Improved FFT Approximations of Probability Functions Based on Modified Quadrature Rules

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Abstract

In the last ten years, the traditional Simpson quadrature rule for numerical integration has been improved to an optimal 3-point quadrature formula and a modified Simpson rule that takes additionally the first derivative of the approximated integrand at the two end-points of integration into account. The impact of the improved quadrature rules on the fast Fourier transform (FFT) approximation of probability density functions with known characteristic functions is discussed. The quality of the approximations is measured with a discrete version of the total variation distance. Numerical examples suggest that a discrete total variation distance of approximately 0.25 is the best possible attainable value across all the considered FFT approximations.

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1. Introduction

The discrete Fourier transform (DFT) approximation of probability density functions with known characteristic functions is especially useful when analytical expressions for the density functions are not available. This is the case for tempered stable and related distributions (e.g. Rachev et al. [14], Scherer et al. [15], Jelonek [7], etc.). Usually, the DFT approximation is based on a simple numerical integration quadrature rule like the mid-point rule (MPR) or the

Simpson rule (SR). However, the quality of this approximation is seldom questioned. In the last ten years, some improvements of the Simpson quadrature rule have been discovered. For example, an *optimal 3-point quadrature* (O3Q) formula of closed type, similar in simplicity to the Simpson rule, has been revealed by Ujevic [18]. Another attempt to improve Simpson's rule is due to Ujevic and Roberts [19]. They have proposed a *modified Simpson rule* (MSR), which takes additionally the first derivative of the approximating integrand at the two end-points into account. A brief account of these developments is presented in Section 2. The corresponding formulas required to compute the fast Fourier transform (FFT) approximations of probability density functions are presented in Section 3. Section 4 investigates the quality of the obtained FFT approximations for a number of elementary probability distributions. As quality measure, we use a *discrete* version of the *total variation* (DTV) distance, which is known to be an upper bound for the Kolmogorov distance. It is shown that a DTV of 0.25 is the "best" attainable value for the considered FFT approximations, at least for the standard normal and standard Laplace distributions. The variance gamma distribution, which is a popular distribution used in financial applications, and the one-sided exponential distribution, are also examined. Though with less precision, a DTV of approximately 0.25 seems the best possible attainable value across all FFT approximations. The O3Q approximation implies in general the best improved FFT approximation, which is followed by the MSR approximation. Our observations suggest that there is possibly an "optimal" symmetric interval with best attainable DTV value for the O3Q approximation, and that the rate of convergence to this value is faster for the O3Q approximation than the alternative ones. It is open for future research to analyse whether this holds more generally and whether exact quantifications of these numerical phenomena can be given.

2. Numerical integration, quadrature rules and improvement.

Simple and popular integral approximations of a function $f(x)$ defined over a finite interval $[a, b]$ are the rectangular rules and the Simpson rule:

$$\text{Left-point rule (LPR)} \quad \int_a^b f(x) dx \approx f(a) \cdot (b - a),$$

$$\text{Mid-point rule (MPR)} \quad \int_a^b f(x) dx \approx f\left(\frac{a+b}{2}\right) \cdot (b - a),$$

$$\text{Right-point rule (RPR)} \quad \int_a^b f(x) dx \approx f(b) \cdot (b - a),$$

$$\text{Simpson rule (SR)} \quad \int_a^b f(x) dx \approx \frac{1}{6} \{ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \} \cdot (b - a).$$

Clearly, the Simpson rule is a linear combination of the first three quadrature rules, a so-called *3-point quadrature formula of closed type*. Ujevic [18] has

derived an *optimal* 3-point quadrature formula of closed type, which has a better error estimate than the Simpson rule. It is given by

$$\text{Optimal 3-point rule (O3R):} \\ \int_a^b f(x)dx \approx \frac{1}{4}\{f(a) + 2f(\frac{a+b}{2}) + f(b)\} \cdot (b-a). \quad (2.1)$$

If $f(x)$ is a differentiable function with $f'(x) \in L_2[a, b]$, then an error estimate in terms of the L_2 -norm is ([18], Theorem 6)

$$\int_a^b f(x)dx \approx \left| \frac{1}{4}\{f(a) + 2f(\frac{a+b}{2}) + f(b)\} \cdot (b-a) - \int_a^b f(x)dx \right| \leq \frac{\|f'\|}{4\sqrt{3}}(b-a)^{3/2}. \quad (2.2)$$

The error estimate for the Simpson rule is worse. It is of the form (2.2) with the constant $1/6$ instead of $1/4\sqrt{3}$ ([18], Remark 3).

Other attempts to modify and improve Simpson's rule have been undertaken. For example, Ujevic and Roberts [19] have proposed a generalized approximation in terms of the first derivative $f'(x)$ at the two end-points of the form

$$\text{Modified Simpson rule (MSR): } \int_a^b f(x)dx \approx \\ \frac{1}{30}\{7f(a) + 16f(\frac{a+b}{2}) + 7f(b)\} \cdot (b-a) - \frac{1}{60}\{f'(b) - f'(a)\} \cdot (b-a)^2 \quad (2.3)$$

Applied to numerical integration, the error of the MSR is of sixth order in grid spacing (see Ujevic and Roberts [19], Corollary 2). The purpose of the present note is to investigate the impact of the O3R and MSR modified quadrature formulas on the fast Fourier transform approximation of probability functions.

3. Modified FFT approximations of probability functions.

The relationship between the characteristic function $\phi_X(z)$ of a probability density function (pdf) $f_X(x)$ associated to a random variable X is given by the inverse Fourier transform. For a sufficiently large interval $[-c, c]$, it is possible to approximate a pdf by means of a numerical Fourier inversion as follows

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} \cdot \phi_X(z) dz \approx \frac{1}{2\pi} \int_{-c}^c e^{-izx} \cdot \phi_X(z) dz. \quad (3.1)$$

Based on the quadrature rules of Section 2, the finite integral in (3.1) can be approximated in different ways. Consider an interval $[a, b]$ that is divided into N disjoints subintervals of equal length $h = (b-a)N^{-1}$ and assume that the

random variable X with pdf $f_X(x)$ has a known characteristic function (chf) $\phi_X(z)$, $z \in \mathbb{C}$. For $k = 0, \dots, N-1$ set $x_k = a + hk$. For N sufficiently large the constant $c = \pi \cdot h^{-1}$ is also large and (3.1) implies the pdf approximation

$$f_X(x_k) \approx \frac{1}{2\pi} \int_{-c}^c e^{-izx_k} \cdot \phi_X(z) dz = \int_{-N/2(b-a)}^{N/2(b-a)} e^{-2\pi i \cdot u x_k} \cdot \phi_X(2\pi \cdot u) du. \quad (3.2)$$

In a first step, we consider Discrete Fourier Transform (DFT) approximations of the pdf $f_X(x)$ for the LPR, MPR and RPR rectangular rules, from which we get approximations for the SR and O3R rules. In a second step, we derive approximations for the MSR rule.

In the first step, set $u_j = (j - \frac{N}{2})(b-a)^{-1}$, $j = 0, \dots, N$, and consider the mid-points

$$m_j = \frac{1}{2}(u_j + u_{j+1}) = (j - \frac{N-1}{2})(b-a)^{-1}, \quad j = 0, \dots, N-1.$$

Applying the different quadrature rules to the right-hand side integral in (3.2), one obtains the following finite sum approximations of $f_X(x_k)$:

LPR:

$$(b-a)^{-1} \cdot \sum_{j=0}^{N-1} e^{-2\pi i \cdot u_j x_k} \cdot \phi_X(2\pi \cdot u_j) = (b-a)^{-1} \cdot \sum_{j=0}^{N-1} e^{-2\pi i \cdot (\frac{a}{b-a} + k)(j - \frac{N}{2}) \frac{1}{N}} \cdot \phi_X(\frac{2\pi}{b-a}(j - \frac{N}{2})),$$

MPR:

$$(b-a)^{-1} \cdot \sum_{j=0}^{N-1} e^{-2\pi i \cdot m_j x_k} \cdot \phi_X(2\pi \cdot m_j) = (b-a)^{-1} \cdot \sum_{j=0}^{N-1} e^{-2\pi i \cdot (\frac{a}{b-a} + k)(j - \frac{N-1}{2}) \frac{1}{N}} \cdot \phi_X(\frac{2\pi}{b-a}(j - \frac{N-1}{2})),$$

RPR:

$$(b-a)^{-1} \cdot \sum_{j=0}^{N-1} e^{-2\pi i \cdot u_{j+1} x_k} \cdot \phi_X(2\pi \cdot u_{j+1}) = (b-a)^{-1} \cdot \sum_{j=0}^{N-1} e^{-2\pi i \cdot (\frac{a}{b-a} + k)(j - \frac{N-2}{2}) \frac{1}{N}} \cdot \phi_X(\frac{2\pi}{b-a}(j - \frac{N-2}{2})).$$

Since $e^{\pi i} = -1$, one has $e^{-2\pi i \cdot (\frac{a}{b-a} + k)(j - \frac{N-\varepsilon}{2}) \frac{1}{N}} = (-1)^{(\frac{a}{b-a} + \frac{k}{N})(N-\varepsilon)} \cdot (-1)^{(\frac{2a}{b-a}) \cdot j} \cdot e^{-2\pi i \cdot k \frac{j}{N}}$, $\varepsilon = 0, 1, 2$. Inserted into the above, one gets approximations of $f_X(x_k)$:

$$\text{LPR: } f_k^L := (b-a)^{-1} \cdot (-1)^{(\frac{a}{b-a} + \frac{k}{N})N} \cdot \sum_{j=0}^{N-1} (-1)^{(\frac{2a}{b-a}) \cdot j} \cdot \phi_X(\frac{2\pi}{b-a}(j - \frac{N}{2})) \cdot e^{-2\pi i \cdot k \frac{j}{N}},$$

$$\text{MPR: } f_k^M := (b-a)^{-1} \cdot (-1)^{(\frac{a}{b-a} + \frac{k}{N})(N-1)} \cdot \sum_{j=0}^{N-1} (-1)^{(\frac{2a}{b-a}) \cdot j} \cdot \phi_X(\frac{2\pi}{b-a}(j - \frac{N-1}{2})) \cdot e^{-2\pi i \cdot k \frac{j}{N}},$$

$$\text{RPR: } f_k^R := (b-a)^{-1} \cdot (-1)^{(\frac{a}{b-a} + \frac{k}{N})(N-2)} \cdot \sum_{j=0}^{N-1} (-1)^{(\frac{2a}{b-a}) \cdot j} \cdot \phi_X(\frac{2\pi}{b-a}(j - \frac{N-2}{2})) \cdot e^{-2\pi i \cdot k \frac{j}{N}},$$

which one interprets as the k -th components of DFT vectors associated to resp. $y^L = (y_0^L, \dots, y_{N-1}^L)$, $y^M = (y_0^M, \dots, y_{N-1}^M)$ and $y^R = (y_0^R, \dots, y_{N-1}^R)$, such that

$$\begin{aligned}
\text{LPR:} \quad f_k^L &= C_k^L \cdot DFT(y^L)_k, \quad C_k^L = (b-a)^{-1} \cdot (-1)^{\left(\frac{a}{b-a} + \frac{k}{N}\right)N}, \\
y_j^L &= (-1)^{\left(\frac{2a}{b-a}\right) \cdot j} \cdot \phi_X\left(\frac{2\pi}{b-a}\left(j - \frac{N}{2}\right)\right), \\
\text{MPR:} \quad f_k^M &= C_k^M \cdot DFT(y^M)_k, \quad C_k^M = (b-a)^{-1} \cdot (-1)^{\left(\frac{a}{b-a} + \frac{k}{N}\right)(N-1)}, \\
y_j^M &= (-1)^{\left(\frac{2a}{b-a}\right) \cdot j} \cdot \phi_X\left(\frac{2\pi}{b-a}\left(j - \frac{N-1}{2}\right)\right), \\
\text{RPR:} \quad f_k^R &= C_k^R \cdot DFT(y^R)_k, \quad C_k^R = (b-a)^{-1} \cdot (-1)^{\left(\frac{a}{b-a} + \frac{k}{N}\right)(N-2)}, \\
y_j^R &= (-1)^{\left(\frac{2a}{b-a}\right) \cdot j} \cdot \phi_X\left(\frac{2\pi}{b-a}\left(j - \frac{N-2}{2}\right)\right).
\end{aligned}$$

In this setting, the SR and O3R quadrature rules imply the DFT approximations:

$$\text{SR:} \quad f_X(x_k) \approx \frac{1}{6} \{f_k^L + 4f_k^M + f_k^R\} \quad (3.3)$$

$$\text{O3R:} \quad f_X(x_k) \approx \frac{1}{4} \{f_k^L + 2f_k^M + f_k^R\} \quad (3.4)$$

To derive MSR based DFT approximations, a calculation of the first derivative $f'(z) = \frac{d}{dz} e^{-izx} \cdot \phi_X(z) = e^{-izx} \cdot \{-ix\phi_X(z) + \phi_X'(z)\}$ at u_j and u_{j+1} is necessary in order to get similarly to the above the left- and right-point derivative contributions to the overall MSR approximation of $f_X(x_k)$:

$$\begin{aligned}
df_k^L &:= \frac{1}{60} (b-a)^{-2} \cdot \sum_{j=0}^{N-1} e^{-2\pi i u_j x_k} f'(2\pi \cdot u_j) \\
&= \frac{1}{60} (b-a)^{-1} C_k^L \cdot \{-2\pi i \cdot x_k \cdot DFT(y^L)_k + DFT(dy^L)_k\} \\
&= \frac{1}{60} (b-a)^{-1} \cdot \{C_k^L \cdot DFT(dy^L)_k - 2\pi i \cdot x_k \cdot f_k^L\}, \\
dy^L &= (dy_0^L, \dots, dy_{N-1}^L), \quad dy_j^L = (-1)^{\left(\frac{2a}{b-a}\right) \cdot j} \cdot \phi_X'\left(\frac{2\pi}{b-a}\left(j - \frac{N}{2}\right)\right), \\
df_k^R &:= \frac{1}{60} (b-a)^{-2} \cdot \sum_{j=0}^{N-1} e^{-2\pi i u_{j+1} x_k} f'(2\pi \cdot u_{j+1}) \\
&= \frac{1}{60} (b-a)^{-1} C_k^R \cdot \{-2\pi i \cdot x_k \cdot DFT(y^R)_k + DFT(dy^R)_k\} \\
&= \frac{1}{60} (b-a)^{-1} \cdot \{C_k^R \cdot DFT(dy^R)_k - 2\pi i \cdot x_k \cdot f_k^R\}, \\
dy^R &= (dy_0^R, \dots, dy_{N-1}^R), \quad dy_j^R = (-1)^{\left(\frac{2a}{b-a}\right) \cdot j} \cdot \phi_X'\left(\frac{2\pi}{b-a}\left(j - \frac{N-2}{2}\right)\right).
\end{aligned}$$

One obtains the following MSR approximation

$$f_X(x_k) \approx \frac{1}{30} \{7f_k^L + 16f_k^M + 7f_k^R\} - \{df_k^R - df_k^L\}. \quad (3.5)$$

An efficient software implementation of the DFT is based on the Fast Fourier Transform (FFT) algorithm by Cooley and Tukey [3] (see also Schwartz [16], [17], Heideman et al. [6], Duhamel and Vetterli [4], Batenkov [1], among others). For numerical approximation of the distribution function $F_X(x) = \int_{-\infty}^x f_X(t) dt$, one

derives a similar DFT approximation in terms of the characteristic function (e.g. Kim et al. [9], Proposition 1) or one uses the recursive formula

$$F_X(x_k) = F_X(x_{k-1}) + hf_X(x_{k-1}), \quad k = 1, \dots, N-1, \quad F_X(x_0) = 0, \quad (3.6)$$

and a simple piecewise linear interpolation for intermediate values:

$$F_X(x) = F_X(x_{k-1}) + h^{-1}(x - x_{k-1})\{F_X(x_k) - F_X(x_{k-1})\}, \quad x \in [x_{k-1}, x_k], \quad k = 1, \dots, N-1.$$

4. Discrete total variation distance and FFT approximations.

It is instructive to investigate the quality of the FFT approximations for a number of elementary probability distributions. Given is an absolutely continuous random variable X with pdf $f_X(x)$ and distribution function $F_X(x)$. The quality of a discrete approximation $F_X^a(x_k)$ to $F_X(x_k)$, $k = 0, \dots, N-1$, is measured by the *discrete total variation* (DTV) distance defined by

$$\Delta_{DTV}(F_X^a, F_X) = \frac{1}{2} \cdot \sum_{k=0}^{N-1} |F_X(x_k) - F_X^a(x_k)|. \quad (4.1)$$

One knows that the discrete total variation distance is an upper bound to the *discrete Kolmogorov distance*

$$\Delta_{DK}(F_X^a, F_X) = \max_{k=0, \dots, N-1} |F_X(x_k) - F_X^a(x_k)| \leq \Delta_{DTV}(F_X^a, F_X). \quad (4.2)$$

Therefore, the smaller the DTV, the better is the quality of approximation. In the next examples, the DTV is computed for the FFT approximations based on the O3Q, MSR, SR and MPR approximations, as presented in Section 3. The FFT is evaluated over a finite interval $[a, b]$ with a number $N = 2^q$, $q = 10, 11, 12, 13, 14$ of equally spaced subintervals. For a symmetric distribution we choose $b = -a$.

Example 4.1: standard normal distribution

The chf is $\phi_X(z) = \exp\{-\frac{1}{2}z^2\}$ and its derivative is $\phi_X'(z) = -z \cdot \phi_X(z)$. The DTV for the FFT approximations differs from 0.25 only marginally when $b \geq 6$. Setting $b = 6$ one has $\text{DTV} = 0.25001$ for the SR and MPR approximations in case $q = 13$, and for the O3Q, MSR and SR approximations in case $q = 14$. One obtains $\text{DTV} = 0.25002$ for the MPR approximation in case $q = 14$. It seems that $\text{DTV} = 0.25$ is the best attainable quality of approximation.

Example 4.2: standard Laplace distribution

The chf is $\phi_X(z) = 2/\{2 + z^2\}$ and its derivative is $\phi'_X(z) = -2z/\{2 + z^2\}^2$. Table 4.1 below summarizes calculations. Though the DTV can be slightly smaller than 0.25 (O3R approximation with $b = 8$), the constant 0.25 must also be viewed as the best attainable quality in FFT approximation. This statement holds for the O3R approximation with a sufficiently large symmetric interval, here with $b = 10, 12$. The DTV distance is second best for the MSR approximation, but differs slightly from 0.25. For smaller values of b the SR and MPR approximations deteriorate rapidly by increasing value of q . For example, with $b = 8$ and $q = 14$ one obtains $DTV \approx 0.3$ for the MPR approximation.

Example 4.3: variance gamma logarithmic return distribution

One may ask whether a DTV of 0.25 can also be attained for distributions in the real-life world. Consider the daily closing prices of the Swiss Market and the Standard & Poors 500 stock market indices over the 3 years 2010-2012. The observed sample logarithmic returns are negatively skewed and have a much higher excess kurtosis than is allowed by a normal distribution. In this situation, the four parameter shifted variance gamma distribution with chf

$$\phi_X(z) = e^{\nu iz} \cdot \left\{ \frac{\alpha\beta}{\alpha\beta + (\alpha - \beta)iz + z^2} \right\}^\rho \quad (4.3)$$

fits quite well the logarithmic returns of these two data sets. Note that the variance gamma distribution has been introduced in finance by Madan and Seneta [11] and Madan and Milne [13]. The important variance gamma process has been studied at many places (e.g. Madan et al. [12], Madan [10], Kotz et al. [8], Section 8.4, Carr et al. [2], Geman [5], etc.). The parameters $\nu = 0.0015$, $\rho = 0.85$, $\alpha = 125$, $\beta = 110$, are representative of a typical variance gamma logarithmic return distribution. Table 4.2 below summarizes calculations. It turns out that all the quadrature rules lead to FFT approximations with a DTV rather close to 0.25. Though the overall preference goes to the O3R approximation, the other ones do not differ much, and can result in a marginally smaller DTV in some specific cases. Again, the MSR approximation is second best in the overall.

Table 4.1: DTV distance of FFT approximations to standard Laplace distribution

a	b	q	quadrature rule			
			O3R	MSR	SR	MPR
-8	8	10	0.24996	0.25017	0.25099	0.25305
		11	0.24996	0.25038	0.25203	0.25617
		12	0.24996	0.25079	0.25412	0.26242
		13	0.24996	0.25163	0.25828	0.27492
		14	0.24996	0.25329	0.26661	0.29991
-10	10	10	0.25000	0.25001	0.25006	0.25018
		11	0.25000	0.25002	0.25012	0.25036
		12	0.25000	0.25005	0.25024	0.25073
		13	0.25000	0.25010	0.25049	0.25147
		14	0.25000	0.25019	0.25098	0.25295
-12	12	10	0.250000	0.250001	0.250003	0.250011
		11	0.250000	0.250001	0.250007	0.250021
		12	0.250000	0.250003	0.250014	0.250043
		13	0.250000	0.250006	0.250029	0.250087
		14	0.250000	0.250011	0.250058	0.250174

Table 4.2: DTV distance of FFT approximations to variance gamma return

a	b	q	quadrature rule			
			O3R	MSR	SR	MPR
-0.1	0.1	10	0.24972	0.24973	0.24982	0.25030
		11	0.24953	0.24954	0.24968	0.25062
		12	0.24930	0.24931	0.24950	0.25128
		13	0.24907	0.24906	0.24925	0.25260
		14	0.24908	0.24897	0.24898	0.25523
-0.125	0.125	10	0.249947	0.249950	0.249953	0.249969
		11	0.249947	0.249949	0.249953	0.249969
		12	0.249939	0.249941	0.249945	0.249964
		13	0.249936	0.249936	0.249939	0.249962
		14	0.249981	0.249979	0.249973	0.249991
-0.15	0.15	10	0.249998	0.249996	0.249995	0.249991
		11	0.249995	0.249997	0.249998	0.250004
		12	0.250014	0.250015	0.250016	0.250019
		13	0.250043	0.250043	0.250043	0.250047
		14	0.250105	0.250105	0.250105	0.250111

Example 4.4: standard exponential distribution

The preceding examples are all distributions defined over the whole real line. What about the FFT approximations for one-sided distributions? A simple example is the standard exponential distribution with chf $\phi_X(z) = \{1 - iz\}^{-1}$. The obtained DTV values are reported in the Table 4.3. First of all, since the exponential distribution is only defined for positive numbers, it is not clear a priori whether the FFT approximations should be calculated over an interval of the form $[0, b]$ or not. The choice $b = 16$ shows that the DTV of approximately 0.54 in this case is far away from a best attainable quality in FFT approximation. The situation changes if asymmetric intervals with values of $a < 0$ are allowed and the DTV distance is replaced by the distance

$$\Delta_{DTV}(F_X^a, F_X) = \frac{1}{2} \cdot \sum_{k=-aN/(b-a)}^{N-1} |F_X(x_k) - F_X^a(x_k)|, \quad (4.4)$$

which measures the quality in FFT approximation over the positive numbers only, as should be. As our examples show, the O3Q approximation is best in all considered cases and all DTV values remain stable and rather close to 0.25. The comparison between the intervals $[-4, 12]$ and $[-8, 8]$ of equal length suggests that a symmetric interval results in smaller DTV values in this case. Second best is again the MSR approximation. The SR and MPR approximations can deteriorate rapidly as q increases, as calculations with the symmetric interval $[-8, 8]$ show. Moreover, with increasing value of q the DTV of the O3Q approximation increases and becomes closer to 0.25. For larger symmetric intervals like $[-16, 16]$ the differences in DTV values between the different FFT approximations becomes smaller and almost negligible. The DTV value of the O3Q approximation with $q = 14$ cannot be improved when enlarging the interval $[-12, 12]$ to $[-16, 16]$. These observations suggest that there is possibly an “optimal” interval $[-a, a]$ with best attainable DTV value for the O3Q approximation, and that the rate of convergence to this value is faster for the O3Q approximation than for the other ones. A closer look at the Table 4.1 shows that these observations remain true for the standard Laplace distribution. It is open for future research to analyse whether this holds more generally and whether exact analytical quantifications of these numerical phenomena can be given.

Table 4.3: DTV distance of FFT approximations to standard exponential

a	b	q	quadrature rule			
			O3R	MSR	SR	MPR
0	16	10	0.54077	0.54076	0.54071	0.54058
		11	0.54175	0.54174	0.54170	0.54159
		12	0.54224	0.54223	0.54217	0.54202
		13	0.54248	0.54246	0.54236	0.54211
		14	0.54261	0.54256	0.54236	0.54188
-4	12	10	0.24897	0.24906	0.24941	0.25090
		11	0.24942	0.24961	0.25078	0.25397
		12	0.24968	0.25023	0.25274	0.25904
		13	0.24982	0.25106	0.25608	0.26865
		14	0.24990	0.25241	0.26246	0.28758
-8	8	10	0.24910	0.25486	0.27760	0.33461
		11	0.24951	0.26095	0.30658	0.42074
		12	0.24971	0.27259	0.36401	0.59262
		13	0.24981	0.29558	0.47860	0.93618
		14	0.24986	0.34143	0.70765	1.62321
-12	12	10	0.24905	0.24924	0.24954	0.25079
		11	0.24949	0.24970	0.25050	0.25275
		12	0.24973	0.25014	0.25182	0.25608
		13	0.24986	0.25070	0.25405	0.26247
		14	0.24992	0.25161	0.25831	0.27510
-16	16	10	0.24909	0.24923	0.24926	0.24961
		11	0.24950	0.24957	0.24960	0.24979
		12	0.24973	0.24977	0.24979	0.24993
		13	0.24985	0.24988	0.24993	0.25010
		14	0.24992	0.24995	0.25006	0.25040

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