The Generalization of GCD Matrices

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Abstract

Let \( S_1, S_2, \ldots, S_n \) be \( n \) finite sets of positive integers and \( S = S_1 \times S_2 \times \cdots \times S_n \). We have got a \( q \times q \) matrix \( \langle S \rangle = \) \((\gcd(d_i, d_j))_{pq}\), where \( s_{ij} = \gcd(d_i, d_j) \). In this paper, we study the bounds of the determinant of \( \langle S \rangle \), and the value of the determinant of them in special condition. Finally, we generalize the GCD matrices to the direct product of general posets and obtain some results.

Keywords: the GCD matrix; the meet semi-lattice; the generalized Euler's function; the mobius inversion.

0 Introduction

The greatest common divisor matrices are a kind of special matrices defined in the positive integer set. And the properties of their determinants have been always the hot research. Beslin and Ligh have defined the greatest common divisor matrix\(^{[1]}\) on the positive integer set \( S = \{x_1, x_2, \ldots, x_n\} \), take as \( \langle S \rangle = (s_{ij})_{nn} \), \( s_{ij} = \gcd(x_i, x_j) \), it is shorten to the GCD matrix and to proved \( \langle S \rangle \) that is positive
definite. In [2] has proved: if $S$ is the FC set then $\det \langle S \rangle = \phi(x_1)\phi(x_2)\cdots\phi(x_n)$, where $\phi$ is the Euler function. In [3], it has defined as the matrix $\langle S \rangle _f = (s_{ij})_{n\times n}$, where $s_{ij} = f(\gcd(x_i, x_j))(1 \leq i, j \leq n)$. H.J.S.Smith has proved that if $S$ is FC set, then $\det \langle S \rangle _f = (f * \mu)(x_1)(f * \mu)(x_2)\cdots(f * \mu)(x_n)$, where $f * \mu$ is the convolution. In [5-6], some good results have been obtained. In this paper, we have gotten some new generalization in this paper for the GCD matrix. In order to the convenient for the introduction, it is defined as follows.

In [5], It is the definition about the meet semi-lattice, the meet matrix LC and MC set, and LC and MC are the corresponding generalization of FC and GCDC set.

**Definition 1** let $S$ be a subset of meet semi-lattice $(U, p)$, defined as $\Psi_{S,f}(x_j) = f(x_j) - \sum_{x_j, p_j, s_j \neq s_j} \Psi_{S,f}(x_j)$, its value is 0 when the sum item is empty, where $f$ is the real function in $U$, $\Psi_{S,f}$ be called the generalized Euler function on $S$.

**Remark 1** By the Mobius inversion[7],
\[
f(x_j) = \sum_{x_j, p_j} \Psi_{S,f}(x_j).
\]

Now we have generalized the concept of GCD matrix. $\forall 1 \leq i \leq n$, let $p_i$ be the partially ordered relation on $S_i$, that is $\forall x_j, y_j \in S_i, x_j, p_j, y_j \Leftrightarrow x_j \mid y_j$. Then $(S_i, p_j)$ is the poset. So $(S_i, p_i)$ is a subset of the meet semi-lattice. And let $S = S_1 \times S_2 \times \cdots \times S_n$, defined a partially ordered relation $p$ for $S$. $\forall x, y \in S, xpy \Leftrightarrow x \mid y, 1 \leq i \leq n$. $(S, p)$ is a finite poset and the meet semi-lattice too. $\forall x, y \in S$, we have the two conclusions:

1) $x \mid y \Leftrightarrow xpy$, $x$ is called the divisor of $y$; 2) $\gcd(x, y) = \prod_{i=1}^{n} \gcd(x_i, y_i)$. Then the meet matrix on $S$ is $\langle S \rangle = (\gcd(d_i, d_j))_{q \times q}$.

Let $\Psi_S$ be the generalized Euler function[5] on $S_i$. It can be defined the generalized Euler function on $S$ as $\Psi_S(d_j) = d_j^{(1)}d_j^{(2)}\cdots d_j^{(n)} = \sum_{d_j \mid d_j, d_j \neq d_j} \Psi_S(d_j)$ where $d_j^{(1)}d_j^{(2)}\cdots d_j^{(n)}$ note the product of the component in $d_j$. Suppose $\forall d_j \in S_i = \{d_j, d_{jj} \cdots d_j\}, d_j = (d_j^{(1)}, d_j^{(2)}, \cdots, d_j^{(n)})$.

We have gotten the GCD matrix on $S$ as follows: $\langle S \rangle_{q \times q} = (\gcd(d_i, d_j))_{q \times q}$. 

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1 Main results

\(\forall 1 \leq i \leq n, T_i\) is the minimum GCDC\(^6\) set containing \(S_i\). Let \(T\) is MC set and \(T = T_1 \times T_2 \times \cdots \times T_n = \{f_1, f_2, \ldots, f_n\}\), it is clearly to \(T \supseteq S\). We have some conclusions as follows:

**Theorem 1** Let \(S\) and \(T\) be given above, then \(\langle S \rangle = AA^T = EAE^T\), so \(\langle S \rangle\) is the positive definite. Where \(A = (a_1), E = (e_i)_{\text{sym}}, \Lambda = \text{diag}(\Psi_T(f_1), \Psi_T(f_2), \ldots, \Psi_T(f_1))\). \(A^T\) and \(E^T\) are respectively the transpose, \(\Psi_T\) is the generalized Euler function on \(T\). And \(e_i = \begin{cases} 1 & f_{\mu i} \\ 0 & \text{otherwise} \end{cases}\)

**Theorem 2** Let \(S, T\) and \(E\) be given above, then \(\det \langle S \rangle = \sum_{1 \leq k_1 < k_2 < \cdots < k_q \leq q} (d_{k_1} E_{k_2} \cdots E_{k_q}) \times \Psi_T(f_{k_1}) \Psi_T(f_{k_2}) \cdots \Psi_T(f_{k_q})\) is the matrix which has formed by \(k_1\text{th}, k_2\text{th}, \ldots, k_q\text{th}\) columns taken out \(E\) according to the original position.

We can discuss the upper or lower bound of the determinants for \(\langle S \rangle\) in the following theorem:

**Theorem 3** Let \(S, T\) and \(E\) be given above, then \(1) \det \langle S \rangle \geq \prod_{d \in S} \Psi_T(d)\), the equality holds if and only if \(S\) is MC set and the element’s divisor in \(S\) has no existed in \(T \setminus S\);

\(2) \det \langle S \rangle \leq \prod_{i=1}^{q} (d_1^{(i)}d_2^{(i)}\cdots d_q^{(i)})\);

\(3)\) When each \(S_i\) is FC set, \(\det \langle S \rangle = \prod_{(x_1, x_2, \ldots, x_n) \in S} [\phi(x_1)\phi(x_2)\cdots\phi(x_n)].\)

2 Proofs of the main results

In this section, we have mainly proved above theorems. If the positive integer set \(S = \{x_1, x_2, \ldots, x_n\}\) is the GCDC set, then the generalized Euler function
Ψ_S(x) ≥ φ(x), ∀ x ∈ S. If S is the FC set, then Ψ_S(x) = φ(x), ∀ x ∈ S, S = T,
V_{x_j} = \{z ∈ T : \min \{y ∈ S : z = x_j \} = \{x_j \} \} We have given the following lemma.

Lemma 1 Let S and T be given above, then Ψ_S(x) = \prod_{i=1}^{n} Ψ_S(x_i).

Now, we have proved the main results as follows:

The proof of Theorem 1: Take
\Lambda = diag(\Psi_T(f_1), \Psi_T(f_2), \cdots, \Psi_T(f_r)) \), then D = EL = \left( e_{qy} \Psi_T(f_j) \right)_{qy} \), So the
\begin{align*}
(i, j) \text{– array in } EAE^T & = \sum_k e_k \Psi_T(f_k) e_k = \sum_k e_k e_k \Psi_T(f_k) \\
& = \sum_{f_k \in \Lambda} \sum_{f_k \in \Lambda} \Psi_T(f_k) \\
& = \gcd(d_i, d_j).
\end{align*}
So we have obtained < S >= EAE^T. By lemma 1, it is easy to Ψ_S(x) = \prod_{i=1}^{n} Ψ_S(x_i).

From [8], ∀ 1 ≤ i ≤ n, T_k is GCDC set, but \Psi_T(y_k) > 0 , therefore Ψ_T(y) > 0 .

Suppose \Lambda = E\Lambda^{\frac{1}{2}} = Ediag\left( \left[ \Psi_T(f_1) \right]^{\frac{1}{2}}, \left[ \Psi_T(f_2) \right]^{\frac{1}{2}}, \cdots, \left[ \Psi_T(f_r) \right]^{\frac{1}{2}} \right), then < S >= AA^T = EAE^T. So is well to the theorem.

The proof of theorem 2: from the proof of Theorem 1 , it is easy to known that
< S >= EAE^T. Take D = EL = \left( e_{qy} \Psi_T(f_j) \right)_{qy} \).

Use the Cauchy-Binet formula[9] to < S >= EAE^T, the result is right.

The reason that the third equality has established is that each column has
extracted the common divisor in the determinant. So the theorem 2 holds.

Corollary 1 If S = S_1 × S_2 × \cdots × S_n is MC set, then det(S) = \prod_{i=1}^{q} \Psi_S(d_i).

The proof of Theorem 3: By theorem 2, we have known < S >= EAE^T and
\Lambda = diag(\Psi_T(f_1), \Psi_T(f_2), \cdots, \Psi_T(f_r)) \).

Because T_k is GCDC set, it is to known that Ψ_T(y_k) = φ(y_k) > 0 , that is Ψ_T(y) > 0 by
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proposition 1. Then $\langle S \rangle$ is positive definite. From theorem 2, $\det \langle S \rangle \geq \prod_{d \in S} \Psi_\tau(d) = \prod_{d \in S} \Psi_\tau(d_i)$. Suppose

$T = T_1 \times T_2 \times \cdots \times T_n = \{ f_1, f_2, \cdots, f_1 \}$, it is apparently to $t > q$.

From theorem 1, $\langle S \rangle = E \Lambda E^T$, $\Lambda = \text{diag}(\Psi_\tau(f_1), \Psi_\tau(f_2), \cdots, \Psi_\tau(f_1))$.

We separate the matrix into the block $E = (E_1, E_2)$. Let

$\Lambda_1 = \text{diag}(\Psi_\tau(f_1), \Psi_\tau(f_2), \cdots, \Psi_\tau(f_1))$,

$\Lambda_2 = \text{diag}(\Psi_\tau(f_q), \Psi_\tau(f_q), \cdots, \Psi_\tau(f_q))$. We have the equality $C = E_1 \Lambda_1 E_1^T$ and $D = E_2 \Lambda_2 E_2^T$. So $\langle S \rangle = E \Lambda E^T = C + D$

Because of $\Psi_\tau(d_i) > 0, \forall d_i \in S$, we have known that $\Lambda_1$ is positive definite. But $\det C = 0$, it is positive definite to $C$ and positive semi-definite to $D$.

$\det \langle S \rangle = \det(C + D) \geq \det C + \det D \geq \det C$. That is $\det \langle S \rangle \geq \prod_{d \in S} \Psi_\tau(d)$, the equality has established if and only if $D = 0$.

While $\Lambda_2$ is positive definite, there must be $E_2 = 0$, that is if and only if $S$ is MC set and the element’s divisor in $S$ has no existed in $T \setminus S$. $T$ is the GDC set, so is $S$. (1) is correct.

(2) $\langle S \rangle$ is positive definite by theorem 2, we have known that

$\det \langle S \rangle \leq \prod_{i=1}^{q} \gcd(d_i, d_j) = \prod_{i=1}^{q} (d_i^{(1)} d_i^{(2)} \cdots d_i^{(n)})$ (3) If $S$ is FC set, then $S = T$. We know $\Psi_\tau(y_k) = \phi(y_k)$, therefore

$\Psi_\tau(y) = \prod_{k=1}^{n} \Psi_\tau(y_k) = \prod_{k=1}^{n} \phi(y_k)$. So

$\det \langle S \rangle = \prod_{(x_1, x_2, \cdots, x_n) \in S} [\phi(x_1) \phi(x_2) \cdots \phi(x_n)]$, To sum up, (1) (2) (3) are all hold.

3 The generalized GCD matrix

Let $P_1, P_2, \cdots, P_n$ be the meet semi-lattice, $\forall 1 \leq i \leq n$, $P_i$ is the partially ordered relation of $P_i$. Let $P = P_1 \times P_2 \times \cdots \times P_n$, $\forall x, y \in P$

$(x, y) = ((x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n))$.

Let $g_i : P_i \rightarrow R$ be the function who take the value in the commutative ring $R$.

$\forall x = (x_1, x_2, \cdots, x_n) \in P$ defined $g(x) = \prod_{i=1}^{n} g_i(x_i)$. 
Let $T_i$ be the smallest finite MC subset of $\mathcal{P}_i$ include in $S_i$.

If $T$ is MC subset include in $S$ and 

$\Psi_{S_i, g_i} > 0$, then $\forall d = (d_1, d_2, \cdots, d_n) \in S$,

$\Psi_{S, g}(d) = \prod_{i=1}^{\eta} \Psi_{S_i, g_i}(d_i)$ and $\Psi_{S, g} > 0$.

**Theorem 4** Let $S$ and $T$ be given above, $T$ is a MC subset contained $S$, if $\Psi_{S, g} > 0$, then (1) $\det (S) \geq \prod_{d \in S} \Psi_{T, g}(d)$ ; (2) $\det (S) < \prod_{d \in S} g(d)$.

The following theorem is not required $\Psi_{S, g} > 0$.

**Theorem 5** Let $S = S_1 \times S_2 \times \cdots \times S_n$, if $S$ is MC set, then we have the following conclusion: $\det (S) = \prod_{d \in S} \Psi_{T, g}(d)$

As a result, Theorem 4 is similar to the theorem 3, and Theorem 5 is similar to the theorem 13 in [4], the proof omits.

We have gotten the certain relations between generalized GCD matrices and the generalized Euler function. It is also available to consider the invariant divisor and characteristic divisor or the eigenvalues of the greatest common divisor matrices for studying them, which is the next working target to us.

**References**


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