On a Class of Nowhere Commutative Semigroup Rings

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Abstract

For a finite algebraic structure A the commutativity degree of A , denoted by $P_r(A)$, is the probability that two elements of A commute. Since 1973, $P_r(A)$ studied for finite groups showing that $P_r(A) \leq \frac{5}{8}$. The study of "almost commutative" semigroups in 2011 showed that for some finite semigroup A , $P_r(A) > \frac{5}{8}$ and even $P_r(A)$ may be arbitrarily close to 1. Since $0 \leq P_r(A) \leq 1$ then looking for algebraic structures A such that $P_r(A) \rightarrow 0$, is of interest, for, the centralizer of every element of such A should be a singleton. In this paper for every integer $n \geq 2$ and every prime $p \geq 3$ we give an infinite class of finite semigroup rings $A_{n,p} = Z_p(P_n)$ of order p^n where, P_n is a non-commutative semigroup and show that $P_r(A_{n,p}) \rightarrow 0$, for sufficiently large values of n and p . We name such semigroup rings as "extremely non-commutative" semigroup rings.

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1 Introduction

For a given finite algebraic structure A (group, ring, semigroup or semigroup ring), the commutativity degree of A is defined to be

$$P_r(A) = \frac{|\{(x, y) \in A^2 | xy = yx\}|}{|A^2|}.$$

The study of this probability for finite groups started in 1973 by Gustafson [5] by showing that $P_r(A) \leq \frac{5}{8}$ for every non-abelian group A . The commutativity for finite rings defined and studied by McHale in 1976 (see [8]). The article [7] studied the case when $P_r(A) \geq \frac{1}{2}$ in 1995 for a finite group A . In the article [4] the case $P_r(A) < \frac{1}{2}$ and specially when $\frac{1}{4} < P_r(A) < \frac{1}{2}$ studied in 2008 by giving certain infinite classes of finite groups. One of the three studied classes in this article satisfies the condition $P_r(A) \rightarrow 0$ for the sufficiently large group A .

The study of this probability for semigroups started in 2011 by Ahmadi and Delir [1] where the authors proved that for some semigroup A , $P_r(A) > \frac{5}{8}$. The natural question that "for which algebraic structure A (except the groups), $P_r(A) \rightarrow 0$ " may be posed here and in what follows we construct a class of two parametric semigroup rings $A_{n,p}$ of order p^n , for every integer $n \geq 2$ and odd prime p , to show that $P_r(A_{n,p}) \rightarrow 0$. We name such an algebraic structure the "extremely non-commutative" algebraic structure.

Our notation is fairly standard and follows [2,3,6,9] on the presentations of groups and semigroups. We may also recall the definition of a group ring $R[S]$ for a commutative ring R and an arbitrary finite semigroup S , consisting of all linear combinations $\sum_{g \in S} \alpha(g)g$, where $\alpha(g) \in R$ and $\alpha(g) = 0$ except for a finite number of coefficients. Of course the sum and multiplication of elements in $R[S]$ are defined as usual by:

$$\begin{aligned} \left(\sum_{g \in S} \alpha(g)g\right) + \left(\sum_{g \in S} \beta(g)g\right) &= \sum_{g \in S} (\alpha(g) + \beta(g))g, \\ \left(\sum_{g \in S} \alpha(g)g\right) \left(\sum_{h \in S} \beta(h)h\right) &= \sum_{g,h \in S} (\alpha(g)\beta(h))gh. \end{aligned}$$

Let $n \geq 2$ be an integer and p an odd prime. Consider the presentation

$$\pi = \langle a_1, a_2, \dots, a_n | a_i^2 = a_i, a_{i+1}a_i = a_{i+1}, a_i a_{i+1} = a_i, (1 \leq i \leq n) \rangle$$

where, the indices are reduced modulo n . The semigroup defined by π (indeed, $Sg(\pi)$, as a usual notation of the finitely presented structures), will be denoted

by P_n . Finally, by denoting the semigroup ring $Z_p[P_n]$ by $A_{n,p}$ we give our main results as:

Proposition 1.1 *The semigroup ring $A_{n,p}$ is finite of order p^n . Moreover, for every integer $n \geq 2$, the inequalities $\frac{1}{2p^2} \leq P_r(A_{n,p}) \leq \frac{1}{p^2}$ hold, for every odd prime p .*

Corollary 1.2 *The semigroup ring $A_{n,p}$ is extremely non-commutative.*

2 Preliminary results

Every element of $A_{n,p}$ is in the form $x = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n$. For a given values of $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ define the set $A(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ as follows:

$$A(\alpha_1, \alpha_2, \dots, \alpha_{n-1}) = \{(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_i) | \alpha_i \in Z_p\}.$$

Then, as a preliminary result we get:

Lemma 2.1 *For every $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$, the set $A(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ contains exactly one n -tuple $(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \beta)$ satisfying $\beta + \sum_{i=1}^{n-1} \alpha_i \equiv 0 \pmod{p}$, where, $\beta \in Z_p$.*

Proof For two different elements $(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \beta)$ and $(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \gamma)$ of $A(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ we easily get $\beta - \gamma \equiv 0 \pmod{p}$ which gives the contradiction $\beta = \gamma$.

Lemma 2.2 *The semigroup ring $A_{n,p}$ contains exactly p^{n-1} elements satisfying $\sum_{i=1}^n \alpha_i \equiv 0 \pmod{p}$.*

Proof There are exactly p^{n-1} sets as $A(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$. On the other hand, there are a one-one correspondence between the elements of the set $A_{n,p}$ and the set

$$\bigcup_{(\alpha_1, \alpha_2, \dots, \alpha_{n-1}) \in Z_p^{n-1}} A(\alpha_1, \alpha_2, \dots, \alpha_{n-1}).$$

So, the result follows at once by using the Lemma 2.1.

Lemma 2.3 *Two different elements $x = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n$ and $y = \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_n a_n$ of $A_{n,p}$ commute if and only if $\sum_{i=1}^n \alpha_i \equiv 0 \pmod{p}$ and $\sum_{i=1}^n \beta_i \equiv 0 \pmod{p}$.*

Proof Considering the relators of the semigroup P_n we get:

$$\begin{aligned} xy = yx &\Leftrightarrow \sum_{i=1}^n \alpha_i (\beta_1 + \beta_2 + \dots + \beta_n) a_i = \sum_{i=1}^n \beta_i (\alpha_1 + \alpha_2 + \dots + \alpha_n) a_i \\ xy = yx &\Leftrightarrow (\beta_1 + \beta_2 + \dots + \beta_n) x = (\alpha_1 + \alpha_2 + \dots + \alpha_n) y. \end{aligned}$$

Consequently, the last equation holds if and only if $\sum_{i=1}^n \alpha_i \equiv 0 \pmod{p}$ and $\sum_{i=1}^n \beta_i \equiv 0 \pmod{p}$, for, $x \neq y$.

3 Results and Discussion

By using the results of Section 2 we give here the proofs of Proposition 1.1 and corollary 1.2.

Proof of Proposition 1.1 The first part is obvious because of the representation of the elements of $A_{n,p}$. For the second part we have to compute $P_r(A_{n,p})$, for a given n and p . By the definition we get that

$$P_r(A_{n,p}) = \frac{|A_{n,p}| + |\{(x, y) \in A_{n,p}^2 | x \neq y, xy = yx\}|}{|A_{n,p}^2|}.$$

Considering the Lemma 2.3 gives us the cardinal of the set

$$\{(x, y) \in A_{n,p}^2 | x \neq y, xy = yx\}$$

as $\frac{p^{n-1}(p^{n-1}-1)}{2}$. So,

$$P_r(A_{n,p}) = \frac{2p^n + p^{n-1}(p^{n-1} - 1)}{2p^{2n}}.$$

Or simply,

$$P_r(A_{n,p}) = \frac{2p + p^{n-1} - 1}{2p^{n+1}}.$$

Consequently, for a given value of n , we may check the inequalities

$$\frac{1}{2p^2} \leq \frac{2p + p^{n-1} - 1}{2p^{n+1}} \leq \frac{1}{p^2},$$

for every prime $p \geq 3$.

Proof of Corollary 1.2 Since $\lim_{n \rightarrow \infty} \frac{2p + p^{n-1} - 1}{2p^{n+1}} = \frac{1}{p^2}$ then, for the sufficiently large values of p , $P_r(A_{n,p}) \rightarrow 0$ showing that $A_{n,p}$ is an extremely non-commutative semigroup ring.

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