The Application of Tensor Product in the Meet Matrix

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Abstract

By means of Mobius inversion and the theory of tensor product, we have studied the structure and the bounds for the determinants of GCD matrices over the direct product. We have obtained some results to the GCD matrices over the finite positive integer set. Finally, we have generalized the results to the direct product of the general poset.

Keywords: the tensor product; the meet semi-lattice; the generalized Euler's function; the meet matrix

1 Introduction

The greatest common divisor matrix and the least common multiple matrix are respectively denoted into $\langle S \rangle = (s_{ij})_{n \times n}$, $[S] = (t_{ij})_{n \times n}$. Where $s_{ij} = \gcd(x_i, x_j)$ is the greatest common divisor(GCD) and $t_{ij} = \text{lcm}(x_i, x_j)$ is the least common multiple(LCM). It has given the calculation result to the determinant of the GCD matrix over the set $E(n) = \{1, 2, \dots, n\}$ ^[1], as follows:

$$\det \langle E(n) \rangle = \phi(1)\phi(2)\cdots\phi(n) \quad (1)$$

When S is the FC set, the following two equality have established^[2-3]:

$$\det[S] = \phi(x_1)\phi(x_2)\cdots\phi(x_n)\pi(x_1)\pi(x_2)\cdots\pi(x_n) \quad (2)$$

$$\det \langle S \rangle = (f*\mu)(x_1)(f*\mu)(x_2)\cdots(f*\mu)(x_n) \quad (3)$$

Next, we have generalized the properties of the GCD matrix $\langle S \rangle_{n \times n}$ from the positive integer set S to the direct product of the positive integer set.

It has proved that the GCD matrix on the general positive integer set $S = \{x_1, x_2, \dots, x_n\}$ is positive definite^[4], and it has the following conclusion^[5]:

$$\det \langle S \rangle = \prod_{x \in S} \phi(x) \Leftrightarrow S \text{ is the FC set, and } \det \langle S \rangle \leq x_1 x_2 \cdots x_n - \frac{n!}{2}$$

Let S be a positive integer set with the divisibility as the partially ordered relation. We have inductively defined a generalized Euler function^[6]

$$\Psi_S(x_j) = x_j - \sum_{x_i | x_j, x_i \neq x_j} \Psi_S(x_i). \text{ If } S \text{ is the FC set, then } \Psi_S(x) = \phi(x) \quad \forall x \in S.$$

$$\text{Similar, } T \text{ is the GCDC set including } S, \text{ so } \det \langle S \rangle \geq \prod_{x \in S} \Psi_T(x) \quad (5)$$

The equality has holden if and only if S is a GCDC set and $T \setminus S$ has not contained the divisor of the element in S ^[6]. Keith Bourque and Steve Ling have found that if S is the FC set, then $\langle S \rangle$ is reversible^[7], and $[S]\langle S \rangle^{-1}$ has the integer arrays.

Let $S_1, S_2, \dots, S_n$ be n finite positive integer sets, $\forall 1 \leq i \leq n$, let p_i be the partially ordered relation on S_i , that is $\forall x_i, y_i \in S_i, x_i p_i y_i \Leftrightarrow x_i | y_i$. Then (S_i, p_i) be the poset.

Let $S = S_1 \times S_2 \times \cdots \times S_n$, defined a partially ordered relation p for S . $\forall x = (x_1, x_2, \dots, x_n) \in S, y = (y_1, y_2, \dots, y_n) \in S$,

make $xpy \Leftrightarrow x_i | y_i, 1 \leq i \leq n$. (S, p) is a poset too. We have: $\forall x, y \in S$

$$1) x | y \Leftrightarrow xpy, x \text{ is called the divisor of } y; 2) \gcd(x, y) = \prod_{i=1}^n \gcd(x_i, y_i),$$

(S, p) is a subset of the meet semi-lattice $N \times N \times \cdots \times N$, satisfied

$$(x, y)_p = ((x_1, y_1)_{p_1}, (x_2, y_2)_{p_2}, \dots, (x_n, y_n)_{p_n}) = (\gcd(x_1, y_1), \gcd(x_2, y_2), \dots, \gcd(x_n, y_n))$$

The meet matrix on S is $\langle S \rangle = (\gcd(d_i, d_j))_{q \times q}$

The concept of the meet semi-lattice is given below.

Definition 1 Let U be a poset with the partially ordered relationship p . U is called to the meet semi-lattice, if $\forall x, y \in U$, it exists the unique $\omega \in U$. It is satisfied: 1)

ωpx and ωpy ;

2) If there is $z \in U$ and zpx and zpy , then $z\omega$. It is called that ω is the meet with x and y , and denoted by $\omega = (x, y)_p$

By definition 1, when the positive integer set N has the divisibility as a partially ordered relation p , it has become a meet semi-lattice, written as (N, p) and $xpy \Leftrightarrow x|y$ (6)

So is $(x, y)_p = \gcd(x, y)$. If $xpy \Leftrightarrow y|x$ then $(x, y)_p = \text{lcm}(x, y)$.

Definition 2 Let S be a subset of the meet semi-lattice U , S is meet closed, if $\forall x, y \in S$, then $(x, y)_p \in S$, S is called the MC set. Similarly, that S is low closed, if $x \in S, \forall y \in U$ and ypx , then $y \in S$. S has been abbreviated to LC.

Definition 3 Let $S = \{x_1, x_2, \dots, x_n\}$ be a finite subset of the meet semi-lattice U , f are defined as the function on U , there is $f: U \rightarrow R$, where R is a commutative ring, it denoted by $\langle S \rangle_f = (s_{ij})_{n \times n}$, $s_{ij} = f((x_i, x_j)_p)$ ($1 \leq i, j \leq n$). $\langle S \rangle_f$ is Called the meet matrix about f on S . In general, R is the real number field.

Definition 4 let $S = \{x_1, x_2, \dots, x_n\}$ be a subset of meet semi-lattice (U, p) , defined as

$$\Psi_{S,f}(x_j) = f(x_j) - \sum_{x_i p x_j, x_i \neq x_j} \Psi_{S,f}(x_i), \text{ its value is 0 when the sum item is empty,}$$

where f is the real function in U , $\Psi_{S,f}$ be called the generalized Euler function on S .

Remark 1 $f(x_j) = \sum_{x_i p x_j} \Psi_{S,f}(x_i)$, and S is a subset of U , then^[8] by the Mobius

inversion $\Psi_{S,f}(x_j) = \sum_{x_i p x_j} f(x_i) \mu(x_i, x_j)$. Let

$$S = S_1 \times S_2 \times \dots \times S_n = \{d_1, d_2, \dots, d_q\},$$

$$\Psi_S(d_j) = d_j^{(1)} d_j^{(2)} \dots d_j^{(n)} - \sum_{d_i p d_j, d_i \neq d_j} \Psi_S(d_i) \text{ where } d_j^{(1)} d_j^{(2)} \dots d_j^{(n)} \text{ note the product of}$$

the component in d_j .

2 Main results

Let $A = (a_{ij})_{n \times n}$ and $B = B_{m \times m}$ is defined as: $A \otimes B = (a_{ij} B) = C_{mn \times mn}$.

Where $a_{ij} B$ is a block of the block matrix $A \otimes B$. Let $S = S_1 \times S_2 \times \dots \times S_n$

$=\{d_1, d_2, \dots, d_q\}$ combined with the tensor product theory.

Theorem 1 let $S = S_1 \times S_2 \times \dots \times S_n$, and

$\langle S_i \rangle (1 \leq i \leq n)$ be the GCD matrix over S_i ,

then $\langle S \rangle = \langle S_1 \rangle \otimes \langle S_2 \rangle \otimes \dots \otimes \langle S_n \rangle$.

Proof: Use the induction to n . When $n = 2$, and $S = S_1 \times S_2$, we have

$\langle S \rangle = ((d_i, d_j)_p)_{q \times q}$ where $q = |S_1| |S_2|$. It

is easy to obtain $\langle S \rangle = \langle S_1 \rangle \otimes \langle S_2 \rangle$.

It holds that the number of the direct product item n is less than n as a hypothesis, let $\bar{S} = S_1 \times S_2 \times \dots \times S_{n-1}$, by the induction hypothesis, we have obtained $\langle \bar{S} \rangle = \langle S_1 \rangle \otimes \langle S_2 \rangle \otimes \dots \otimes \langle S_{n-1} \rangle$. Let $B = \langle S_n \rangle$ then $S = \bar{S} \times S_n$. We have got $\langle S \rangle = \langle \bar{S} \rangle \otimes B$ by above deduction. The theorem is correct by the definition of the tensor product.

Let $T_1, T_2, \dots, T_n$ be the positive integer finite set, $\forall 1 \leq i \leq n, T_i$ is the minimum GCD set containing S_i . Let $T = T_1 \times T_2 \times \dots \times T_n = \{f_1, f_2, \dots, f_t\}$, it is clearly to $T \supseteq S$, and T is MC set.

Theorem 2 Let S and T be given above, then $\langle S \rangle = AA^T = E \Lambda E^T$, so $\langle S \rangle$ is the positive definite. Where $A = (a_{ij}) E = (e_{ij})_{q \times t}$

$\Lambda = \text{diag}(\Psi_T(f_1), \Psi_T(f_2), \dots, \Psi_T(f_t))$.

Ψ_T is the generalized Euler function on T . And $e_{ij} = \begin{cases} 1 & f_i p d_i \\ 0 & \text{otherwise} \end{cases}, a_{ij} = e_{ij} \{\Psi_T(f_j)\}^{1/2}$

Theorem 3 Let S and T be given above, then $\det \langle S \rangle = \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_q \leq t} (\det E(k_1, k_2, \dots, k_q))^2 \times \Psi_T(f_{k_1}) \times \Psi_T(f_{k_2}) \times \dots \times \Psi_T(f_{k_q})$ where $E(k_1, k_2, \dots, k_q)$ is the matrix which has formed by k_1 th, k_2 th, $\dots$, k_q th columns taken out E according to the original position after the composition.

Theorem 4 Let S and T be given above, then 1) $\det \langle S \rangle \geq \prod_{d \in S} \Psi_T(d)$, the equality holds if and only if S is MC set and the element's divisor in S has no existed

in $T \setminus S$; 2) $\det \langle S \rangle \leq \prod_{i=1}^q (d_i^{(1)} d_i^{(2)} \dots d_i^{(n)})$;

3) When each S_i is FC set, $\det \langle S \rangle =$

$$\prod_{(x_1, x_2, \dots, x_n) \in S} [\phi(x_1)\phi(x_2)\cdots\phi(x_n)].$$

Firstly, we give the lemma.

Lemma 1 Let $A = A_{m \times m}$ and $B = B_{n \times n}$ Then $\det(A \otimes B) = (\det A)^n (\det B)^m$.

Theorem 5 Let $S = S_1 \times S_2 \times \cdots \times S_n$ then

$$\det \langle S \rangle = (\det \langle S_1 \rangle)^{L_1} (\det \langle S_2 \rangle)^{L_2} \cdots (\det \langle S_n \rangle)^{L_n}, \text{ where}$$

$$L_i = |S_1| |S_2| \cdots |S_n| / |S_i| \quad (1 \leq i \leq n).$$

The proof is easy and omits

Let S and T be given above. we have obtained the following equality:

$$\prod_{i=1}^q (\Psi_{T_i}(d_i^{(1)}) \Psi_{T_2}(d_i^{(2)}) \cdots \Psi_{T_n}(d_i^{(n)})) =$$

$$\left(\prod_{x_1 \in S_1} x_1 \right)^{L_1} \left(\prod_{x_2 \in S_2} x_2 \right)^{L_2} \cdots \left(\prod_{x_n \in S_n} x_n \right)^{L_n}$$

Where L_i is as meaning as theorem 5.

It is easy to verify that theorem 2, theorem 3, and theorem 4 are correct. It has avoided the complex Mobius inversion. Their proof process is omitted.

3 The meet matrix

Let $P_1, P_2, \dots, P_n$ be the meet semi-lattice, $\forall 1 \leq i \leq n$, p_i is the partially ordered relation of P_i , then $P = P_1 \times P_2 \times \cdots \times P_n$ is also a poset and the meet semi-lattice with the partially ordered relation p .

Also let $g_i : P_i \rightarrow R$ be the function who take the value in the commutative ring R . Let $g : P \rightarrow R$ be the function in the meet semi-lattice P ,

$$\forall x = (x_1, x_2, \dots, x_n) \in P \text{ defined } g(x) = \prod_{i=1}^n g_i(x_i),$$

where the product is in the commutative ring. Let S_i be the finite subset of P_i , T_i be the smallest finite MC subset of P_i include in S_i .

Let Ψ_{S_i, g_i} be the generalized Euler function on S_i , then $\Psi_{S, g}$ is the generalized Euler function on S . then

$$\Psi_{S, g}(d_j) = g(d_j) - \sum_{d_i p d_j, d_i \neq d_j} \Psi_{S, g}(d_i). \text{ Let}$$

$$\langle S \rangle_g = (m_{ij})_{q \times q} \text{ on } S, \text{ where } m_{ij} = g((d_i, d_j)_p)$$

We have required that R is the real field for $g_i : P_i \rightarrow R$, let

$$\Psi_{S_i, g_i} > 0. \forall d = (d_1, d_2, \dots, d_n) \in S, \Psi_{S, g}(d_j) = \prod_{i=1}^n \Psi_{S_i, g_i}(d_i), \text{ then } \Psi_{S, g} > 0.$$

Theorem 6 Let T be a MC subset contained S , if $\Psi_{S, g} > 0$, then

- 1) $\det \langle S \rangle_g \geq \prod_{d \in S} \Psi_{T, g}(d)$, the equality holds if and only if S is MC set, and there is no the elements in $T \setminus S$ that have the relationships p with the elements in S ;
- 2) $\det \langle S \rangle_g < \prod_{d \in S} g(d)$.

The following theorem is not required $\Psi_{S, g} > 0$.

Theorem 7 Let $S = S_1 \times S_2 \times \dots \times S_n$ be a MC set, then we have the following conclusion: $\det \langle S \rangle_g = \prod_{d \in S} \Psi_{T, g}(d)$

Theorem 6 is similar to the theorem 4 by the tensor product theory, so their proof is to omit. The properties of the meet matrix have also been studied from the invariant divisor, the characteristic divisor or the engenvalue, which is the next working target to us.

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