The Generalization of GCD Matrices

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Abstract

Let $S_1, S_2, \dots, S_n$ be n finite sets of positive integers and $S = S_1 \times S_2 \times \dots \times S_n$. We have got a $q \times q$ matrix $\langle S \rangle = (\gcd(d_i, d_j))_{q \times q}$, where $s_{ij} = \gcd(d_i, d_j)$. In this paper, we study the bounds of the determinant of $\langle S \rangle$, and the value of the determinant of them in special condition. Finally, we generalize the GCD matrices to the direct product of general posets and obtain some results.

Keywords: the GCD matrix; the meet semi-lattice; the generalized Euler's function; the mobius inversion.

0 Introduction

The greatest common divisor matrices are a kind of special matrices defined in the positive integer set. And the properties of their determinants have been always the hot research. Beslin and Ligh have defined the greatest common divisor matrix^[1] on the positive integer set $S = \{x_1, x_2, \dots, x_n\}$, take as $\langle S \rangle = (s_{ij})_{n \times n}$, $s_{ij} = \gcd(x_i, x_j)$, it is shorten to the GCD matrix and to proved $\langle S \rangle$ that is positive

definite. In [2] has proved: if S is the FC set then $\det \langle S \rangle = \phi(x_1)\phi(x_2)\cdots\phi(x_n)$, where ϕ is the Euler function. In [3], it has defined as the matrix $\langle S \rangle_f = (s_{ij})_{n \times n}$, $s_{ij} = f(\gcd(x_i, x_j)) (1 \leq i, j \leq n)$. H.J.S.Smith has proved that if S is FC set, then^[4] $\det \langle S \rangle_f = (f * \mu)(x_1)(f * \mu)(x_2)\cdots(f * \mu)(x_n)$. Where $f * \mu$ is the convolution. In [5-6], some good results have been obtained. In this paper, we have gotten some new generalization in this paper for the GCD matrix. In order to the convenient for the introduction, it is defined as follows.

In[5], It is the definition about the meet semi-lattice, the meet matrix LC and MC set, and LC and MC are the corresponding generalization of FC and GCD set.

Definition 1 let S be a subset of meet semi-lattice (U, p) , defined as $\Psi_{S,f}(x_j) = f(x_j) - \sum_{x_i p x_j, x_i \neq x_j} \Psi_{S,f}(x_i)$, its value is 0 when the sum item is empty, where f is the real function in U , $\Psi_{S,f}$ be called the generalized Euler function on S .

Remark 1 By the Mobius inversion^[7],

$$f(x_j) = \sum_{x_i p x_j} \Psi_{S,f}(x_i).$$

Now we have generalized the concept of GCD matrix. $\forall 1 \leq i \leq n$, let p_i be the partially ordered relation on S_i , that is

$\forall x_i, y_i \in S_i, x_i p_i y_i \Leftrightarrow x_i \mid y_i$. Then (S_i, p_i) is the poset. So (S_i, p_i) is a subset of the meet semi-lattice. And let $S = S_1 \times S_2 \times \cdots \times S_n$, defined a partially ordered relation p for S . $\forall x, y \in S, xpy \Leftrightarrow x_i \mid y_i, 1 \leq i \leq n$. (S, p) is a finite poset and the meet semi-lattice too. $\forall x, y \in S$, we have the two conclusions:

1) $x \mid y \Leftrightarrow xpy$, x is called the divisor of y ; 2) $\gcd(x, y) = \prod_{i=1}^n \gcd(x_i, y_i)$. Then

the meet matrix on S is $\langle S \rangle = (\gcd(d_i, d_j))_{q \times q}$.

Let Ψ_{S_i} be the generalized Euler function^[5] on S_i . It can be defined the generalized Euler function on S as

$$\Psi_S(d_j) = d_j^{(1)} d_j^{(2)} \cdots d_j^{(n)} - \sum_{d_i p d_j, d_i \neq d_j} \Psi_S(d_i) \text{ where } d_j^{(1)} d_j^{(2)} \cdots d_j^{(n)} \text{ note the product of}$$

the component in d_j . Suppose

$$\forall d_i \in S = \{d_1, d_2, \cdots, d_q\}, d_i = (d_i^{(1)}, d_i^{(2)}, \cdots, d_i^{(n)}).$$

We have gotten the GCD matrix on S as follows: $\langle S \rangle_{q \times q} = (\gcd(d_i, d_j))_{q \times q}$.

1 Main results

$\forall 1 \leq i \leq n, T_i$ is the minimum GCDC^[6] set containing S_i . Let T is MC set and $T = T_1 \times T_2 \times \cdots \times T_n = \{f_1, f_2, \dots, f_i\}$, it is clearly to $T \supseteq S$. We have some conclusions as follows:

Theorem 1 Let S and T be given above, then $\langle S \rangle = AA^T = E\Lambda E^T$, so $\langle S \rangle$ is the positive definite. Where $A = (a_{ij})$, $E = (e_{ij})_{q \times n}$, $\Lambda = \text{diag}(\Psi_T(f_1), \Psi_T(f_2), \dots, \Psi_T(f_i))$. A^T and E^T are respectively the transpose, Ψ_T is the generalized Euler function on T .

$$\text{And } e_{ij} = \begin{cases} 1 & f_i \text{ divides } f_j \\ 0 & \text{otherwise} \end{cases} \quad a_{ij} = e_{ij} \{\Psi_T(f_j)\}^{1/2}$$

Theorem 2 Let S, T and E be given above, then $\det \langle S \rangle =$

$\sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_q} (\det E(k_1, k_2, \dots, k_q))^2 \times \Psi_T(f_{k_1}) \Psi_T(f_{k_2}) \dots \Psi_T(f_{k_q})$ $E(k_1, k_2, \dots, k_q)$ is the matrix which has formed by k_1 th, k_2 th, $\dots$, k_q th columns taken out E according to the original position.

We can discuss the upper or lower bound of the determinants for $\langle S \rangle$ in the following theorem:

Theorem 3 Let S, T and E be given above, then (1) $\det \langle S \rangle \geq \prod_{d \in S} \Psi_T(d)$, the equality holds if and only if S is MC set and the element's divisor in S has no existed in $T \setminus S$;

$$(2) \det \langle S \rangle \leq \prod_{i=1}^q (d_i^{(1)} d_i^{(2)} \dots d_i^{(n)});$$

(3) When each S_i is FC set,

$$\det \langle S \rangle = \prod_{(x_1, x_2, \dots, x_n) \in S} [\phi(x_1) \phi(x_2) \dots \phi(x_n)].$$

2 Proofs of the main results

In this section, we have mainly proved above theorems. If the positive integer set $S = \{x_1, x_2, \dots, x_n\}$ is the GCDC set, then the generalized Euler function

$\Psi_S(x) \geq \phi(x), \forall x \in S$. If S is the FC set, then $\Psi_S(x) = \phi(x), \forall x \in S, S=T$,
 $V_{x_j} = \{z \in T : \min\{y \in S : z|y\} = x_j\} = \{x_j\}$ We have given the following lemma.

Lemma 1 Let S and T be given above, then $\Psi_S(x) = \prod_{i=1}^n \Psi_{S_i}(x_i)$.

Now, we have proved the main results as follows:

The proof of Theorem 1: Take

$\Lambda = \text{diag}(\Psi_T(f_1), \Psi_T(f_2), \dots, \Psi_T(f_t))$, then $D = E\Lambda = (e_{ij}\Psi_T(f_j))_{q \times t}$, So the

(i, j) -array in $E\Lambda E^T$ is

$$\begin{aligned} \sum_k e_{ik} \Psi_T(f_k) e_{jk} &= \sum_k e_{ik} e_{jk} \Psi_T(f_k) \\ &= \sum_{f_k p d_i, f_k p d_j} \Psi_T(f_k) = \sum_{f_k p(d_i, d_j)_p} \Psi_T(f_k) \\ &= \gcd(d_i, d_j). \end{aligned}$$

So we have obtained $\langle S \rangle = E\Lambda E^T$. By lemma 1, it is easy to $\Psi_S(x) = \prod_{i=1}^n \Psi_{S_i}(x_i)$.

From [8], $\forall 1 \leq i \leq n, T_k$ is GCDC set, but $\Psi_{T_k} \geq \phi(y_k) > 0$, therefore $\Psi_T(y) > 0$.

Suppose $A = E\Lambda^{1/2} =$

$E \text{diag}(\{\Psi_T(f_1)\}^{1/2}, \{\Psi_T(f_2)\}^{1/2}, \dots, \{\Psi_T(f_t)\}^{1/2})$, then $\langle S \rangle = AA^T$, that is $\langle S \rangle = AA^T = E\Lambda E^T$. So is well to the theorem.

The proof of theorem 2: from the proof of Theorem 1, it is easy to known that

$$\langle S \rangle = E\Lambda E^T. \text{ Take } D = E\Lambda = (e_{ij}\Psi_T(f_j))_{q \times t}.$$

Use the Cauchy-Binet formula^[9] to $\langle S \rangle = E\Lambda E^T$, the result is right.

The reason that the third equality has established is that each column has extracted the common divisor in the determinant. So the theorem 2 holds.

Corollary 1 If $S = S_1 \times S_2 \times \dots \times S_n$ is MC set, then $\det[S] = \prod_{i=1}^q \Psi_{S_i}(d_i)$.

The proof of Theorem 3: By theorem 2, we have known $\langle S \rangle = E\Lambda E^T$ and

$$\Lambda = \text{diag}(\Psi_T(f_1), \Psi_T(f_2), \dots, \Psi_T(f_t)).$$

Because T_k is GCDC set, it is to know that $\Psi_{T_k} \geq \phi(y_k) > 0$, that is $\Psi_T(y) > 0$ by

proposition 1. Then $\langle S \rangle$ is positive definite. From theorem 2, $\det \langle S \rangle \geq$

$$\prod_{d \in S} \Psi_T(d) = \prod_{i=1}^q \Psi_T(d_i). \quad \text{Suppose}$$

$T = T_1 \times T_2 \times \cdots \times T_n = \{f_1, f_2, \dots, f_t\}$, it is apparently to $t > q$.

From theorem 1, $\langle S \rangle = E \Lambda E^T$, $\Lambda = \text{diag}(\Psi_T(f_1), \Psi_T(f_2), \dots, \Psi_T(f_t))$.

We have separate the matrix into the block $E = (E_1, E_2)$. Let

$$\Lambda_1 = \text{diag}(\Psi_T(f_1), \Psi_T(f_2), \dots, \Psi_T(f_q)) \quad ,$$

$\Lambda_2 = \text{diag}(\Psi_T(f_{q+1}), \Psi_T(f_{q+2}), \dots, \Psi_T(f_t))$. We have the equality $C = E_1 \Lambda_1 E_1^T$ and

$$D = E_2 \Lambda_2 E_2^T. \text{ So } \langle S \rangle = E \Lambda E^T = C + D$$

Because of $\Psi_T(d_i) > 0, \forall d_i \in S$, we have known that Λ_1 is positive definite.

But $\det C \neq 0$, it is positive definite to C and positive semi-definite to D .

$\det \langle S \rangle = \det(C + D) \geq \det C + \det D \geq \det C$. That is $\det \langle S \rangle \geq \prod_{d \in S} \Psi_T(d)$, the equality

has established if and only if $D = 0$.

While Λ_2 is positive definite, there must be $E_2 = 0$, that is if and only if S is MC set and the element's divisor in S has no existed in $T \setminus S$. T is the GCD set, so is S . (1) is correct.

(2) $\langle S \rangle$ is positive definite by theorem 2, we have known that

$$\det \langle S \rangle \leq \prod_{i=1}^q \gcd(d_i, d_i) = \prod_{i=1}^q (d_i^{(1)} d_i^{(2)} \cdots d_i^{(n)}) \quad (3) \text{ If } S \text{ is FC set, then } S = T. \text{ We}$$

know $\Psi_{T_k}(y_k) = \phi(y_k)$, therefore

$$\Psi_T(y) = \prod_{k=1}^n \Psi_{T_k}(y_k) = \prod_{k=1}^n \phi(y_k). \text{ So}$$

$$\det \langle S \rangle = \prod_{(x_1, x_2, \dots, x_n) \in S} [\phi(x_1) \phi(x_2) \cdots \phi(x_n)], \text{ To sum up, (1) (2) (3) are all hold.}$$

3 The generalized GCD matrix

Let $P_1, P_2, \dots, P_n$ be the meet semi-lattice, $\forall 1 \leq i \leq n$, p_i is the partially ordered relation of P_i . Let $P = P_1 \times P_2 \times \cdots \times P_n, \forall x, y \in P$

$$(x, y)_p = ((x_1, y_1)_{p_1}, (x_2, y_2)_{p_2}, \dots, (x_n, y_n)_{p_n}).$$

Let $g_i: P_i \rightarrow R$ be the function who take the value in the commutative ring R .

$$\forall x = (x_1, x_2, \dots, x_n) \in P \text{ defined } g(x) = \prod_{i=1}^n g_i(x_i).$$

Let T_i be the smallest finite MC subset of P_i include in S_i .

If T is MC subset include in S and

$\Psi_{S_i, g_i} > 0$, then $\forall d = (d_1, d_2, \dots, d_n) \in S$,

$$\Psi_{S, g}(d_j) = \prod_{i=1}^n \Psi_{S_i, g_i}(d_i) \text{ and } \Psi_{S, g} > 0.$$

Theorem 4 Let S and T be given above, T is a MC subset contained S , if $\Psi_{S, g} > 0$, then (1) $\det \langle S \rangle_g \geq \prod_{d \in S} \Psi_{T, g}(d)$; (2) $\det \langle S \rangle_g < \prod_{d \in S} g(d)$.

The following theorem is not required $\Psi_{S, g} > 0$.

Theorem 5 Let $S = S_1 \times S_2 \times \dots \times S_n$, if S is MC set, then we have the following conclusion: $\det \langle S \rangle_g = \prod_{d \in S} \Psi_{T, g}(d)$

As a result, Theorem 4 is similar to the theorem 3, and Theorem 5 is similar to the theorem 13 in [4], the proof omits.

We have gotten the certain relations between generalized GCD matrices and the generalized Euler function. It is also available to consider the invariant divisor and characteristic divisor or the eigenvalues of the greatest common divisor matrices for studying them, which is the next working target to us.

References

- [1] S.Beslin and S.Ligh, Greatest common divisor matrices[J]. Linear Algebra Appl. 118 (1989):69-76.
- [2] Z.Li, The determinants of GCD matrices[J]. Linear Algebra Appl. 134 (1990):137-143.
- [3] K.Bourque and S.Ligh, Matrices associated with arithmetical functions[J]. Linear and Multilinear Algebra. 34 (1993):261-267.
- [4] H.J.S.Smith, On the Value of a certain arithmetical determinant[J]. Proc. London Math.Soc. 7 (1875-1876):208-212.
- [5] S.Hong, Bounds for determinants of Matrices associated with classes of arithmetical functions[J]. Linear Algebra Appl. 281 (1998):311-322.
- [6] Ismo.Korkee. Pentti Haukkanen, Bounds for determinants of meet matrices associated with incidence functions[J]. Linear Algebra Appl. 329 (2001):77-88.

- [7] R.P Stanley, Enumerative combinatorics (Vol.I). Wadsworth and Brooks/cole.Monterey, California,1986
- [8] N.Jacobson, Basic Algebra I [M].W.H.Freeman and company, San Francisco1974
- [9] Fan Yun, Liu Hongwei, Group and Combination coding [M]. Wuhan University Press, Wuhan,China, 2002

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