Geometric Odd Extension to $\Pi TM_0$

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Abstract

We introduce a natural process for (odd) extensions of ordinary fields from an ordinary manifold $M_0$ to the whole super manifold $\Pi TM_0$ associated to the tangent bundle $TM_0$. Super-connection and super-curvature on the super manifold $\Pi TM_0$ extending the ordinary connection and curvature of $M_0$ are described and the relation with the notion of super-geodesic is also discussed.

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1 Introduction

Differential geometry on super-manifolds has been developed in the framework of super-gravity and super-Yang-Mills theories. In super-gravity theories one starts by an appropriate set of non-zero torsion constraints and super-connection and super-curvature tensors are described accordingly [3]. Super-Yang-Mills theories are based on putting some constraints on the curvature of a connection in such a way that it becomes flat along some odd distributions [4]. Both of these theories are developed on the super-manifold associated to the spin bundle of a spin manifold in restricted appropriate dimensions.
Chiral fields as the main ingredient of these theories are constructed through extensions of ordinary fields on an ordinary spin manifold $M_0$ to the whole super-manifold $M$ associated to the spin bundle $S$ on $M_0$. The odd extension is described by some odd differential equations provided by integrable foliations obtained from representations of super-Poincaré group on the super manifold $M$. This special foliation associated to the super-Poincaré group seems to restrict all the above theories to the special super-manifold associated to the spin bundle. Although it seems that the geometry behind these constructions can be reproduced for general super-manifolds. In the present note we would like to generate a natural extension process for super-manifold associated to the tangent bundle of an ordinary manifold. The approach described here seems to be simpler than what is applied in super-gravity and super-Yang Mills theories. However it can be considered as a first extension process which is, as can be seen, not only very natural, but also leads to an extension of the geometry of $M_0$ to the supermanifold $\Pi TM_0$. As an application we will discuss the relation between the induced super-geometry on $\Pi TM_0$ and the notion of super-geodesics in [1] [2].

2 Some preliminary notations

We first recall that if $M_0$ is a smooth manifold of dimension $m$ the super manifold $M := \Pi TM_0$ associated to the tangent bundle of $M_0$ is defined as a ringed space whose structure sheaf $\mathcal{O}_M$ consists of the sheaf of differential forms on $M_0$. So by definition, for an open subset $U \subset M_0$, we have $\mathcal{O}_M(U) = \Gamma(\Lambda^*(T^*M_0))|_U$. Now if $\mathcal{T}$ denotes the tangent super-bundle of $M$ then we claim that there exists a natural decomposition like

$$\mathcal{T} = \mathcal{T}^{ev} \oplus \mathcal{T}^{odd} \simeq (TM_0 \oplus TM_0) \hat{\otimes} \mathcal{O}_M$$

as a module over $\mathcal{O}_M$. To see this identification take a vector $\xi \in T_xM$ at some point $x \in M$. The even derivation $\xi : \mathcal{O}_x \to \mathbb{R}$ associated to this vector acts on an $\alpha \in \Lambda^*(T^*M)(U)$, defined in some neighborhood $U$ of $x$, as:

$$\xi.\alpha|_{M_0} := d(\alpha_0)(\xi)$$  \hfill (1)

where $\alpha$ is decomposed as $\alpha = \alpha^0 + \alpha^+$ into zero degree component $\alpha_0$, which is a real function on $U$ and the components of positive degree $\alpha^+$. The action of $\xi \in TM_0$ as an odd vector on $\alpha$ is nothing but:

$$\xi(\alpha) := i_{\nu} \alpha$$  \hfill (2)
If $\xi \in T_x M_0$ we will use the notation $\tilde{\xi}$ when the vector $\xi$ is considering to be an element of $T^{odd}$.

Assume that the set $\mathcal{B} = \{\xi_i | i = 1, \ldots, m\}$ constitute a basis for $T_x M_0$ and let $T^+_s$ and $T^-_s$ denote the subspaces of $T$ generated by $\mathcal{B}^+_s = \{\xi_i + \tilde{\xi}_i | i = 1, \ldots, m\}$ and $\mathcal{B}^-_s = \{\xi_i - \tilde{\xi}_i | i = 1, \ldots, m\}$, respectively. Thus we get the following decomposition:

$$ T = T^+_s \oplus T^-_s $$

It is clear that this decomposition does not depend on the choice of the basis $\mathcal{B}$.

Set $\partial^\pm_{\xi_i} := \xi_i \pm \tilde{\xi}_i$, for $i = 1, \ldots, m$ and let $\{(\partial^\pm_{\xi_i})^* | i = 1, \ldots, m\}$ denote its dual basis. So one can write $$(\partial^\pm_{\xi_i})^* = \frac{1}{2}((\xi_i)^* \mp (\tilde{\xi}_i)^*)$$ where $\{(\xi_i)^* | i = 1, \ldots, m\}$ is the dual basis of $\mathcal{B}$. Define the operators $d^\pm_s$ by:

$$ d^\pm_s = \sum_i \partial^\pm_{\xi_i} (\partial^\pm_{\xi_i})^* $$

these operators act on $\hat{\Omega}^*(\Pi T M_0)$ consisting of the space of $\mathcal{H}^\infty$ differential forms on $\Pi T M_0$ and the de-Rham operator $d$ can be decomposed as:

$$ d = d^+_s + d^-_s $$

### 3 Natural extension to $\Pi T M_0$

Let $M_0$ and $N_0$ be two closed smooth manifolds of dimensions $m$ and $n$, respectively, and let $g : M_0 \to N_0$ be a smooth function. Then the following lemma holds:

**Lemma 1** There exists a natural extension $\tilde{g}$:

$$ of g$ acting by pullback on the sheaf of sections of the supermanifold $\Pi T N_0$. 
Now we can ask the main question whose answer suggests a natural method for the extension of functions and differential forms from $M_0$ to $\Pi TM_0$. Let $f : N_0 \to \mathbb{R}$ be a smooth function and let $y \in \mathbb{R}$ be a regular value of $f$. We denote by $M_0 = f^{-1}\{y\}$ the level set of the function $f$ associated to $y$. Let $j : M_0 \hookrightarrow N_0$ be the inclusion map of $M_0$ into $N_0$. According to the above lemma $\tilde{j} : \Pi TM_0 \hookrightarrow \Pi TN_0$ provides an embedding of $\Pi TM_0$ into $\Pi TN_0$. Here is the question: is there any $\mathcal{H}^\infty$ function $F \in \Gamma(N, \Lambda^*(T^*N_0))$ extending $f$ to the whole super-manifolds $\Pi TN_0$ whose level set $F^{-1}(\{y\})$ coincides with the sub-manifold $\tilde{j}(\Pi TM_0)$?

In order to answer the above question we need the following definition:

**Definition 1** For a smooth real function $f \in C^\infty(N_0)$ the super symmetric extensions $S^\pm(f) \in \mathcal{H}^\infty(\Pi TN_0)$ are defined to be equal to $S^\pm(f) = f \pm \tilde{df}$, where the notation $\tilde{df}$ means that we are considering the differential form $df$ as an $\mathcal{H}^\infty$ function on $\Pi TN_0$.

**Lemma 2** With the above hypothesis an extension $F \in \Gamma(M, \Lambda^*(T^*N_0))$ of $f$ has the property $F^{-1}(\{y\}) = \tilde{j}(\Pi TM_0)$ if $$d_s^{-}F|_{M_0} = 0$$

Moreover $S^+(f)$ is the unique extension of $f$ upto first order to the odd part and satisfying the above constraint.

*Proof.* Straightforward.

Similarly we can also define natural odd extension of differential forms as follows. Define a canonical 1-1 map:

$$Odd : \Omega^{k+1}(N_0) \to \mathcal{H}^\infty(\Pi TN_0) \otimes \Omega^k(N_0) \subset \hat{\Omega}^k(\Pi TN_0)$$

by

$$Odd(\beta)(X_1, ..., X_k) = (-1)^k i_{X_k} \circ ... \circ i_{X_1}(\beta) \in \Omega^1(N_0) \subset \mathcal{H}^\infty(\Pi TN_0)$$

Where $\Omega^{k+1}(N_0)$ denotes the space of differential $k + 1$-forms on $N_0$ and by $\hat{\Omega}^k(\Pi TN_0)$ we mean the space of $\mathcal{H}^\infty$ differential $k$ forms on $\Pi TN_0$. Also $\beta \in \Omega^{k+1}(N_0)$ for some $k \in \mathbb{N} \cup \{0\}$. 
**Definition 2** The (plus or minus) odd extension $S^\pm(\alpha)$ of a differential form $\alpha \in \Omega^*(N_0)$ are defined by:

$$S^\pm : \Gamma(\Lambda^k(T^*N_0)) \to \Gamma(\Lambda^k((T_s^\pm)^*))$$

$$S^\pm(\alpha) = \alpha \pm \text{Odd}(d\alpha)$$

The following fundamental property holds for $S^\pm(\alpha)$:

**Proposition 1** $S^+(\alpha)$ is the unique element of $\Gamma(\Lambda^k((T^+_s)^*))$ extending $\alpha$ up to first order to the odd part and satisfying:

$$d^-_s(S^+(\alpha))|_{M_0} = 0$$

equivalently we have $dS^+(\alpha)|_{M_0} \in \Gamma(\Lambda^{k+1}((T^+_s)^*))$.

**Proof:** First note that for any point $y \in N_0$ and for any vector $v \in T_yN_0$ we have

$$i_v d(\alpha + \text{Odd}(d\alpha))|_{M_0} = 0$$

This means that $d(\alpha + \text{Odd}(d\alpha))|_{M_0} \in \Lambda^{k+1}((T^+_s)^*)$ or equivalently $d^-_s(\alpha + \text{Odd}(d\alpha))|_{M_0} = 0$. The converse is also straightforward.

Now we would like to repeat the above procedure for differential forms with values in vector bundles over $N_0$. So let $\pi : E \to N_0$ be a real smooth vector bundle over $N_0$ equipped with a connection $\nabla$. Let also $p : \Pi TN_0 \to N_0$ be the projection map from $\Pi TN_0$ onto $N_0$. We denote by $\mathcal{E}$ the pullback of the (even) vector bundle $E$ by $p$ to the whole $\Pi TN_0$ i.e. $\mathcal{E} := p^*E$.

As before the operator $d^\nabla : C^\infty(N) \to \Omega^1(E)$ has natural super-symmetric extensions $d^\nabla,^\pm : \mathcal{H}^\infty(\mathcal{E}) \to \hat{\Omega}^1(\mathcal{E})$ defined by:

$$d^\nabla,^\pm = \sum_i (\nabla_{\xi_i} \pm \tilde{\xi}_i)(\partial_{\xi_i}^\pm)^*$$

We recall that $\{\xi_i|i = 1, ..., n\}$ is a basis for $T_yN_0$ at a point $y \in N_0$ and $\tilde{\xi}_i$ is the odd derivation associated to $\xi_i$. We also have

$$d^\nabla = d^\nabla,^+ + d^\nabla,^-$$

And the following lemma holds,
Lemma 3  With the above hypothesis given a section $u \in \Gamma(N_0, E)$ there exists a unique $S^+(u) \in \Gamma(\PiTN_0, \mathcal{E})$ extending $u$ up to first order to the odd part such that, $(d^\nabla_{s^-})(S^+(u))|_{M_0} = 0$.

Proof: It suffices to set

$$S^+(u) := u + d\nabla u$$

where again we use the notation $d\nabla u$ to emphasize that we are considering it as an odd $\mathcal{H}\infty$ section of $\mathcal{E}$ on $\PiTN_0$.

In order to define natural odd extension of vector bundle-valued differential forms $\alpha \in \Omega^k(E)$ to the whole $\mathcal{E}$, we define $Odd$ operator in the same manner as before:

$$Odd : \Omega^{k+1}(E) \rightarrow \mathcal{H}\infty(\PiTN_0) \otimes \Omega^k(E)$$

$$Odd(\beta)(X_1, ..., X_k) = (-1)^k i_{X_k} \circ ... \circ i_{X_1}(\beta) \in \Omega^1(E) \subset \mathcal{H}\infty(\PiTN_0, \mathcal{E})$$

Where $\beta \in \Omega^{k+1}(E)$ is a smooth $E$-valued differential $k + 1$-form on $N_0$, with $k \in \mathbb{N} \cup \{0\}$.

Definition 3 The (plus or minus) odd extension $S^\pm(\alpha)$ of a vector bundle valued differential form $\alpha \in \Omega^*(E)$ are defined through the following applications

$$S^\pm : \Gamma(\Lambda^k(T^*N_0) \otimes E) \rightarrow \Gamma(\Lambda^k((T^s_\pm)^*) \otimes \mathcal{E})$$

$$S^\pm(\alpha) = \alpha \pm Odd(d\nabla \alpha)$$

Similarly we can prove,

Proposition 2  With the above notations $S^+(\alpha)$ is the unique element of $\Gamma(\Lambda^k((T^s_+)^*) \otimes \mathcal{E})$ extending $\alpha$ up to first order to the odd part and satisfying:

$$d^\nabla_{s^-}(S^+(\alpha))|_{M_0} = 0$$

equivalently we have $d\nabla S^+(\alpha)|_{M_0} \in \Gamma(\Lambda^{k+1}((T^+_s)^*) \otimes \mathcal{E})$. 
4 Super-connection and super curvature.

We now apply the above method to naturally extend the differential geometric objects from $M_0$ to $\Pi TM_0$. So assume that the smooth manifold $M_0$ is equipped with a riemannian metric $g_0$ and let $\nabla$ be the Levi-Civita connection associated to $g_0$. In a local coordinate system defined on an open set $U \subset M_0$ the connection $\nabla$ can be written as $\nabla := d + A$ with $A \in \Omega^1(\text{End}(TM_0|_U))$.

We can thus define natural extensions of the connection matrix $A$ to $\Pi TM_0$ as follows:

$$S^\pm A = A \pm \text{Odd}(d^\nabla A) = A \pm \text{Odd}(R^\nabla) \in \hat{\Omega}^1(\text{End}(\text{Tev}|_U))$$

where $R^\nabla \in \Omega^2(\text{End}(TM_0))$ is the curvature tensor associated to the Levi-Civita connection $\nabla$. Due to the transformation of $R^\nabla$ under the action of structure group of $TM_0$ it is obvious that $S^\pm A$ is indeed a connection form on $\Pi TM_0$. We denote this super connection by $\hat{\nabla}$.

The super-curvature of $\hat{\nabla}$ is given in local coordinates by,

$$R^{\hat{\nabla}} = d(S^+ A) + S^+ A \wedge S^+ A = R^\nabla + 2A \wedge \text{Odd}(R^\nabla) + \text{Odd}(R^\nabla) \wedge \text{Odd}(R^\nabla)$$

Geodesic curvature of super-curves in $\Pi TM_0$. As an application of the above odd differential geometry on $\Pi TM_0$ we associate a geodesic super-curvature to super-curves living in $\Pi TM_0$ which is an extension of the ordinary geodesic curvature to its odd part. We then describe the even part of the equation of super-geodesics (discussed in [2][1] and references therein) in terms of this notion of geodesic super-curvature.

Suppose that $M_0$ is a 2-dimensional manifold and let $J_0$ be a complex structure on $M_0$ compatible with the riemannian metric $g_0$ of $M_0$. Let $I \subset \mathbb{R}^{1|1}$ be a super-interval in $\mathbb{R}^{1|1}$ with $I_0 = [0,l]$ for some real positive $l \in \mathbb{R}$. A super-curve $\gamma : I \to M$ on $M$ can be described by the data of an ordinary curve $\gamma_0 : I_0 \to M_0$ along with a vector field $\psi : \mathbb{R} \to TM_0$ which is a lift of the curve $\gamma_0$ to the vector bundle $TM_0$. Assume that $\gamma_0$ is parameterized by length and let $(t, \tau)$ be a coordinates system on $I$. Thus we can define the geodesic super-curvature of $\gamma$ as the pullback of

$$<\hat{\nabla} \dot{\gamma}, J \dot{\gamma}> \in \mathcal{H}^\infty(\Pi TM_0)$$
by $\gamma$ into $I$ which leads to the following extension of geodesic curvature to the odd part

$$
\hat{\kappa}_g = \kappa^e_g + \tau \kappa^o_g := \kappa_g(t) + \tau < R(\dot{\gamma}, \psi)\dot{\gamma}, J\dot{\gamma} >
$$

where $\kappa_g$ is the geodesic curvature of the curve $\gamma_0$ and $R$ denotes the sectional curvature of the surface $M_0$. Two cases are interesting

i) If $\psi = \dot{\gamma}$ then the super-curve $\gamma$ is a natural extension of $\gamma_0$ to $\Pi TM_0$ in the sense of section 2 and in this case we have: $\hat{\kappa}_g = \kappa_g$.

ii) If $\psi = J\dot{\gamma}$ then we have $\hat{\kappa}_g = K(\gamma)$, where $K$ is the gaussian curvature of $M_0$.

In this case if $\kappa^e_g(t) = c\kappa^o_g$ for some constant $c \in \mathbb{R}$ then the even part $\gamma_0$ describes a super-geodesics in the sense of [2] and [1].

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References


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