p} \in (\sqrt{Ann(N)})_P$, for $r \in R, p \notin P$. Then $q^n r^n = (qr)^n \in Ann(N)$, for some positive integer n and some $q \notin P$. So that $q^n r^n N = 0$. If $\frac{m}{u} \in N_P$ is any element, where $m \in M, u \notin P$, then $vm \in N$, for some $v \notin P$, so that $(\frac{r}{p})^n \frac{m}{u} = \frac{r^n}{p^n}\frac{m}{u} = \frac{q^n}{q^n}\frac{r^n}{p^n}\frac{v}{v}\frac{m}{u} = \frac{q^n r^n vm}{q^n p^n vu} = 0$, that means $(\frac{r}{p})^n \in Ann(N_P)$, which gives that $\frac{r}{p} \in \sqrt{Ann(N_P)}$. Thus $(\sqrt{Ann(N)})_P \subseteq \sqrt{Ann(N_P)}$. Hence we have $\sqrt{Ann(N_P)} = (\sqrt{Ann(N)})_P$.

Proposition2.11. Let M be an $R-$module, N a proper submodule of M and P is a maximal ideal of R. If L is a prime submodule of M for which $N \nsubseteq L$ and $S_M(L) \subseteq P$, then L_P is a prime submodule of M_P such that $N_P \nsubseteq L_P$.

Proof. By [3, Proposition 2.17], we have L_P is a proper submodule of M_P. Let $\frac{r}{p}\frac{m}{q} \in L_P$, where $r \in R, m \in M$ and $p, q \notin P$. Then $urm \in L$, for some $u \notin P$ and so $u \notin S_M(L)$, from which we get $rm \in L$. As L is prime we get $m \in L$ or $rM \subseteq L$. The first case gives $\frac{m}{q} \in L_P$ and by using [3, Corollary 2.9], the second case leads to $\frac{r}{p}M_P = (rM)_P \subseteq L_P$. Hence L_P is a prime submodule of M_P. If possible suppose that $N_P \subseteq L_P$. Let $x \in N$, then $\frac{x}{1} \in L_P$, from which, by [4, Lemma 2.1], we get $x \in L$, so $N \subseteq L$, which is a contradiction and thus $N_P \nsubseteq L_P$.

Proposition2.12. Let M be an $R-$module and N a proper submodule of M and P is a maximal ideal of R. If $\overline{L}$ is a prime submodule of M_P such that $N_P \nsubseteq \overline{L}$, then there exists a prime submodule L of M with $\overline{L} = L_P$ for which $N \nsubseteq L$ and $S_M(L) \subseteq P$.

Proof. By [4, Lemma 2.27], we have $\overline{L} = L_P$, for the prime submodule $L = \{x \in M : \frac{x}{1} \in \overline{L}\}$ of M and $S_M(L) \subseteq P$. So we have $N_P \nsubseteq L_P$, which gives $N \nsubseteq L$.

Combining Proposition 2.11 and Proposition 2.12, we get the following theorem.

Theorem2.13. Let M be an $R-$module and N a proper submodule of M. If P is a maximal ideal of R, then there is a one to one correspondence between the prime submodules L of M, for which $N \nsubseteq L$ and $S_M(L) \subseteq P$ and the prime submodules of M_P that not contain N_P.

Proof. Let $D_P(N) = \{L : L$ is a prime submodule of M such that $N \nsubseteq L$ and $S_M(L) \subseteq P\}$ and $D(N_P) = \{\overline{L} : \overline{L}$ is a prime submodule of M_P such that $N_P \nsubseteq \overline{L}\}$. Then $f : D_P(N) \to D(N_P)$, defined by $f(L) = L_P$ is the required correspondence.

Proposition2.14. Let M be an $R-$module and P a maximal ideal of R. If N is a proper submodule of M, then $\cap D(N_P) = (\cap D_P(N))_P$.

Proof. Let $\frac{x}{p} \in \cap D(N_P)$, for $x \in M, p \notin P$. Let $L \in D_P(N)$, then L is a prime submodule of M with $N \nsubseteq L$ and $S_M(L) \subseteq P$. By Proposition 2.11, we have L_P is a prime submodule of M_P with $N_P \nsubseteq L_P$, so that $L_P \subseteq D(N_P)$ and thus $\frac{x}{p} \in L_P$ and then by [4, Lemma 2.1], we get $x \in L$, so that $x \in \cap D_P(N)$, which gives that $\frac{x}{p} \in (\cap D_P(N))_P$. Hence $\cap D(N_P) \subseteq (\cap D_P(N))_P$. Conversely, let $\frac{x}{p} \in (\cap D_P(N))_P$, for $x \in M, p \notin P$. Then $qx \in \cap D_P(N)$, for some $q \notin P$. Let $\overline{L} \in D(N_P)$, that is, $\overline{L}$ is a prime submodule of M_P and $N_P \nsubseteq \overline{L}$, so by Proposition 2.12, $\overline{L} = L_P$, where $L = \{x \in M : \frac{x}{1} \in \overline{L}\}$ is a prime submodule of M for which $N \nsubseteq L$ and $S_M(L) \subseteq P$, that means $L \subseteq D_P(N)$ and thus we have $qx \in L$. If $x \notin L$, then $q \in S_M(L) \subseteq P$, which is a contradiction, so we must have $x \in L$, which gives that $\frac{x}{1} \in \overline{L}$ and then $\frac{x}{p} = \frac{1}{p}\frac{x}{1} \in \overline{L}$, thus $\frac{x}{p} \in \cap D(N_P)$, so that $(\cap D_P(N))_P \subseteq \cap D(N_P)$. Hence $\cap D(N_P) = (\cap D_P(N))_P$.

Lemma2.15. Let M be an $R-$module and P a maximal ideal of R. If N and L are submodule of M such that $S_M(N) \subseteq P$ and $S_M(L) \subseteq P$, then $N = L$ if and only if $N_P = L_P$.

Proof. Let $N = L$, then it is obvious that $N_P = L_P$. Conversely, suppose that $N_P = L_P$. Let $x \in N$, then $\frac{x}{1} \in L_P$ and by [4, Lemma 2.1], we get $x \in L$, so that $N \subseteq L$. In a similar argument we can prove that $L \subseteq N$ and thus we get $N = L$.

Proposition2.16. Let M be a non zero locally reduced and a locally multiplication $R-$module with $S_M(0) \subseteq J(R)$ and let N be a proper submodule of M. If $S_M(Ann(N)M) \subseteq J(R), S_M(\cap D_P(N)) \subseteq J(R)$, then $\cap D_P(N) = Ann(N)M$, for each maximal ideal P of R.

Proof. Let P be any maximal ideal of R. Then M_P is a reduced and a multiplication R_P-module and as $M \neq 0$, we get $M_P \neq 0$, so by [7, Lemma 2.4], we have $Ann(N_P)M_P = \cap D(N_P)$. Then by using [4, Proposition 2.5] and Proposition 2.14, we get $(Ann(N)M)_P = (Ann(N))_P M_P = Ann(N_P)M_P = \cap D(N_P) = (\cap D_P(N))_P$. As we have $S_M(Ann(N)M) \subseteq J(R) \subseteq P, S_M(\cap D_P(N)) \subseteq J(R) \subseteq P$, so by Lemma 2.15, we get $\cap D_P(N) = Ann(N)M$.

Proposition2.17. Let M be a non zero locally reduced and a locally multiplication $R-$module and N is a proper submodule of M. If $S_M(0) \subseteq J(R)$, then $\sqrt{Ann(N)} = Ann(N)$.

Proof. Let P be any maximal ideal of R. Then as $S_M(0) \subseteq J(R)$, we have $S_M(0) \subseteq P$ and M_P is a reduced and a multiplication $R-$module and since $M \neq 0$, so $M_P \neq 0$. Thus by [7, Lemma 2.4], we have $\sqrt{Ann(N_P)} = Ann(N_P)$, then by Proposition 2.10 and [4, Proposition 2.5], we have $(\sqrt{Ann(N)})_P = \sqrt{Ann(N_P)} = Ann(N_P) = (Ann(N))_P$. Hence by [3, Corollary 2.2], we get $\sqrt{Ann(N)} = Ann(N)$.

Proposition2.18. Let M be a non zero locally reduced and a locally multiplication $R-$module and N, L are proper submodules of M with $N \subseteq L, S_M(L) \subseteq J(R)$ and $S_M(0) \subseteq J(R)$, then N is essential in L if and only if $Ann(N) = Ann(L)$.

Proof. Let P be any maximal ideal of R, so that M is both a reduced and a multiplication $R-$module and as $S_M(0) \subseteq J(R)$, we have $S_M(0) \subseteq P$ and $S_M(L) \subseteq P$. Now, $N \subseteq L$ gives $N_P \subseteq L_P$ and as $M \neq 0$, $S_M(0) \subseteq P$, we have $M_P \neq 0$. Now, suppose that N is essential in L. First, we claim that N_P is essential in L_P. Let $\overline{K}$ be a submodule of L_P such that $N_P \cap \overline{K} = 0$, then $\overline{K} = K_P$, for some submodule K of L. Let $x \in \ N \cap K$, then $\frac{x}{1} \in (N \cap K)_P = N_P \cap K_P = 0$, so that $\frac{x}{1} = 0$ and thus by [4, Lemma 2.1], we get $x = 0$, so that $N \cap K = 0$. Then as N is essential in L, we get $K = 0$ which gives $\overline{K} = K_P = 0$. Hence N_P is essential in L_P. Then by [7, Theorem 2.5], we have $Ann(N_P) = Ann(L_P)$, so by [4, Proposition 2.5], we get $(Ann(N))_P = (Ann(L))_P$ and by [3, Corollary 2.2], we get $Ann(N) = Ann(L)$. Conversely, suppose that $Ann(N) = Ann(L)$. Then by [4, Proposition 2.5], we have $Ann(N_P) = (Ann(N))_P = (Ann(L))_P = Ann(L_P)$ and as $N_P \subseteq L_P$, by [7, Theorem 2.5], we have N_P is essential in L_P. Let K be any submodule of L such that $N \cap K = 0$, then K_P is a submodule of L_P and then we have $N_P \cap K_P = (N \cap K)_P = 0$, so we get $K_P = 0$, and as $S_M(0) \subseteq P$, by [4, Lemma 2.1], we get $K = 0$, so N is essential in L.

Definition2.19. Let M be an $R-$module and P be a maximal ideal of R. For a submodule N of M, we define $V_P(N) = \{L \in Spec(M) : N \subseteq L$ and $S_M(L) \subseteq P\}$.

Proposition2.20. Let M be an $R-$module and N a proper submodule of M. If P is a maximal ideal of R, then $V_P(N) = \phi$ if and only if $V(N_P) = \phi$.

Proof. Let $V_P(N) = \phi$. If $V(N_P) \neq \phi$, then there exists $\overline{L} \in V(N_P)$, that is $\overline{L}$ is a prime submodule of M_P such that $N_P \subseteq \overline{L}$. Then by [3, Proposition 2.16], $\overline{L} = L_P$, for the prime submodule $L = \{x \in M : \frac{x}{1} \in \overline{L}\}$ of M and by [4, Lemma 2.27], we have $S_M(L) \subseteq P$. If $N \nsubseteq L$, then there exists $x \in N$ and $x \notin L$, then $\frac{x}{1} \in N_P \subseteq \overline{L}$, from which we get $x \in L$, that is a contradiction, so that $N \subseteq L$, which implies that $L \in V_P(N)$ and thus $V_P(N) \neq \phi$, that is a contradiction. Hence $V(N_P) = \phi$. Conversely, suppose that $V(N_P) = \phi$. If $V_P(N) \neq \phi$, then there exists $L \in V_P(N)$, that is, L is a prime submodule of M with $N \subseteq L$ and $S_M(L) \subseteq P$, this gives $N_P \subseteq L_P$ and by [4, Proposition 2.21], we get L_P is a prime submodule of M_P with $N_P \subseteq L_P$, so that $L_P \in V(N_P)$ and thus $V(N_P) \neq \phi$, which is a contradiction, so that $V_P(N) = \phi$.

Proposition2.21. Let M be a $R-$module and N a proper submodule of M. If P is a maximal ideal of R such that $S_M(0) \subseteq P$, then N is essential in M if and only if N_P is essential in M_P.

Proof. Let N be essential in M. To show N_P is essential in M_P. Let $\overline{K}$ be a submodule of M_P such that $N_P \cap \overline{K} = 0$, then $\overline{K} = K_P$, for some submodule

K of M. Let $x \in N \cap K$, then $\frac{x}{1} \in (N \cap K)_P = N_P \cap K_P = 0$, so that $\frac{x}{1} = 0$ and thus by [4, Lemma 2.1], we get $x = 0$, so that $N \cap K = 0$. Then as N is essential in M, we get $K = 0$ which gives $\overline{K} = K_P = 0$. Hence N_P is essential in M_P. Conversely, suppose that N_P is essential in M_P. Let K be any submodule of M such that $N \cap K = 0$, then we get $N_P \cap K_P = (N \cap K)_P = 0$, this gives $K_P = 0$ and then by [4, Lemma 2.1], we have $K = 0$. Hence N is essential in M.

Proposition2.22. Let M be a non zero locally reduced and a locally multiplication $R-$module and P is a maximal ideal of R. If N a non zero proper submodule of M such that $S_M(N) \subseteq P$ and $S_M(0) \subseteq P$, then N is essential in M if and only if $V_P(N) = \phi$.

Proof. As $M \neq 0$ and $S_M(0) \subseteq P$, we have $M_P \neq 0$. Let N be essential in M, then by Proposition 2.21, N_P is essential in M_P and as $N \neq 0$, we have $N_P \neq 0$ (otherwise, as $S_M(N) \subseteq P$, by [4, Lemma 2.1], we get $N = 0$). Hence by [7, Corollary 2.6], we get $V(N_P) = \phi$ and so by Proposition 2.20, we get $V_P(N) = \phi$. Conversely, if $V_P(N) = \phi$, then by Proposition 2.20, we have $V(N_P) = \phi$ and then by [7, Corollary 2.6], we have N_P is essential in M_P and by Proposition 2.21, N is essential in M.

References

[1] S. E. Atani and A. Y. Darani : Notes on the Primal Submodules, Chiang Mai J. Sci. 35(3),2008, pp 399-410.

[2] S. E. Atani and F. Farzalipour : On Weakly Prime Submodules, Tamkang Journal of Mathematics, Vol. 38, No. 3, 2007, 247-252.

[3] A. K. Jabbar : A generalization of Prime and Weakly Prime Submodules, Pure Mathematical Sciences, Vol. 2, no. 1, 2013, pp 1-11.

[4] A. K. Jabbar : On Locally Artinian Modules, International Journal of Algebra, Vol. 6, no. 27, 2012, pp 1325-1334.

[5] M. D. Larsen and P. J. McCarthy : Multiplicative Theory of Ideals, Academic Press, 1971.

[6] H. Li :An introduction to Commutative Algebra, World Scientific Publishing Co. Pte. Ltd,2004.

[7] K. Samei: Reduced Multiplication Modules, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 2, 2011, pp 121-132.

[8] A. J. Taherizadeh : Prime Submodules of Modules Satisfying dccr, International Journal of Algebra, Vol. 5, no. 20, 2011, 977-983.

[9] R. Wisbauer: Foundations of Module and Ring Theory, Gordon and Breach Science publishers, 1991.

Received: March 13, 2013