International Mathematical Forum, Vol. 8, 2013, no. 20, 973 - 981
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Weak Connes Amenability of the Even Duals of a Banach Algebra

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Abstract

Let $\mathcal{A}$ be a Banach algebra. In this paper, we introduce the concept of weak Connes amenability of $\mathcal{A}$. Next, by some conditions, we show that how weak amenability of $\mathcal{A}^{(2n)}$ implies weak Connes amenability of $\mathcal{A}^{(2n+2)}$. Finally, we relate this notion to the factorization property of odd duals of $\mathcal{A}$.

Mathematics Subject Classification: 46H20, 46H25

Keywords: Banach algebra, Arens Regularity, Weak Amenability, Connes Amenability, factorization

1 Introduction and Preliminaries

An Amenable Banach algebra was first introduced by B.E. Johnson in [3]. This notion also appeared in the work of A.Ya. Helemskii [2] which was published in the same year. Then, different generalizations were developed from this structure by other mathematicians. For instance, Dales, Ghahramani and Gronbaek started the concept of $n-$weak amenability of Banach algebras in [5]. Other notion from this disscusion, studied by V. Runde in [8], is Connes amenability of dual Banach algebras.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^*$, $\mathcal{A}^{**}$, $\ldots$ and $\mathcal{A}^{(n)}$ are the first, second, $\ldots$ and $n-$th dual of $\mathcal{A}$, respectively. For $a \in \mathcal{A}$ and $a^{'} \in \mathcal{A}^*$, we denote by $a^{'}a$ and $aa^{'}$ respectively, the functionals on $\mathcal{A}^*$ defined by $\langle a^{'}a, b\rangle = \langle a^{'}, ab\rangle = a^{'}(ab)$ and $\langle aa^{'}, b\rangle = \langle a^{'}, ba\rangle = a^{'}(ba)$ for all $b \in \mathcal{A}$.

R. Arens [4] defined two products on the second dual of Banach algebra $\mathcal{A}$ and represented them with symbols $\Box$ and $\Diamond$. These products are defined in three steps:

$$\begin{aligned}\langle a^{'}a, b\rangle &= \langle a^{'}, ab\rangle,\\ \langle a^{''}a^{'}, a\rangle &= \langle a^{''}, a^{'}a\rangle,\\ \langle a^{''}\Box b^{''}, a^{'}\rangle &= \langle a^{''}, b^{''}a^{'}\rangle,\end{aligned}$$

where $a, b \in \mathcal{A}$, $a^{'} \in \mathcal{A}^*$ and $a^{''}, b^{''} \in \mathcal{A}^{**}$. In this case, $a^{''}\Box b^{''}$ is called *the first Arens product* of $a^{''}$ and $b^{''}$. Similarly

$$\begin{aligned}\langle aa^{'}, b\rangle &= \langle a^{'}, ba\rangle,\\ \langle a^{'}a^{''}, a\rangle &= \langle a^{''}, aa^{'}\rangle,\\ \langle a^{''}\Diamond b^{''}, a^{'}\rangle &= \langle b^{''}, a^{'}a^{''}\rangle,\end{aligned}$$

for all $a, b \in \mathcal{A}$, $a^{'} \in \mathcal{A}^*$ and $a^{''}, b^{''} \in \mathcal{A}^{**}$. Then $a^{''}\Diamond b^{''}$ is called *the second Arens product* of $a^{''}$ and $b^{''}$. The Banach algebra $\mathcal{A}$ is said an *Arens regular* if and only if $a^{''}\Box b^{''} = a^{''}\Diamond b^{''}$ for every $a^{''}, b^{''} \in \mathcal{A}^{**}$. For more details about Arens product, We refer to [4].
Throughout of this paper, for each $n \geq 0$, we consider $(2n)-$th dual of Banach algebra $\mathcal{A}$, $\mathcal{A}^{(2n)}$, with the first Arens product.

By *Goldstine's theorem* [4, p.425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\mathcal{A}$ such that $a^{''} = weak^* - \lim_\alpha a_\alpha$ and $b^{''} = weak^* - \lim_\beta b_\beta$. So, it is easy to see that,

$$a^{''}\Box b^{''} = w^* - \lim_\alpha w^* - \lim_\beta (a_\alpha b_\beta)$$

and

$$a^{''}\Diamond b^{''} = w^* - \lim_\beta w^* - \lim_\alpha (a_\alpha b_\beta).$$

Thus, $\mathcal{A}$ is Arens regular if and only if we have

$$w^* - \lim_\alpha w^* - \lim_\beta (a_\alpha b_\beta) = w^* - \lim_\beta w^* - \lim_\alpha (a_\alpha b_\beta)$$

Also, we can find the *first* and *second topological centers* of $\mathcal{A}^{**}$, which are

$$\begin{aligned} Z_{\mathcal{A}}(\mathcal{A}^{**}) &= \{a'' \in \mathcal{A}^{**} : b'' \mapsto a''\Box b'' \textit{ is weak}^* - \textit{weak}^* - \textit{continuous}\}, \\ Z^t_{\mathcal{A}}(\mathcal{A}^{**}) &= \{b'' \in \mathcal{A}^{**} : a'' \mapsto a''\Diamond b'' \textit{ is weak}^* - \textit{weak}^* - \textit{continuous}\}. \end{aligned}$$

A Banach algebra $\mathcal{A}$ is Arens regular when $Z_{\mathcal{A}}(\mathcal{A}^{**}) = \mathcal{A}^{**}$, or equivalently, $Z^t_{\mathcal{A}}(\mathcal{A}^{**}) = \mathcal{A}^{**}$. (see, [6]).
Let $\mathcal{A}$ be a Banach algebra and X is a Banach $\mathcal{A}-$bimodule. A *derivation* $D : \mathcal{A} \longrightarrow X$ is a continuous linear map such that for all $a, b \in \mathcal{A}$,

$$D(ab) = a \cdot D(b) + D(a) \cdot b.$$

For each $x \in X$, we define a map $D_x : \mathcal{A} \longrightarrow X$ by

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}).$$

It is easily seen that D_x is a derivation. Such derivations are called *inner derivations*. $\mathcal{Z}^1(\mathcal{A}, X)$ is the Banach space of all continuous derivations from $\mathcal{A}$ into $X, \mathcal{B}^1(\mathcal{A}, X)$ is the normed space of all inner derivations from $\mathcal{A}$ into X. Then the first Hochschild cohomology group of $\mathcal{A}$ with coefficients in X is the quotient space

$$\mathcal{H}^1(\mathcal{A}, X) = \frac{\mathcal{Z}^1(\mathcal{A}, X)}{\mathcal{B}^1(\mathcal{A}, X)}.$$

With the above notations, a Banach algebra $\mathcal{A}$ is called *amenable* if $\mathcal{H}^1(\mathcal{A}, X^*) = 0$ for every Banach $\mathcal{A}-$bimodule X. In fact, $\mathcal{A}$ is amenable if every derivation from $\mathcal{A}$ into X^*, is inner for every Banach $\mathcal{A}-$bimodule X. Also, Banach algebra $\mathcal{A}$ is *weakly amenable* ($n-$*weakly amenable*) if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}(\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\})$, see [4, 5] for more details.

A Banach algebra $\mathcal{A}$ is *dual* if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = (\mathcal{A}_*)^*$. Note that if $\mathcal{A}$ be a dual Banach algebra, then it is not necessary that the predual $\mathcal{A}_*$ is unique [8].

For example, if $\mathcal{A}$ be an Arens regular Banach algebra, then $\mathcal{A}^{**}$ (with the first Arens product) is a dual Banach algebra with predual $\mathcal{A}^*$; if G be a locally compact topological group then measure algebra $M(G)$ is a dual Banach algebra with predual $C_0(G)$; or if E be a reflexive Banach algebra then $\mathcal{L}(E)$ is a dual Banach algebra with predual $E\widehat{\otimes}E^*$, where $\widehat{\otimes}$ is projective tensor product.

Now, let $\mathcal{A}$ be a dual Banach algebra and let X^* be a dual Banach $\mathcal{A}-$bimodule. Then $f \in X^*$ is *normal* whenever for all $a \in \mathcal{A}$, mappings $a \mapsto a \cdot f$ and $a \mapsto f \cdot a$ be *weak*$^* -$ *weak*$^*-$continuous from $\mathcal{A}$ into X^*. Also, X^* is called normal if every element $f \in X^*$ is normal [8].

The dual Banach algebra $\mathcal{A}$ is called *Connes amenable* if every $weak^*-weak^*-$ continuous derivation from $\mathcal{A}$ into each normal dual Banach $\mathcal{A}-$bimodule X^* is inner; i.e. $\mathcal{H}^1_{w^*}(\mathcal{A}, X^*) = \{0\}$. This concept was introduced by V. Runde, [8].

Definition 1.1. *A dual Banach algebra $\mathcal{A}$ is called weakly Connes amenable if every $weak^*-weak^*-$continuous derivation from $\mathcal{A}$ into normal dual Banach $\mathcal{A}-$bimodule $\mathcal{A}^*$ is inner; that is $\mathcal{H}^1_{w^*}(\mathcal{A}, \mathcal{A}^*) = \{0\}$.*

In this paper, We will study the weak Connes Amenability of the even duals of an Arens regular Banach algebra $\mathcal{A}$. We will do this work by induction.

2 Weak Connes Amenability

Let $\mathcal{A}$ be an Arens regular Banach algebra and $\mathcal{A}^{(2n)}$ be its $(2n)-$th dual for all $n \geq 0$. It is clear that $\mathcal{A}^0 = \mathcal{A}$. In this section, we will show that how weak amenability of $\mathcal{A}^{(2n)}$ and weak Connes amenability of $\mathcal{A}^{(2n+2)}$ are related together. First, we present a general law about normality of modular structures.

Lemma 2.1. *Let $\mathcal{A}$ be an Arens regular Banach algebra. Then $\mathcal{A}^*$ is a normal Banach $\mathcal{A}-$bimodule.*

Proof. Let $(a_\alpha)_\alpha$ is a net in $\mathcal{A}$ and $a' \in \mathcal{A}^*$. Then, by the Arens regularity of $\mathcal{A}$, we have

$$\begin{aligned} \langle (w^* - \lim_\alpha a_\alpha) \cdot a', b\rangle &= \langle a', b \cdot (w^* - \lim_\alpha a_\alpha)\rangle \\ &= \lim_\alpha \langle a', b \cdot a_\alpha\rangle \\ &= \lim_\alpha \langle a_\alpha \cdot a', b\rangle \\ &= \langle w^* - \lim_\alpha (a_\alpha \cdot a'), b\rangle, \end{aligned}$$

for every $b \in \mathcal{A}$. Also,

$$\begin{aligned} \langle a' \cdot (w^* - \lim_\alpha a_\alpha), b\rangle &= \langle a', w^* - \lim_\alpha a_\alpha \cdot b\rangle \\ &= \lim_\alpha \langle a', a_\alpha \cdot b\rangle \\ &= \lim_\alpha \langle a' \cdot a_\alpha, b\rangle \\ &= \langle w^* - \lim_\alpha (a' \cdot a_\alpha), b\rangle. \end{aligned}$$

Hence, mappings $a \mapsto a \cdot a'$ and $a \mapsto a' \cdot a$ are $weak^* - weak^*-$continuous from $\mathcal{A}$ into $\mathcal{A}^*$. Consequently, $\mathcal{A}^*$ is a normal Banach $\mathcal{A}-$bimodule. $\square$

Similarly, we have the following extended result.

Corollary 2.2. *Let $\mathcal{A}$ be a Banach algebra and $n \geq 0$. If $\mathcal{A}^{(2n)}$ is an Arens regular, then $\mathcal{A}^{(2n+1)}$ is a normal Banach $\mathcal{A}^{(2n)}-$bimodule.*

Now, we prove the first theorem of this paper.

Theorem 2.3. *Let $\mathcal{A}$ be an Arens regular Banach algebra such that every map from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact. If $\mathcal{A}^{**}$ is weakly Connes amenable, then $\mathcal{A}$ is a weakly Connes amenable.*

Proof. Let $D : \mathcal{A} \to \mathcal{A}^*$ is $weak^* - weak^*-$continuous derivation which $\mathcal{A}^*$ is a normal Banach $\mathcal{A}-$bimodule, by lemma 2.1. It is easily seen that $D^{**} : \mathcal{A}^{**} \to \mathcal{A}^{(3)}$ is $weak^* - weak^*-$continuous derivation by [4, theorem 2.8.59]. Since $\mathcal{A}^{**}$ is weakly Connes amenable, so D^{**} is inner. Then there is $a^{(3)} \in \mathcal{A}^{(3)}$ such that for all $a^{''} \in \mathcal{A}^{**}$

$$D^{**}(a^{''}) = a^{''} \cdot a^{(3)} - a^{(3)} \cdot a^{''}.$$

Now, suppose that $K : \mathcal{A} \to \mathcal{A}^{**}$ be a canonical map. Let $a^{'} = K^*(a^{(3)})$. Then for each $a \in \mathcal{A}$

$$D(a) = a \cdot a^{'} - a^{'} \cdot a.$$

Therefore D is inner and $\mathcal{A}$ is weakly Connes amenable. □

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra such that every map from $\mathcal{A}^{**}$ into $\mathcal{A}^{(3)}$ is weakly compact. If $\mathcal{A}^{**}$ is an Arens regular and $\mathcal{A}^{(4)}$ is weakly Connes amenable, then $\mathcal{A}^{**}$ is a weakly Connes amenable.*

Proof. Let $D : \mathcal{A}^{**} \to \mathcal{A}^{(3)}$ is $weak^* - weak^*-$continuous derivation which $\mathcal{A}^{(3)}$ is a normal Banach $\mathcal{A}^{**}-$bimodule, by corollary 2.2. It is easily seen that $D^{**} : \mathcal{A}^{(4)} \to \mathcal{A}^{(5)}$ is $weak^* - weak^*-$continuous derivation [4]. Since $\mathcal{A}^{(4)}$ is weakly Connes amenable, so D^{**} is inner. Then there is $a^{(5)} \in \mathcal{A}^{(5)}$ such that for all $a^{(4)} \in \mathcal{A}^{(4)}$

$$D^{**}(a^{(4)}) = a^{(4)} \cdot a^{(5)} - a^{(5)} \cdot a^{(4)}.$$

Now, suppose that $K : \mathcal{A}^{**} \to \mathcal{A}^{(4)}$ be a canonical map. Let $a^{(3)} = K^*(a^{(5)})$. Then for each $a^{''} \in \mathcal{A}^{**}$

$$D(a^{''}) = a^{''} \cdot a^{(3)} - a^{(3)} \cdot a^{''}.$$

Therefore D is inner and $\mathcal{A}^{**}$ is weakly Connes amenable. □

Now, by the two theorems 2.3 and 2.4, we can state the next theorem inductively.

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra and $n \geq 0$. Also, let every map from $\mathcal{A}^{(2n)}$ into $\mathcal{A}^{(2n+1)}$ is weakly compact. If $\mathcal{A}^{(2n)}$ is an Arens regular and $\mathcal{A}^{(2n+2)}$ is weakly Connes amenable, then $\mathcal{A}^{(2n)}$ is a weakly Connes amenable.*

Proof. Proof is similar to proofs of two last theorems. □

In the next, we can see that weak amenability of an arbitrary Banach algebra implies weak Connes amenability of its even duals.

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^{**}$ is an Arens regular. If $\mathcal{A}$ is weakly amenable, then $\mathcal{A}^{**}$ is weakly Connes amenable.*

Proof. Let $D : \mathcal{A}^{**} \to \mathcal{A}^{(3)}$ be a $weak^* - weak^*-$continuous derivation which by corollary 2.2, $\mathcal{A}^{(3)}$ is a normal Banach $\mathcal{A}^{**}-$bimodule. For each $a \in \mathcal{A}$, we define $\tilde{D} : \mathcal{A} \to \mathcal{A}^*$ by

$$\tilde{D}(a) = D(\hat{a})|_{\mathcal{A}}.$$

For every $a, b \in \mathcal{A}$, we have

$$\tilde{D}(ab) = D(\widehat{ab}) = D(\hat{a}\Box\hat{b}) = aD(\hat{b}) + D(\hat{a})b = a\tilde{D}(b) + \tilde{D}(a)b.$$

Thus $\tilde{D}$ is a continuous derivation from $\mathcal{A}$ into $\mathcal{A}^*$. By hypothesis, since $\mathcal{A}$ is weakly amenable, thus $\tilde{D}$ is inner. Then

$$D(\hat{a}) = \tilde{D}(a) = a \cdot a^{(3)}|_{\mathcal{A}} - a^{(3)}|_{\mathcal{A}} \cdot a = \hat{a} \cdot a^{(3)}|_{\mathcal{A}} - a^{(3)}|_{\mathcal{A}} \cdot \hat{a}\ .$$

Considering natural mapping $K : \mathcal{A}^* \to \mathcal{A}^{(3)}$, there is $b^{(3)} \in \mathcal{A}^{(3)}$ such that $K(a^{(3)}|_{\mathcal{A}}) = b^{(3)}$. So

$$D(\hat{a}) = \hat{a} \cdot b^{(3)} - b^{(3)} \cdot \hat{a}.$$

Therefore D is inner. Consequently, $\mathcal{A}^{**}$ is weakly Connes amenable. □

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra. Also, suppose that $\mathcal{A}^{(4)}$ is an Arens regular. If $\mathcal{A}^{**}$ is weakly amenable, then $\mathcal{A}^{(4)}$ is weakly Connes amenable.*

Proof. Let $D : \mathcal{A}^{(4)} \to \mathcal{A}^{(5)}$ be a $weak^* - weak^*-$continuous derivation which by corollary 2.2, $\mathcal{A}^{(5)}$ is a normal Banach $\mathcal{A}^{(4)}-$bimodule. For each $a^{''} \in \mathcal{A}^{**}$, we define $\tilde{D} : \mathcal{A}^{**} \to \mathcal{A}^{(3)}$ by follow

$$\tilde{D}(a^{''}) = D(\widehat{a^{''}})|_{\mathcal{A}^{**}}.$$

For every $a^{''}, b^{''} \in \mathcal{A}^{**}$, we have

$$\tilde{D}(a^{''}\Box b^{''}) = D(\widehat{a^{''}b^{''}}) = D(\widehat{a^{''}}\Box\widehat{b^{''}}) = a^{''}D(\widehat{b^{''}}) + D(\widehat{a^{''}})b^{''} = a^{''}\tilde{D}(b^{''}) + \tilde{D}(a^{''})b^{''}.$$

Thus $\tilde{D}$ is a continuous derivation from $\mathcal{A}^{**}$ into $\mathcal{A}^{(3)}$. By hypothesis, since $\mathcal{A}^{**}$ is weakly amenable, thus $\tilde{D}$ is inner. Then

$$D(\widehat{a''}) = \tilde{D}(a'') = a'' \cdot a^{(5)}|_{\mathcal{A}^{**}} - a^{(5)}|_{\mathcal{A}^{**}} \cdot a'' = \widehat{a''} \cdot a^{(5)}|_{\mathcal{A}^{**}} - a^{(5)}|_{\mathcal{A}^{**}} \cdot \widehat{a''} .$$

Considering natural mapping $K : \mathcal{A}^{(3)} \to \mathcal{A}^{(5)}$, there is $b^{(5)} \in \mathcal{A}^{(5)}$ such that $K(a^{(5)}|_{\mathcal{A}^{**}}) = b^{(5)}$. So

$$D(\widehat{a''}) = \widehat{a''} \cdot b^{(5)} - b^{(5)} \cdot \widehat{a''}.$$

Therefore D is inner. Consequently, $\mathcal{A}^{(4)}$ is weakly Connes amenable. □

According to the above theorems, we have the following general theorem. So, we omit the proof.

Theorem 2.8. *Let $\mathcal{A}$ be a Banach algebra. Also, suppose that $\mathcal{A}^{(2n+2)}$ is an Arens regular. If $\mathcal{A}^{(2n)}$ is weakly amenable, then $\mathcal{A}^{(2n+2)}$ is weakly Connes amenable.*

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^*$ is a Banach $\mathcal{A}-$bimodule. We say that $\mathcal{A}^*$ *factors* on the left (right) if $\mathcal{A}^* = \mathcal{A}^* \cdot \mathcal{A}$ ($\mathcal{A}^* = \mathcal{A} \cdot \mathcal{A}^*$) and $\mathcal{A}^*$ factors if both equalities $\mathcal{A} \cdot \mathcal{A}^* = \mathcal{A}^* = \mathcal{A}^* \cdot \mathcal{A}$ hold [1].
In general, If $\mathcal{A}^{(2n+1)}$ be a Banach $\mathcal{A}^{(2n)}-$bimodule, then we say that $\mathcal{A}^{(2n+1)}$ *factors* if $\mathcal{A}^{(2n)} \cdot \mathcal{A}^{(2n+1)} = \mathcal{A}^{(2n+1)} = \mathcal{A}^{(2n+1)} \cdot \mathcal{A}^{(2n)}$. Now, by applying this property, we show that even duals of $\mathcal{A}$ is weakly Connes amenable, under some conditions.

Theorem 2.9. *Let $\mathcal{A}$ be weakly amenable Banach algebra. If $\mathcal{A}^{**}$ is the left ideal in $\mathcal{A}^{(4)}$ and $\mathcal{A}^{(3)}$ factors on the right, then $\mathcal{A}^{**}$ is weakly Connes amenable.*

Proof. Consider $\mathcal{A}^{(3)}$ as a Banach $\mathcal{A}^{**}-$bimodule. Let $b^{(4)} \in \mathcal{A}^{(4)}$. Also, let $(a''_\alpha)_\alpha$ be a net in $\mathcal{A}^{**}$ such that $\widehat{a''_\alpha} \xrightarrow{w^*} \widehat{a''}$ in $\mathcal{A}^{(4)}$. We show that $b^{(4)}\widehat{a''_\alpha} \xrightarrow{w^*} b^{(4)}\widehat{a''}$ in $\mathcal{A}^{(4)}$. Let $a^{(3)} \in \mathcal{A}^{(3)}$. Then, since $\mathcal{A}^{(3)}$ factors on the right, there are $b'' \in \mathcal{A}^{**}$ and $b^{(3)} \in \mathcal{A}^{(3)}$ such that $a^{(3)} = b''b^{(3)}$. Since $\mathcal{A}^{**}$ is the left ideal in $\mathcal{A}^{(4)}$, thus we have $\widehat{a''_\alpha b''} \xrightarrow{w^*} \widehat{a''}b''$ in $\mathcal{A}^{(4)}$ if and only if $\widehat{a''_\alpha b''} \xrightarrow{w^*} \widehat{a''}b''$ in $\mathcal{A}^{**}$. Also. Since $b'' \mapsto b''b^{(3)}; \mathcal{A}^{**} \to \mathcal{A}^{(3)}$ is $weak^* - weak^*-$continuous mapping, it is follows that $\widehat{a''_\alpha}b''b^{(3)} \xrightarrow{w^*} \widehat{a''}b''b^{(3)}$ in $\mathcal{A}^{(3)}$. Therefore

$$\begin{aligned}
\lim_\alpha \langle b^{(4)}\widehat{a''_\alpha}, a^{(3)}\rangle &= \lim_\alpha \langle b^{(4)}\widehat{a''_\alpha}, b''b^{(3)}\rangle \\
&= \lim_\alpha \langle b^{(4)}, \widehat{a''_\alpha}b''b^{(3)}\rangle \\
&= \langle b^{(4)}, \widehat{a''}b''b^{(3)}\rangle \\
&= \langle b^{(4)}\widehat{a''}, b''b^{(3)}\rangle \\
&= \langle b^{(4)}\widehat{a''}, a^{(3)}\rangle.
\end{aligned}$$

This shows that $b^{(4)} \in Z_{\mathcal{A}^{**}}(\mathcal{A}^{(4)})$. Hence $\mathcal{A}^{**}$ is Arens regular. By the hypothesis, since $\mathcal{A}$ is weakly amenable, so imediately theorem 2.6 implies that $\mathcal{A}^{**}$ is weakly Connes amenable. $\square$

Theorem 2.10. *Let $\mathcal{A}$ be weakly amenable Banach algebra. If $\mathcal{A}^{**}$ is the right ideal in $\mathcal{A}^{(4)}$ and $\mathcal{A}^{(3)}$ factors on the left, then $\mathcal{A}^{**}$ is weakly Connes amenable.*

Proof. Consider $\mathcal{A}^{(3)}$ as a Banach $\mathcal{A}^{**}-$bimodule. Let $b^{(4)} \in \mathcal{A}^{(4)}$. Also, let $(a''_\alpha)_\alpha$ be a net in $\mathcal{A}^{**}$ such that $\widehat{a''_\alpha} \xrightarrow{w^*} \widehat{a''}$ in $\mathcal{A}^{(4)}$. We show that $b^{(4)}\widehat{a''_\alpha} \xrightarrow{w^*} b^{(4)}\widehat{a''}$ in $\mathcal{A}^{(4)}$. Let $a^{(3)} \in \mathcal{A}^{(3)}$. Then, since $\mathcal{A}^{(3)}$ factors on the left, there are $b'' \in \mathcal{A}^{**}$ and $b^{(3)} \in \mathcal{A}^{(3)}$ such that $a^{(3)} = b^{(3)}b''$. Since $\mathcal{A}^{**}$ is the right ideal in $\mathcal{A}^{(4)}$, thus we have $b''\widehat{a''_\alpha} \xrightarrow{w^*} b''\widehat{a''}$ in $\mathcal{A}^{(4)}$ if and only if $b''\widehat{a''_\alpha} \xrightarrow{w^*} b''\widehat{a''}$ in $\mathcal{A}^{**}$. Also. Since $b'' \mapsto b^{(3)}b''; \mathcal{A}^{**} \to \mathcal{A}^{(3)}$ is $weak^* - weak^*-$continuous mapping, it is follows that $\widehat{a''_\alpha}b^{(3)}b'' \xrightarrow{w^*} \widehat{a''}b^{(3)}b''$ in $\mathcal{A}^{(3)}$. Therefore

$$\begin{aligned}
\lim_\alpha \langle b^{(4)}\widehat{a''_\alpha}, a^{(3)}\rangle &= \lim_\alpha \langle b^{(4)}\widehat{a''_\alpha}, b^{(3)}b''\rangle \\
&= \lim_\alpha \langle b^{(4)}, \widehat{a''_\alpha}b^{(3)}b''\rangle \\
&= \langle b^{(4)}, \widehat{a''}b^{(3)}b''\rangle \\
&= \langle b^{(4)}\widehat{a''}, b^{(3)}b''\rangle \\
&= \langle b^{(4)}\widehat{a''}, a^{(3)}\rangle.
\end{aligned}$$

This shows that $b^{(4)} \in Z_{\mathcal{A}^{**}}(\mathcal{A}^{(4)})$. Hence $\mathcal{A}^{**}$ is Arens regular. By the hypothesis, since $\mathcal{A}$ is weakly amenable, so imediately theorem 2.6 implies that $\mathcal{A}^{**}$ is weakly Connes amenable. $\square$

Corollary 2.11. *Let $\mathcal{A}$ is weakly amenable Banach algebra. If $\mathcal{A}^{**}$ is a two-sided ideal in $\mathcal{A}^{(4)}$ and $\mathcal{A}^{(3)}$ factors, then $\mathcal{A}^{**}$ is weakly Connes amenable.*

Proof. It is easily followed from Theorems 2.9 and 2.10 . $\square$

Note that condition of factorization of Banach $\mathcal{A}^{**}-$bimodule $\mathcal{A}^{(3)}$ is necessary in above results. [7]
Finally, by according to corollary 2.11 we can bring the following theorem in general. Of course, note that n is a non-negative number.

Theorem 2.12. *Let $\mathcal{A}^{(2n)}$ be weakly amenable Banach algebra. If $\mathcal{A}^{(n+2)}$ is a two-sided ideal in $\mathcal{A}^{(n+4)}$ and $\mathcal{A}^{(n+3)}$ factors, then $\mathcal{A}^{(2n+2)}$ is weakly Connes amenable.*

Proof. It is easily followed from the above theorems. $\square$

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Received: March 16, 2013