On Caccioppoli-Kannan Type Fixed Point Principle in Generalized Metric Spaces

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Abstract

In this Paper we prove a Caccioppoli type Kannan's fixed point theorem in g.m.s. that includes Kannan's fixed point theorem in g.m.s. of P.K. Saha (Theorem 2.1.[9]) and Dorel Mihet (Theorem 2.1.[8]).

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1. Introduction

Recently Branciari [1] introduced the concept of a generalized metric space such that any metric space is a generalized metric space but the converse is not true and he presented the well known Banach's fixed point theorem in such a space. It is now reasonable to consider if some other well known fixed point theorem, not necessarily involving contraction type of mapping, can be obtained in generalized metric spaces. This is supported by the recent works of P.K. Saha and R. Tiwary in [9], Dorel Mihet [8]. Further it is known that in metric spaces Banach's fixed point theorem follows from Caccioppoli fixed point theorem [7]. H. Dasgupta [4] introduced and proved a Caccioppoli-type fixed point theorem in metric spaces from which Kannan's fixed point theorem in metric space follows. In this paper we extend Caccioppoli-type fixed point theorem of H. Dasgupta with some restrictions in generalized metric spaces of which Kannan's fixed point theorem in g.m.s. ([8], [9]) and hence in metric space [5] follows.

2. Preliminaries

Definition 2.1([1]). Let X be a non-void set and $d : X \times X \rightarrow R^+$ (non-negative reals) be a mapping such that for $x, y \in X$ and for all distinct points $\xi, \eta \in X$, each of them different from x and y , one has

- (i) $d(x, y) = 0$ if and only if $x = y$
- (ii) $d(x, y) = d(y, x)$
- (iii) $d(x, y) \leq d(x, \xi) + d(\xi, \eta) + d(\eta, y)$,

then (X, d) is called a generalized metric space, (shortly g.m.s.).

Definition 2.2([1]). Let (X, d) be a g.m.s. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists a positive integer k such that $d(x_n, x_{n+m}) < \epsilon$ whenever $n \geq k$ and $m = 1, 2, \dots$. A g.m.s. (X, d) is called complete if every Cauchy sequence in X is convergent in X .

Definition 2.3(cf. [2], [3], [6]). Let $T : X \rightarrow X$ be a mapping where X is a g.m.s. For each $x \in X$, let $O(x, \infty) = \{x, Tx, T^2x, T^3x, \dots\}$. Then a g.m.s. X is said to be T -orbitally complete if and only if every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in X$ converges in X .

Remark 2.4. Every complete g.m.s. X (where a mapping $T : X \rightarrow X$ is defined) is T -orbitally complete but a T -orbitally complete g.m.s. need not be complete ([6], Note, p 590).

Theorem 2.5(Kannan fixed point theorem in metric space [5]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping such that $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$ where $0 < \beta < \frac{1}{2}$ and $x, y \in X$. Then T has a unique fixed point in X .

Theorem 2.6(Kannan fixed point theorem in g.m.s. [8], [9]). Let (X, d) be a g.m.s. and let $T : X \rightarrow X$ be a mapping such that

$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$ where $0 < \beta < \frac{1}{2}$ and $x, y \in X$. If X is T -orbitally complete, then T has a unique fixed point in X .

Theorem 2.7(Caccioppoli type fixed point theorem [7]). Let (X, ρ) be a complete metric space and let T be a mapping of X into itself. Suppose that for each positive integer n ,

$$\rho(T^n x, T^n y) \leq a_n[\rho(x, Tx) + \rho(y, Ty)],$$

for all $x, y \in X$ where $a_n(> 0)$ is independent of x, y and $0 < a_1 < 1$. Then if the series $\sum_{n=1}^{\infty} a_n$ is convergent, then T has a unique fixed point in X .

3. New Results

In this section we prove the following theorem and show that Kannan fixed point theorem in g.m.s. ([8], [9]) follows from this.

Theorem 3.1. Let (X, d) be a g.m.s. and let $T : X \rightarrow X$ be a mapping such that

$$d(T^n x, T^n y) \leq a_n[d(x, Tx) + d(y, Ty)] \quad (A),$$

for all $x, y \in X$ where $a_n(> 0)$ is independent of x, y and $0 < a_1 < 1$ and $a_1 + a_n < 1$ for $n \geq 2$. Then if X is T -orbitally complete with respect to the generalized metric d and the series $\sum_{n=1}^{\infty} a_n$ is convergent, then T has a unique fixed point in X .

Proof. We first prove that T has a fixed point. Let $x \in X$. We consider the sequence $\{x_n\}$ of iterates $x_n = T^n x, n = 1, 2, 3, \dots$

We divide the proof into two cases:

Case I. First we assumed that $T^p x \neq T^r x$ for $p, r \in N, p \neq r, N$ being the set of natural numbers. Then for $n \in N$ we have

$$d(T^n x, T^{n+1} x) \leq a_n[d(x, Tx) + d(Tx, T^2 x)].$$

Again $d(Tx, T^2 x) \leq a_1[d(x, Tx) + d(Tx, T^2 x)]$, which gives

$$d(Tx, T^2 x) \leq \frac{a_1}{1 - a_1} d(x, Tx) \quad (1)$$

Hence

$$d(T^n x, T^{n+1} x) \leq a_n[1 + (\frac{a_1}{1 - a_1})]d(x, Tx) \quad (2)$$

Similarly,

$$d(T^n x, T^{n+2} x) \leq a_n [1 + (\frac{a_1}{1-a_1})^2] d(x, Tx) \quad (3)$$

As $\sum_{n=1}^{\infty} a_n$ is convergent, from (2) and (3) it follows that

$$d(x_n, x_{n+m}) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for each } m = 1, 2 \quad (4)$$

Now if $m > 2$, then using the fact that $T^p x \neq T^r x$ for $p, r \in N$ and $p \neq r$ we can write

$$\begin{aligned} d(x_n, x_{n+m}) &= d(T^n x, T^{n+m} x) \\ &= \leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \dots + d(T^{n+m-1} x, T^{n+m} x) \end{aligned}$$

So using (2), we get

$$d(T^n x, T^{n+m} x) \leq \frac{d(x, Tx)}{1-a_1} [a_n + a_{n+1} + \dots + a_{n+m-1}]$$

Now since $\sum_{n=1}^{\infty} a_n$ is convergent so each of $a_n, a_{n+1}, \dots, a_{n+m-1}$ tends to 0 as $n \rightarrow \infty$ for each $m > 2$ and so for each $\epsilon (> 0)$ there is a positive integer n_0 such that

$$d(T^n x, T^{n+m} x) < \epsilon. \text{ for } n \geq n_0 \text{ and for each } m > 2,$$

where $\epsilon (> 0)$ is arbitrary.

So

$$d(x_n, x_{n+m}) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for each } m > 2 \quad (5)$$

(4) and (5) imply that $\{x_n\}$ is a Cauchy sequence in (X, d) . As X is T-orbitally complete, there exists a $u \in X$ such that $\lim_{n \rightarrow \infty} T^n x = u$.

We now show that $Tu = u$. We note that $T^n x \neq u, Tu$ for any $n \in N$, for if $T^k x = u$ or $T^k x = Tu$ for some $k \in N$, then we can easily verify that $\{T^n u\}$ is a sequence in (X, d) with the following properties:

- (i) $\lim_{n \rightarrow \infty} T^n u = u$
- (ii) $T^n u \neq u$ for any $n \in N$
- (iii) $T^p u \neq T^r u$ for $p, r \in N, p \neq r$.

So

$$\begin{aligned} d(T^{n+1} u, Tu) &\leq a_1 [d(T^n u, T^{n+1} u) + d(u, Tu)] \\ &\leq a_1 [1 + \frac{a_n}{1-a_1}] d(u, Tu), \text{ by (2).} \end{aligned}$$

Letting $n \rightarrow \infty$ we then obtain $d(u, Tu) \leq a_1(u, Tu)$. which implies $u = Tu$. This contradicts the property (ii) for $n = 1$. Hence the possibility $T^k x = u$ or $T^k x = Tu$ for some $k \in N$ can not stand.

Now by (iii) of Definition 2.1 and condition (A) of Theorem 3.1, we obtain

$$\begin{aligned} d(u, Tu) &\leq d(u, T^n x) + d(T^n x, T^{n+1} x) + d(T^{n+1} x, Tu) \\ &\leq d(u, T^n x) + d(T^n x, T^{n+1} x) + a_1[d(T^n x, T^{n+1} x) + d(u, Tu)] \\ &\leq \frac{1}{1-a_1} [d(u, T^n x) + (1+a_1) \frac{a_n}{1-a_1} d(x, Tx)], \text{ by (2).} \end{aligned}$$

Letting $n \rightarrow \infty$ we get $u = Tu$. Hence T has a fixed point in X .

Case II. In this case we assume that $T^p x = T^r x$ for some $p, r \in N, p \neq r$. Let $p > r$. Then $T^{p-r}(T^r x) = T^r x$, i.e., $T^k y = y$, where $k = p - r \geq 1$, $y = T^r x \in X$.

Now for $k = 1$, $Ty = y$. For $k > 1$, $d(y, Ty) = d(T^k y, T^{k+1} y)$

$$\leq \frac{a_k}{1-a_1} d(y, Ty), \text{ or, } (1 - \frac{a_k}{1-a_1}) d(y, Ty) \leq 0.$$

Since $k > 1, a_1 + a_k < 1$ and so $d(y, Ty) = 0$, i.e., $Ty = y$ and so T has a fixed point in X .

For uniqueness of the fixed point let $Tu = u$ and $Tv = v; u, v \in X$.

So, $d(u, v) = d(Tu, Tv) \leq a_1[d(u, Tu) + d(v, Tv)] = 0$. Hence $u = v$.

This proves the theorem.

Remark 3.1. We show that Kannan's fixed point theorem in generalized metric space (Theorem 2.6) and in metric space (Theorem 2.5) follow from our Theorem 3.1.

Proof of Theorem 2.6. $d(Tx, Ty) \leq (\frac{\beta}{\alpha})\alpha[d(x, Tx) + d(y, Ty)]$, where

$$0 < \alpha = \frac{\beta}{1-\beta} < 1.$$

Using method of induction we easily prove that

$$d(T^n x, T^n y) \leq (\frac{\beta}{\alpha})\alpha^n [d(x, Tx) + d(y, Ty)] = a_n [d(x, Tx) + d(y, Ty)], \text{ where}$$

$a_n = (\frac{\beta}{\alpha})\alpha^n < \beta < \frac{1}{2}$ as $0 < \alpha < 1$, which implies that all the conditions of Theorem 3.1 are satisfied and hence by Theorem 3.1, T has a unique fixed point in X .

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