Exact Solutions of the Kudryashov-Sinelshchikov Equation by Modified exp-Function Method¹

Yinghui He, Shaolin Li and Yao Long

Department of Mathematics, Honghe University
Mengzi, Yunnan, 661100, P.R. China

Copyright © 2013 Yinghui He et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, the modified exp-function method is used to seek generalized wave solutions of Kudryashov-Sinelshchikov equation. As a result, some new types of exact traveling wave solutions for arbitrary α, β are obtained which include exponential function, hyperbolic function and trigonometric function. The related results are extend. Obtained results clearly indicate the reliability and efficiency of the modified exp-function method.

Keywords: nonlinear wave equation, Kudryashov-Sinelshchikov equation, exact wave solution, modified exp-function method

1 Introduction

The investigation of exact solutions of nonlinear wave equations plays an important role in the study of nonlinear physical phenomena. Recently, many effective methods for obtaining exact solutions of nonlinear wave equations have been proposed, such as bäcklund transformation method, homogeneous balance method, bifurcation method, Hirotas bilinear method, the hyperbolic tangent function expansion method, the Jacobi elliptic function expansion method, F-expansion method and so on. He and Wu [1] developed the exp-function method to seek the solitary, periodic and compaction like solutions of nonlinear differential equations. It is an effective and simple method

¹This research is supported by the Natural Science Foundation of of china (11161020) and Research Foundation of Honghe university (10XJY120).

and is widely used. Based on this method, modified exp-function expansion method is proposed.

In 2010, Kudryashov and Sinelshchikov [2] obtained a more common nonlinear partial differential equation for describing the pressure waves in a mixture liquid and gas bubbles taking into consideration the viscosity of liquid and the heat transfer, that is

$$u_t + \alpha uu_x + u_{xxx} - (uu_{xx})_x - \beta u_x u_{xx} = 0, \quad (1)$$

where, α, β are real parameters. In [3], they derived partial cases of nonlinear evolution equations of the fourth order for describing nonlinear pressure waves in a mixture liquid and gas bubbles. Some exact solutions are found and properties of nonlinear waves in a liquid with gas bubbles are discussed. Eq.(1) is called Kudryashov-Sinelshchikov equation, it is generalization of the KdV and the BKdV equation and similar but not identical to the Camassa-Holm (CH) equation, it has been studied by some authors [2,4-6]. Undistorted waves are governed by a corresponding ordinary differential equation which, for special values of some integration constant, is solved analytically in [1]. Solutions are derived in a more straightforward manner and cast into a simpler form and some new types of solutions which contain solitary wave and periodic wave solutions are presented in [5]. Ryabov [6] obtained some exact solutions for $\beta = -3$ and $\beta = -4$ using a modification of the truncated expansion method [8]. Li and He discussed the equation by the bifurcation method of dynamical systems and the method of phase portraits analysis [9-11]. In [12], the equation is studied by the Lie symmetry method. In [13], the equation is studied by the $\frac{G'}{G}$ -expansion method and its variants. The aim of this work is to further complement the studies on the Kudryashov-Sinelshchikov equations by the new method.

The organization of the paper is as follows: in Section 2, a brief description of the modified Exp-function method is given. In section 3, we will study the Kudryashov-Sinelshchikov equation by the new methods. Finally conclusions are given in Section 4.

2 Modified Exp-function method

The exp-function method was first proposed by He and Wu to solve differential equations [1]. In this paper, we will introduce a modified Exp-function method. The main procedures of this method are as follows.

We consider a general nonlinear PDE in the form

$$H(u, u_x, u_t, u_{xx}, u_{tt}, u_{tx}, \dots) = 0 \quad (2)$$

Using a transformation $\xi = x - ct$, where c are constants, we can rewrite Eq.(2) into the following nonlinear ODE:

$$H_1(u, u', u'', \dots) = 0 \quad (3)$$

where the prime denotes the derivation with respect to ξ .

Let $u = v + s$, where s are constants. Then Eq.(3) becomes

$$H_2(v, v', v'', \dots) = 0 \quad (4)$$

Assume that the solution of Eq.(2) can be expressed in the following form

$$u(\xi) = \frac{b_c g^c + \dots + b_{-d} g^{-d}}{a_p g^p + \dots + a_{-q} g^{-q}}, \quad (5)$$

where $g = e^{k\xi}$ which is the solution of the homogeneous linear equation corresponding to equation (4). In order to determine the constants c, p and q, d , we can balance the linear term of the highest order in Eq.(3) with the highest order nonlinear term.

Substituting Eq.(5) into Eq.(3), we can get polynomial equation on g . Let the coefficient of g^i be zero, and solve the equation set, the a_i, b_i can be determined.

3 Solutions of Kudryashov-Sinelshchikov equation

Using the transformation $\xi = x - ct$, equation (1) can be rewrite as

$$-cu' + \alpha u u' + u''' - (u u'')' - \beta u' u'' = 0 \quad (6)$$

Integrating (6) once and making the integral constant be zero, we have

$$-cu + \frac{\alpha}{2} u^2 + u'' - u u'' - \frac{\beta}{2} u'^2 = 0 \quad (7)$$

Let $u = v + s$, equation (9) can be rewrite as

$$(1-s)v'' + \frac{\beta}{2} v'^2 - v v'' + \frac{\alpha}{2} v^2 + (\alpha s - c)v + \frac{1}{2} \alpha s^2 - cs = 0 \quad (8)$$

Let $\frac{1}{2} \alpha s^2 - cs = 0$, then $s = 0$ or $s = \frac{2c}{\alpha}$.

According to homogeneous balance principle, we get $p = c$ and $q = d$. Here, we take $p = c = 2, q = d = 2$.

Case 1 $s = 0$.

The solution of the linear equation corresponding to equation (8) is

$$g = e^{k\xi}, k = \pm\sqrt{c} \quad (9)$$

So, we can assume

$$u(\xi) = \frac{b_0 + b_1g + b_2g^2 + b_3g^3 + b_4g^4}{a_0 + a_1g + a_2g^2 + a_3g^3 + a_4g^4} \quad (10)$$

Substituting Eq.(9) and Eq.(10) into Eq.(7) yields a set of algebraic equations for $g^i (i = 0, 1, \dots, 16)$. Letting the coefficients of these terms g^i to be zero yields a set of over-determined algebraic equations. Solving the system of algebraic equations, we obtain

Set 1.1

$$a_0 = a_1 = a_2 = a_4 = b_0 = b_1 = b_3 = 0, b_4 = 0, \alpha = c(\beta + 2) \quad (11)$$

Substituting (11) into (10), when $\alpha = c(\beta + 2)$, we can obtain following solutions of the Kudryashov-Sinelshchikov equation as follow.

$$u(x, t) = u(\xi) = \frac{b_2}{a_3} \frac{1}{g} = \frac{b_2}{a_3} e^{\pm\sqrt{c}(x-ct)}, \quad (12)$$

where b_2, a_3 and c are arbitrary constants.

Set 1.2

$$a_0 = a_1 = 0, a_2 = -\frac{4}{3}b_2(\beta+1), a_3 = a_4 = 0, b_0 = -\frac{b_2^2(8+5\beta)}{12\beta b_4}, b_1 = b_3 = 0, \alpha = 4c(\beta+2). \quad (13)$$

Substituting (13) into (10), when $\alpha = 4c(\beta + 2)$ we can obtain following solutions of the Kudryashov-Sinelshchikov equation as follow.

$$u(x, t) = u(\xi) = \frac{1}{16b_2(\beta+1)} \left(\frac{b_2^2(8+5\beta)}{\beta b_4 g^2} + 12b_2 + 12b_4 g^2 \right), \quad (14)$$

where b_2, a_4, α , are arbitrary constants and $\beta \neq -1$.

Set 1.2.1 Let $b_4 = \pm \frac{\sqrt{-3\beta(8+5\beta)}b_2}{6\beta}, \beta(8+5\beta) < 0$.

If $c > 0$, (14) can be reduced as

$$u(x, t) = u(\xi) = \frac{3}{4(\beta+1)} \left(-1 \pm \sqrt{-\frac{24+15\beta}{\beta}} \cosh(\pm 2\sqrt{c}(x-ct)) \right) \quad (15)$$

If $c < 0$, (14) can be reduced as

$$u(x, t) = u(\xi) = \frac{3}{4(\beta+1)} \left(-1 \pm \sqrt{-\frac{24+15\beta}{\beta}} \cos(\pm 2\sqrt{-c}(x-ct)) \right) \quad (16)$$

Set 1.2.2 Let $b_4 = \pm \frac{\sqrt{3\beta(8+5\beta)}b_2}{6\beta}$, $\beta(8+5\beta) > 0$.

If $c > 0$, (14) can be reduced as

$$u(x, t) = u(\xi) = \frac{3}{4(\beta+1)} \left(-1 \pm \sqrt{\frac{24+15\beta}{\beta}} \sinh(\pm 2\sqrt{c}(x-ct)) \right) \quad (17)$$

If $c < 0$, (14) can be reduced as

$$u(x, t) = u(\xi) = \frac{3}{4(\beta+1)} \left(-1 \pm I \sqrt{\frac{24+15\beta}{\beta}} \sin(\pm 2\sqrt{-c}(x-ct)) \right) \quad (18)$$

case 2 $s = \frac{\alpha}{2c}$.

The solution of the linear equation corresponding to equation (8) is

$$g = e^{k\xi}, k = \pm \sqrt{\frac{\alpha c}{2c-\alpha}} \quad (19)$$

So, we can assume

$$u(\xi) = \frac{b_0 + b_1g + b_2g^2 + b_3g^3 + b_4g^4}{a_0 + a_1g + a_2g^2 + a_3g^3 + a_4g^4} \quad (20)$$

Substituting Eq.(20) and Eq.(19) into Eq.(7) yields a set of algebraic equations for $g^i (i = 0, 1, \dots, 16)$. Letting the coefficients of these terms g^i to be zero yields a set of over-determined algebraic equations. Solving the system of algebraic equations, we obtain

Set 2.1

$$a_0 = a_1 = 0, a_2 = -\frac{1}{2}\beta b_2, a_3 = a_4 = b_0 = b_1 = 0, b_4 = 0, \alpha = -c\beta \quad (21)$$

Substituting (21) into (20), when $\alpha = -c\beta$, we can obtain following solutions of the Kudryashov-Sinelshchikov equation as follow.

$$u(x, t) = u(\xi) = -\frac{2}{\beta} - \frac{2b_3}{\beta b_2}g = -\frac{2}{\beta} - \frac{2b_3}{\beta b_2}e^{\pm \sqrt{-\frac{c\beta}{2+\beta}}(x-ct)}, \quad (22)$$

where b_2, b_3, c and $\beta \neq 0, -2$ are arbitrary constants.

Set 2.2

$$a_0 = a_1 = a_2 = 0, a_3 = \frac{1}{2}b_3, b_0 = b_1 = b_2 = b_4 = 0, \alpha = c, \beta = -4. \quad (23)$$

Substituting (23) into (20), when $\beta = -4, \alpha = c$ we can obtain following solutions of the Kudryashov-Sinelshchikov equation as follow.

$$u(x, t) = u(\xi) = \frac{2b_3}{b_3 + 2a_4g} = \frac{2b_3}{b_3 + 2a_4e^{\pm \sqrt{c}(x-ct)}}, \quad (24)$$

where b_3, a_4 and c are arbitrary constants.

Set 2.3

$$a_0 = a_1 = a_3 = 0, a_4 = b_0 = 0, b_1 = -\frac{b_2(2a_2 - b_2)}{4b_3}, b_4 = 0, \alpha = c, \beta = -1. \quad (25)$$

Substituting (25) into (20), when $\beta = -1, \alpha = c$ we can obtain following solutions of the Kudryashov-Sinelshchikov equation as follow.

$$u(x, t) = u(\xi) = \frac{-2b_2a_2 + b_2^2 + 4b_2b_3g + 4b_3^2g^2}{4b_3a_2g}, \quad (26)$$

where $g = e^{\sqrt{c}(x-ct)}$, a_2, b_2, b_3 and c are arbitrary constants.

Set 2.3.1 Let $b_3 = \pm \frac{1}{2} \sqrt{b_2^2 - 2b_2a_2}$, $b_2(b_2 - 2a_2) > 0$.

If $c > 0$, (26) can be reduced as

$$u(x, t) = u(\xi) = \frac{b_2}{2a_2} + (b_2^2 - 2a_2b_2) \cosh(\sqrt{c}(x - ct)) \quad (27)$$

If $c < 0$, (26) can be reduced as

$$u(x, t) = u(\xi) = \frac{b_2}{2a_2} + (b_2^2 - 2a_2b_2) \cos(\sqrt{-c}(x - ct)) \quad (28)$$

Set 2.3.2 Let $b_3 = \frac{1}{2} \sqrt{-b_2^2 + 2b_2a_2}$, $b_2(b_2 - 2a_2) < 0$.

If $c > 0$, (26) can be reduced as

$$u(x, t) = u(\xi) = \frac{b_2}{2a_2} + (-b_2^2 + 2a_2b_2) \sinh(\sqrt{c}(x - ct)) \quad (29)$$

If $c < 0$, (26) can be reduced as

$$u(x, t) = u(\xi) = \frac{b_2}{2a_2} + (-b_2^2 + 2a_2b_2) \sin(\sqrt{-c}(x - ct)) \quad (30)$$

Set 2.4

$$a_0 = a_1 = 0, a_2 = \frac{a_3^2}{4a_4}, b_0 = b_1 = 0, b_2 = \frac{a_3^2}{3a_4}, b_3 = 0, b_4 = \frac{4a_4}{3}, \alpha = \frac{3c}{2}, \beta = -3. \quad (31)$$

Substituting (31) into (20), when $\beta = -3, \alpha = \frac{3c}{2}$ we can obtain following solutions of the Kudryashov-Sinelshchikov equation as follow.

$$u(x, t) = u(\xi) = \frac{4}{3} \frac{a_3^2 + 4a_4^2g^2}{(a_3 + 2a_4g)^2}, \quad (32)$$

where $g = e^{\sqrt{3c}(x-ct)}$, a_3, a_4 and c are arbitrary constants.

Set 2.4.1 Let $a_4 = \pm \frac{a_3}{2}$.

If $c > 0$, (32) can be reduced as

$$u(x, t) = u(\xi) = \frac{2}{3} \left(1 + \tanh^2 \left(\frac{\sqrt{3}c}{2} (x - ct) \right) \right) \quad (33)$$

If $c < 0$, (32) can be reduced as

$$u(x, t) = u(\xi) = \frac{2}{3} \left(1 - \tanh^2 \left(\frac{\sqrt{-3}c}{2} (x - ct) \right) \right) \quad (34)$$

Remark 1. We can also get other similarly solutions of Equation (1) which are omitted for convenience.

Remark 2. The validity of the solutions we obtained are verified.

Remark 3. We discussed the situation for arbitrary α and β , so the results of the related literature are extended.

4 Conclusions

In this paper, we have obtained many solitary solutions of the Kudryashov-Sinelshchikov equation by using the modified exp-function method. It shows that the new method is easy and straightforward for nonlinear differential equations. It is said that this method can be applied to other kinds of nonlinear problems.

References

- [1] X.H. Wu, J.H. He, Exp-function method and its application to nonlinear equations, *Chaos, Solitons and Fractals*. 2008; 38 (3):903-910.
- [2] Kudryashov NA, Sinelshchikov DI. Nonlinear waves in bubbly liquids with consideration for viscosity and heat transfer. *Phys Lett A* 2010; 374:2011-2016.
- [3] Kudryashov NA, Sinelshchikov DI. Nonlinear evolution equations for describing waves in bubbly liquids with viscosity and heat transfer consideration. *Appl Math Comput* 2010; 217:414-421.
- [4] Kudryashov NA, Sinelshchikov DI. Nonlinear waves in liquids with gas bubbles with account of viscosity and heat transfer. *Fluid Dynam* 2010; 45:96-112.
- [5] Randrüüt M. On the Kudryashov-Sinelshchikov equation for waves in bubbly liquids. *Phys Lett A* 2011; 375:3687-3692.

- [6] Ryabov PN, Exact solutions of the Kudryashov and Sinelshchikov equation, *Appl Math Comput* 2010; 217:3585-3590.
- [7] Kudryashov NA. One method for finding exact solutions of nonlinear differential equations. *Commun Nonlinear Sci Numer Simulat* 2012; 17:2248-2253.
- [8] Ryabov PN, Sinelshchikov DI, Kochanov MB. Application of the Kudryashov method for finding exact solutions of the high order nonlinear evolution equations. *Appl Math Comput* 2011; 218:3965-3972.
- [9] JIBIN LI. exact travelling wave solutions and their bifurcations for Kudryashov-Sinelshchikov equation. *International Journal of Bifurcation and Chaos* 2012; 5:(1250118)1-19.
- [10] Bin He. The bifurcation and exact peakons solitary and periodic wave solutions for the KudryashovCSinelshchikov equation. *Commun Nonlinear Sci Numer Simulat* 2012; 17:4137-4148.
- [11] Bin He. Periodic Loop Solutions and Their Limit Forms for the Kudryashov-Sinelshchikov Equation. *Mathematical Problems in Engineering*. doi:10.1155/2012/320163.
- [12] Mehdi Nadjafikhah, Vahid Shirvani-Sh. Lie Symmetry Analysis of Kudryashov-Sinelshchikov Equation. *Mathematical Problems in Engineering*. doi:10.1155/2011/457697.
- [13] Yinghui He. Exact Solutions of the Kudryashov-Sinelshchikov Equation Using the Multiple $\frac{G'}{G}$ -Expansion Method. *Mathematical Problems in Engineering*. doi:10.1155/2013/708049.

Received: March 30, 2013