Fuzzy Strong Regular Congruence Triples
for an $E$-inversive Semigroup

Yabing Shi, Zhenji Tian* and Tianjie Zhang

School of Sciences
Lanzhou University of Technology
Lanzhou, Gansu, 730050, P.R. China

Abstract

In this paper, we introduce the concept of fuzzy strong regular congruence triple of a fuzzy strong regular congruence on an $E$-inversive semigroup, and then we show that each fuzzy strong regular congruence on an $E$-inversive semigroup is uniquely determined by its fuzzy strong regular congruence triple. Finally, it is obtained that there exists a one-one correspondence between fuzzy strong regular congruence triples and fuzzy strong regular congruences on an $E$-inversive semigroup.

Keywords: Fuzzy strong regular congruence; Fuzzy strong regular congruence triple; $\mathcal{L}^*(\mathcal{R}^*)$-regular part; Green$^*$ equivalence relations; $E$-inversive semigroup

1 Introduction and preliminaries

The concepts of fuzzy sets and fuzzy congruences on inverse semigroups are introduced by Al-Thukair in [1] and he sets up a one-one correspondence between the set of fuzzy congruences and the set of fuzzy congruence pairs on an inverse

† Supported by the National Natural Science Foundation of China(Grant N. 11261030)
* Corresponding author: Zhenji Tian (E-mail: zjtian@lut.cn)
semigroup. Wang [8] and Tian [6] studied the relationships between fuzzy congruences and fuzzy congruence triples on completely 0-simple semigroup and completely simple semigroup respectively. Francis Pastijn and Mario Petrich [2] studied the congruence triples by using Green’s relations on regular semigroup and concluded that congruence relation is uniquely determined by its associated triple. Li [3] introduced the concepts of fuzzy congruence triples and fuzzy congruences on a regular semigroup and he obtained that there exists a one-one correspondence between the fuzzy congruence and fuzzy congruence triples on a regular semigroup. Luo [5] first generalized Green’s relations on arbitrary semigroups and introduced the strong regular congruence pairs for an $E$-inversive semigroup. It is proved that each strong regular congruence $\rho$ on an $E$-inversive semigroup $S$ is uniquely determined by the strong regular congruence pair. In this paper, we introduce the concept of fuzzy strong regular congruence triple of a fuzzy strong regular congruence on an $E$-inversive semigroup, and then we show that each fuzzy strong regular congruence on an $E$-inversive semigroup is uniquely determined by its fuzzy strong regular congruence triple. Finally, it is obtained that there exists a one-one correspondence between fuzzy strong regular congruence triples and fuzzy strong regular congruences on an $E$-inversive semigroup.

2 Definition and basic results

An element $a$ of a semigroup $S$ is called regular if there exists $x$ such that $axa = a$. A semigroup $S$ is called regular semigroup if its all elements are regular. A semigroup $S$ is called $E$-inversive semigroup if there exists $x \in S$ such that $ax \in E(S)$. As usual, $E(S)$ is the set of all idempotents and $Reg(S)$ is the set of regular elements of $S$. If $a$ is a regular element of $S$, $V(a) = \{x \in S|axa = a, xax = x\}$ is the set of all inverses of $a$. An element $x \in S$ is called weak inverse of $a \in S$ if $xax = x$. Devote by $W(a)$ the set of all weak inverses of $a \in S$.

Let $S$ be a semigroup and $e, f \in E(S)$, and let

$$M(e, f) = \{g \in E(S)| ge = g = fg\}$$

and

$$S(e, f) = \{g \in E(S)| ge = g = fg, egf = ef\}.$$  

$S(e, f)$ is called the sandwich set of $e$ and $f$. Clearly, if $e, f \in E(S)$ and $eRF(eLF)$, then $S(e, f) = \{e\}(S(e, f) = \{f\})$. If $\rho$ is a congruence on $S$ and $h \in S(e, f)$, then $h\rho \in S(ep, fp)$. Note that in an $E$-inversive semigroup $S$, $M(e, f)$ is not empty for any $e, f \in E(S)$.

A congruence on an $E$-inversive semigroup $S$ is called to be strong regular if for any $a \in S$, there exists $a' \in W(a)$ such that $apaa'a$. Clearly, in this case, $S/\rho$ is a regular semigroup.
As the generalization of Green relations on regular semigroup, Green* relations on E-inversive semigroup S are defined by:

**Definition 2.1** Let S be a semigroup and τ be an equivalence relation on E(S). Define the following binary relations on S by, for a, b ∈ S,

\[
aL_\tau^*b \iff (\forall a' \in W(a))(\exists b' \in W(b))(a'a\tau b'b) \\
&\text{&} (\forall b' \in W(b))(\exists a' \in W(a))(a'a\tau b'b);
\]

\[
aR_\tau^*b \iff (\forall a' \in W(a))(\exists b' \in W(b))(aa'\tau b'b) \\
&\text{&} (\forall b' \in W(b))(\exists a' \in W(a))(aa'\tau b'b);
\]

\[
aH_\tau b \iff \begin{cases} (\forall a' \in W(a))(\exists b' \in W(b))(aa'\tau b'b, a'a\tau b'b) \\
&\text{&} (\forall b' \in W(b))(\exists a' \in W(a))(aa'\tau b'b, a'a\tau b'b). \end{cases}
\]

If τ is the identity relation on E(S), we use the notation L* τ, R* τ in place of L* τ, R* τ, respectively. It is clear that L* τ, R* τ, L* τ ∩ R* τ and H τ are equivalence relations on S and H τ ⊆ L* τ ∩ R* τ. Recall from Theorem 2.3 [4] that if a, b are regular elements of a semigroup S, then aL* b(aR* b, a(L* ∩ R*) b) if and only if aLb(aRb, aHb).

Let ρ be a congruence on a semigroup S. The restriction of ρ to E(S) is called the trace of ρ and is denoted by trρ. The subset \{a ∈ S | apa^2 \} of S is called the kernel of ρ and is denoted by ker ρ. From lemma 1.1 of [6], if ρ is a strong regular congruence on an E-inversive semigroup S, then

\[
\ker \rho = \{a ∈ S | \exists e ∈ E(S), aρe \}.
\]

If γ is an equivalence on a semigroup S, the greatest congruence on S contained in γ is denoted by γ^0. If Γ is a family of equivalences on S, then \(\cap_{\gamma \in \Gamma} \gamma^0 = (\cap_{\gamma \in \Gamma})^0(\text{see}[2])\).

**Lemma 2.2** Let ρ be a strong regular congruence on an E-inversive semigroup S with τ = trρ. Then

1. \((eρ)R^* (fρ) \iff e(τR^* τ)f \iff eR^*_τf\ (e, f ∈ E(S));\)
2. \((aρ)R^* (bp) \iff aR^*_τb;\)
3. \(R^*_τ = ρR^*ρ = ρ ∨ R^*;\)
4. \(R^*_τ|_{E(S)} = τR^* τ = (ρ ∨ R^*)|_{E(S)} = τ ∨ (R^*|_{E(S)});\)
5. \(τ = tr(L^*_τ ∩ R^*_τ)^0 = τL^*_τ ∩ τL^* τ.\)

By lemma 2.7 in [2] and lemma 2.2, we can easily obtain the following lemma.

**Lemma 2.3** Let ρ be a strong regular congruence on an E-inversive semigroup S with τ = trρ. Then

1. \(eτR^* τ f;\)
2. \(∅ \neq (eτ)(fτ) ∩ E(S) \subseteq fτ, ∅ \neq (fτ)(eτ) ∩ E(S) \subseteq eτ;\)
3. \(S(e, f) \subseteq eτ, S(f, e) \subseteq fτ,\)

for any \(e, f ∈ E(S).\)
Lemma 2.4 Let $\Gamma$ be a family of strong regular congruences on an $E$-inversive semigroup $S$. Then
\[
((\bigcap_{\rho \in \Gamma} \rho) \vee R^*)|_{E(S)} = (\bigcap_{\rho \in \Gamma} (\rho \vee R^*))|_{E(S)}.
\]

Proof. Let $e, f \in E(S)$ be such that $e((\bigcap_{\rho \in \Gamma} (\rho \vee R^*))f$. Then for every strong regular congruence $\rho \in \Gamma$, we have $e(\rho \vee R^*)f$. Thus, if $g \in S(e, f)$ we have from lemma 2.2 and lemma 2.3 that $gpe$ for all $\rho \in \Gamma$. Further, $(gf)\rho = (g\rho)(f\rho) = (e\rho)(f\rho) = f\rho$ for all $\rho \in \Gamma$ since $(e\rho)R^*(f\rho)$ by lemma 2.2(1),(3). Consequently
\[
e((\bigcap_{\rho \in \Gamma} \rho) \vee R^*)gR^*gf((\bigcap_{\rho \in \Gamma} \rho), f, E(S) = (\bigcap_{\rho \in \Gamma} (\rho \vee R^*))|_{E(S)}.
\]
Hence
\[
((\bigcap_{\rho \in \Gamma} \rho) \vee R^*)|_{E(S)} \supseteq (\bigcap_{\rho \in \Gamma} (\rho \vee R^*))|_{E(S)}
\]
holds and the reverse inclusion obviously holds. □

Lemma 2.5 For any strong regular congruence $\rho$ on an $E$-inversive semigroup $S$, $K = \ker \rho$, $T = \operatorname{tr} \rho$. Let $a, b \in S$, we have
\[
apb \iff a(L^*_r \cap R^*_r)b, ab' \in K \text{ for all } b' \in W(b).
\]

Let $X$ be a non-empty, a map $A : X \to [0,1]$ is called a fuzzy subset of $X$. The set of all fuzzy subsets on $X$ by $F(X)$. Let $A \in F(X)$, $t \in [0,1]$, define $A^t = \{a \in X | A(a) \geq t\}$, we call $A^t$ a $t$-cut set. Let $t \in [0,1]$, define $A^t = \{a \in X | A(a) \geq t\}$, we call $A^t$ a power $t$-cut set. The mapping $\rho : X \times X \to [0,1]$ is called a fuzzy relation on $X$, let $\rho$ and $\sigma$ be fuzzy relations on $X$, $\rho \leq \sigma$ means that $\rho(a, b) \leq \sigma(a, b)$ for all $a, b \in X$. Their composition denoted by $\rho \circ \sigma$ and defined as:
\[
(\rho \circ \sigma)(a, c) = \sup\{\min\{\rho(a, b), \sigma(b, c)\}, b \in X\},
\]
for any $a, c \in X$. We denote $\rho \circ \sigma$ by $\rho\sigma$ for the sake of simplicity.

Let $t \in [0,1]$, define
\[
\rho^t = \{(a, b) \in X \times X | \rho(a, b) \geq t\}.
\]
Let $t \in [0,1]$, define
\[
\rho^t = \{(a, b) \in X \times X | \rho(a, b) \geq t\}.
\]
Let $\Gamma \subseteq F(X)$, define $\bigcap_{A \in \Gamma} A$ as
\[
(\bigcap_{A \in \Gamma} A)(x) = \inf\{A(x) | A \in \Gamma\}
\]
for all $x \in X$.

A fuzzy relation $\rho$ is called a fuzzy equivalence relation on a semigroup $S$ if

1. $\rho(a, a) = 1$ for all $a \in S$;
2. $\rho(a, b) = \rho(b, a)$ for all $a, b \in S$;
3. $\rho(a, c) \geq \min\{\rho(a, b), \rho(b, c)\}$ for all $a, b, c \in S$.

A fuzzy equivalence relation $\rho$ on a semigroup $S$ is called a fuzzy congruence if $\rho(ab, cd) \geq \min\{\rho(a, c), \rho(b, d)\}$ or $\rho(ac, bc) \geq \rho(a, b)$ and $\rho(ca, cb) \geq \rho(a, b))$ for all $a, b, c, d \in S$.

Let $S$ be an $E$-inversive semigroup. A fuzzy congruence $\rho$ is called a fuzzy strong regular congruence on $S$ if there exists $a' \in W(a)$ such that $\rho(a, a'a) = 1$ for any $a \in S$. Denote by $FE(S)$ the set of all fuzzy equivalences of a semigroup $S$, $FC(S)$ the set of all fuzzy congruences and $FRC(S)$ the set of all fuzzy strong regular congruences. If $\rho \in FRC(S)$, then $K_\rho(a) = \sup\{\rho(a, e) | e \in E(S)\}$ is called the fuzzy kernel of $\rho$ for all $a \in S$. If $\Gamma \subseteq FE(S)(\Gamma \subseteq FC(S))$, then $\cap_{\rho \in \Gamma} \rho$ is the least fuzzy equivalence(fuzzy congruence) relation contained in each $\rho \in \Gamma$. A fuzzy relation $A$ is called regular normal on $S$ if there exists $\rho \in FRC(S)$ such that $A = K_\rho$.

For $A \in FRC(S)$, we define $\xi_A$ as:

$$
\xi_A(a, b) = \sup\{\rho(a, b) | K_\rho = A, \rho \in FRC(S)\}.
$$

It is clear that $\xi_A$ is the greatest fuzzy strong regular congruence in $\{\rho \in FRC(S) | K_\rho = A\}$.

**Lemma 2.6**[3] Let $\rho$ be a fuzzy relation on a semigroup $S$, then $\rho$ is a fuzzy equivalence relation if and only if $\rho^t$ is an equivalence relation on $S$ for all $t \in [0, 1)$.

**Lemma 2.7**[3] Let $\mu$ be a fuzzy equivalence relation on a semigroup $S$. $\mu^0$ is defined as $\mu^0(a, b) = \inf\{\mu(xay, xby) | x, y \in S^1\}$, if $\emptyset \neq \Gamma \subseteq FE(S)$, then

1. $\mu^0$ is the greatest fuzzy congruence contained in $\mu$;
2. $\cap_{\mu \in \Gamma} \mu^0 = (\cap_{\mu \in \Gamma} \mu)^0$.

Let $S$ be a semigroup and $\tau$ be an equivalence relation on $E(S)$. Define the following fuzzy binary relations on $S$ by, for $a, b \in S$,

$$
\mathcal{L}_f^*(a, b) = \begin{cases} 
1, & (a, b) \in \mathcal{L}_\tau^*, \\
0, & (a, b) \notin \mathcal{L}_\tau^*.
\end{cases}
$$

$$
\mathcal{R}_f^*(a, b) = \begin{cases} 
1, & (a, b) \in \mathcal{R}_\tau^*, \\
0, & (a, b) \notin \mathcal{R}_\tau^*.
\end{cases}
$$

It is clear that both of $\mathcal{L}_f^*$ and $\mathcal{R}_f^*$ are fuzzy equivalences.
3 Fuzzy strong regular congruence triples

Let $\rho \in FE(S)$. Define fuzzy equivalence $\rho_a$ as $\rho_a(x) = \rho(a, x)$.

**Lemma 3.1** Let $S$ be a semigroup.

(1) If $\rho, \sigma \in FE(S)$ and $\nu_0 = \rho, \nu_1 = \rho \sigma, \nu_2 = \rho \sigma \rho, \cdots$, then

$$\rho \vee \sigma = \sup_{n = 0, 1, 2, \cdots, \infty} \nu_n;$$

(2) If $\rho \in FE(S)(FC(S))$, then $\rho_a = \rho_b$ if and only if $\rho(a, b) = 1$ for $a, b \in S$.

Let $S$ be an $E$-inversive semigroup. Let $\rho, \sigma \in FE(S)$ be such that $\rho \leq \sigma$. The fuzzy relation on $S/\rho = \{\rho_a | a \in S\}$, denote by $\sigma/\rho$, is defined as for all $a, b \in S$, $(\sigma/\rho)(\rho_a, \rho_b) = \sigma(a, b)$ where $\rho_a, \rho_b \in S/\rho$.

**Lemma 3.2** Let $S$ be an $E$-inversive semigroup. Let $\rho, \sigma \in FE(S)$ be such that $\rho \leq \sigma$. Then $\sigma/\rho$ is a fuzzy equivalence relation on $S/\rho$.

**Proof.** Let $\rho, \sigma \in EF(S)$. Suppose that $(a, b)$ and $(c, d)$ in $S \times S$ such that $\sigma(a, b) \neq \sigma(c, d)$, then $(\rho_a, \rho_b) \neq (\rho_c, \rho_d)$. Otherwise, we have $\rho_a = \rho_c$ and $\rho_b = \rho_d$, which leads to $\rho(a, c) = 1$ and $\rho(b, d) = 1$. Since $\rho \leq \sigma$, we have $\rho(a, c) \leq \sigma(a, c) = 1$ and $\rho(b, d) \leq \sigma(b, d) = 1$. That is $\sigma(a, b) = \sigma(c, d)$, which contradicts the assumption. Thus, $\sigma/\rho$ is a mapping. It is a routine matter to show that $\sigma/\rho$ is a fuzzy equivalence relation on $S/\rho$, that is, for all $\rho_a, \rho_b, \rho_c \in S/\rho$, it satisfies the following conditions:

(1) $(\sigma/\rho)(\rho_a, \rho_a) = \sigma(a, a) = 1$;

(2) $(\sigma/\rho)(\rho_a, \rho_b) = \sigma(a, b) = \sigma(b, a) = (\sigma/\rho)(\rho_b, \rho_a)$;

(3) $(\sigma/\rho)(\rho_a, \rho_c) = \sigma(a, c) \geq \min\{\sigma(a, b), \sigma(b, c)\} = \min\{\sigma(a, b), \sigma(b, c)\}$.

Let $S$ be an $E$-inversive semigroup and let $\alpha \in FE(S/\mathcal{L}_f), \beta \in FE(S/\mathcal{R}_f^*)$. Define $\bar{\alpha}, \bar{\beta}$ as follow:

$$\bar{\alpha}(a, b) = \alpha((\mathcal{L}_f^*)_a, (\mathcal{L}_f^*)_b), \bar{\beta}(a, b) = \beta((\mathcal{R}_f^*)_a, (\mathcal{R}_f^*)_b)$$

for all $a, b \in S$. It is easy to verify that $\bar{\alpha}, \bar{\beta} \in FE(S), \bar{\alpha}^0, \bar{\beta}^0 \in FRC(S)$, and $\mathcal{L}_f^* \leq \bar{\alpha}, \mathcal{R}_f^* \leq \bar{\beta}$. A fuzzy equivalence relation $\alpha \in FE(S/\mathcal{L}_f)$ ($\beta \in FE(S/\mathcal{R}_f^*)$) is called regular normal, if $\alpha = (\bar{\alpha}^0 \vee \mathcal{L}_f^*)/(\mathcal{L}_f^*), \beta = (\bar{\beta}^0 \vee \mathcal{R}_f^*)/(\mathcal{R}_f^*)$. Let $\rho$ be a strong regular congruence on $S$. $(\rho \vee \mathcal{L}_f^*)/\mathcal{L}_f^*$ is called an $\mathcal{L}$-star-regular part of $\rho, (\rho \vee \mathcal{R}_f^*)/\mathcal{R}^*_f$ is called a $\mathcal{R}$-star-regular part of $\rho$.

**Lemma 3.3** Let $S$ be an $E$-inversive semigroup and $\beta \in FE(S/\mathcal{R}_f^*)$. Then $\beta$ is regular normal if and only if $\beta$ is $\mathcal{R}^*$-regular part of some fuzzy regular congruences on $S$. 
Lemma 3.4 Let $S$ be an $E$-inversive semigroup. If $\beta \in FRC(S)$, then $\beta \cap L_f^\ast = \beta \cap L_f^\ast$.

Proof. It is easy to verify that $\beta \cap L_f^\ast$ is a fuzzy equivalence relation. Since $\beta \cap L_f^\ast$ is the least fuzzy equivalence relation containing $\beta$ and $L_f^\ast$, we have $\beta \cap L_f^\ast \leq \beta \cap L_f^\ast$. Therefore, $\beta = (\beta \cap L_f^\ast) / R_f^\ast = (\beta \cap L_f^\ast) / R_f^\ast$, and so $\beta$ is regular normal. 

Lemma 3.5 Let $S$ be an $E$-inversive semigroup. Then $\rho \cap L_f^\ast = \rho \cap L_f^\ast$.

Proof. If $\beta$ is regular normal, choose $\beta^0$ in $FRC(S)$ such that $\beta$ is $R^\ast$-regular part of $\beta^0$.

Conversely, let $\beta$ be a $R^\ast$-regular part of $\rho \in FRC(S)$, that is $\beta = (\rho \cap R^\ast_f) / R_f^\ast$. By lemma 3.2, we have $\beta = \rho \cap R^\ast_f$. Clearly, $\rho \leq \beta^0$, thus, $\rho \cap R^\ast_f \leq \beta^0 \cap R^\ast_f$. On the other hand, from lemma 3.1 know that $\beta^0 \leq \rho \cap R^\ast_f$. Further, $\beta^0 \cap R^\ast_f \leq \rho \cap R^\ast_f$. Therefore, $\beta = (\rho \cap R^\ast_f) / R_f^\ast = (\beta^0 \cap R^\ast_f) / R_f^\ast$, and so $\beta$ is regular normal. 

Lemma 3.6 Let $S$ be an $E$-inversive semigroup. Then the mapping 

$$\theta: \rho \rightarrow \rho \cap R^\ast_f \hspace{0.1cm} (\rho \in FRC(S))$$

is a complete lattice homomorphism $FRC(S)$ into $FE(S)$.

Proof. Let $\Gamma \subseteq FRC(S)$ and $a, b \in S$. It clear that $(\cap_{\rho \in \Gamma} \rho \cap R^\ast_f) \cap (\cap_{\rho \in \Gamma} \rho \cap R^\ast_f) = \cap_{\rho \in \Gamma} (\rho \cap R^\ast_f) \cap R_f^\ast$.

On the other hand, since $\cap_{\rho \in \Gamma} \rho \cap R^\ast_f \leq \cap_{\rho \in \Gamma} (\rho \cap R^\ast_f)$, we have

$$\cap_{\rho \in \Gamma} \rho \cap R^\ast_f \leq (\cap_{\rho \in \Gamma} (\rho \cap R^\ast_f)) \cap R_f^\ast = \cap_{\rho \in \Gamma} (\rho \cap R^\ast_f).$$

Therefore

$$\cap_{\rho \in \Gamma} \rho \cap R^\ast_f = \cap_{\rho \in \Gamma} (\rho \cap R^\ast_f).$$

It follows that properties of Green* relations that there exist $e, f \in E(S)$ such that $e \cap R^\ast a$, and $f \cap R^\ast b$, by lemma 3.5, we have

$$(\cap_{\rho \in \Gamma} (\rho \cap R^\ast_f))(a, b) = (\cap_{\rho \in \Gamma} (\rho \cap R^\ast_f))(e, f) = ((\cap_{\rho \in \Gamma} \rho \cap R^\ast_f))(e, f) = ((\cap_{\rho \in \Gamma} \rho \cap R^\ast_f))(a, b),$$

that is

$$(\cap_{\rho \in \Gamma} \rho \cap R^\ast_f) = \cap_{\rho \in \Gamma} (\rho \cap R^\ast_f).$$
We have proved that the mapping $\theta : \rho \to \rho \lor \mathcal{R}_f^*$ is a complete lattice homomorphism. \hfill \Box

**Definition 3.7** A triple $(\alpha, A, \beta)$ consisting of regular normal fuzzy equivalence $\alpha$ in $FE(S/\mathcal{L}_f^*)$ and $\beta$ in $FE(S/\mathcal{R}_f^*)$ and a regular normal fuzzy subset $A$ on $S$, is a fuzzy strong regular congruence triple if

1. $\bar{\alpha} = (\bar{\alpha} \land \bar{\beta})^0 \lor \mathcal{L}_f^*$, $\bar{\beta} = (\bar{\alpha} \land \bar{\beta})^0 \lor \mathcal{R}_f^*$;
2. $A \leq K_\alpha$, $\bar{\alpha} \leq \xi_A \lor \mathcal{L}_f^*$;
3. $A \leq K_{\beta\rho}$, $\beta \leq \xi_A \lor \mathcal{R}_f^*$.

If this is the case, we define

$$\rho_{(\alpha,A,\beta)} = \xi_A \cap (\bar{\alpha} \land \bar{\beta})^0.$$ 

**Lemma 3.8** Let $S$ be an $E$-inversive semigroup. If $\rho \in FRC(S)$, then

$$\rho = \xi_{K_\rho} \cap (\rho \lor \mathcal{L}_f^*)^0 \cap (\rho \lor \mathcal{R}_f^*)^0.$$ 

**Proof.** It is clear that

$$\rho(a, b) \leq (\xi_{K_\rho} \cap (\rho \lor \mathcal{L}_f^*)^0 \cap (\rho \lor \mathcal{R}_f^*)^0)(a, b)$$

$$= \min\{\xi_{K_\rho}(a, b), (\rho \lor \mathcal{L}_f^*)^0(a, b), (\rho \lor \mathcal{R}_f^*)^0(a, b)\}$$

for any $a, b \in S$, $\rho \in FRC(S)$.

On the other hand, let $r = \min\{(\rho \lor \mathcal{L}_f^*)^0(a, b), (\rho \lor \mathcal{R}_f^*)^0(a, b), \xi_{K_\rho}(a, b)\}$. If $r = 0$, then $\rho(a, b) = 0$. Suppose that $r > 0$, from lemma 3.4, for any $\varepsilon > 0$ ($\varepsilon < r$), if $t = r - \varepsilon$, then there exist $g, h \in S$ such that $\min\{\rho(a, g), \mathcal{R}_f^*(g, h), \rho(h, b)\} > t$. Thus, $(\rho \lor \mathcal{R}_f^*)(bp^\ell)$. Similarly, we have $(\rho \lor \mathcal{R}_f^*)(bp^\ell)$. Therefore, $(\rho \lor \mathcal{R}_f^*)(bp^\ell) > t$, where $b' \in W(b)$, then there exists $a = bb' \in E(S)$ such that $\rho(ab', e) > t$. Thus, $ab' \in ker\ell$. It follows from lemma 2.5 that $\rho(a, b) > t = r - \varepsilon$. We obtain $\rho(a, b) \geq r$, by making use of arbitrariness of $\varepsilon$. Thus, $\rho(a, b) = r$. It means

$$\rho = \xi_{K_\rho} \cap (\rho \lor \mathcal{L}_f^*)^0 \cap (\rho \lor \mathcal{R}_f^*)^0.$$ 

\hfill \Box

**Theorem 3.9** Let $S$ be an $E$-inversive semigroup. If $(\alpha, A, \beta)$ is a fuzzy strong regular congruence triple, then $\rho_{(\alpha,A,\beta)}$ is a unique fuzzy strong regular congruence $\rho$ on $S$ such that $\alpha$ is the $\mathcal{L}^*$-regular part of $\rho$, $A = K_{\rho(\alpha,A,\beta)}$ and $\beta$ is the $\mathcal{R}^*$-regular part of $\rho$.

Conversely, if $\rho$ is a fuzzy strong regular congruence on $S$, then $(\alpha, A, \beta) = ((\rho \lor \mathcal{L}_f^*)/\mathcal{L}_f^*, K_{\rho}, (\rho \lor \mathcal{R}_f^*)/\mathcal{R}_f^*)$ is a fuzzy strong regular congruence triple for $S$ and $\rho = \rho_{(\alpha,A,\beta)}$. 

\hfill \Box
Proof. Let \((\alpha, A, \beta)\) be a fuzzy strong regular congruence triple. Then
\[
\rho_{(\alpha, A, \beta)}(a, a'a) = (\xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0)(a, a'a)
\]
\[
= (\xi_A \cap \bar{\alpha}^0 \cap \bar{\beta}^0)(a, a'a)
\]
\[
= \inf\{\xi_A(a, a'a), \bar{\alpha}^0(a, a'a), \bar{\beta}^0(a, a'a)\}
\]
\[
= 1
\]
for any \(a \in S, a' \in W(a)\). This shows that \(\rho_{(\alpha, A, \beta)}\) is a fuzzy strong regular congruence.

By lemma 2.7 and definition 3.7, for all \(x \in S\), we have
\[
K_{\rho_{(\alpha, A, \beta)}}(x) = K_{(\xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0)}(x)
\]
\[
= \sup\{(\xi_A \cap \bar{\alpha}^0 \cap \bar{\beta}^0)(x, e) \mid e \in E(S)\}
\]
\[
= K_{\xi_A}(x) \cap K_{\bar{\alpha}^0}(x) \cap K_{\bar{\beta}^0}(x)
\]
\[
= A(x) \cap K_{\bar{\alpha}^0}(x) \cap K_{\bar{\beta}^0}(x)
\]
\[
= A(x).
\]
Further
\[
\rho_{(\alpha, A, \beta)} \lor R^*_f = (\xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0) \lor R^*_f
\]
\[
= (\xi_A \lor R^*_f) \cap ((\bar{\alpha} \cap \bar{\beta})^0 \lor R^*_f)
\]
\[
= (\xi_A \lor R^*_f) \cap \bar{\beta}
\]
\[
= \bar{\beta},
\]
and thus
\[
(\rho_{(\alpha, A, \beta)} \lor R^*_f) / R^*_f = \beta
\]
is the \(R^*\)-regular part of \(\rho_{(\alpha, A, \beta)}\). Dually, \(\alpha\) is the \(L^*\)-regular part of \(\rho_{(\alpha, A, \beta)}\).

Let \(\rho \in FRC(S)\) be such that \(\alpha\) is the \(L^*\)-regular part of \(\rho\), \(A = K_{\rho}\) and \(\beta\) is the \(R^*\)-regular part of \(\rho\). Then \(K_{\rho} = A = K_{\rho_{(\alpha, A, \beta)}}\). By lemma 3.8, we have
\[
\rho = \xi_{K_{\rho}} \cap (\rho \lor L^*_f)^0 \cap (\rho \lor R^*_f)^0
\]
\[
= \xi_A \cap \bar{\alpha}^0 \cap \bar{\beta}^0
\]
\[
= \xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0
\]
\[
= \rho_{(\alpha, A, \beta)}.
\]

Conversely, let \(\rho \in FRC(S)\) and let \(\alpha = (\rho \lor L^*_f) / L^*_f, A = K_{\rho}\) and \(\beta = (\rho \lor R^*_f) / R^*_f\). By lemma 3.3 and its dual \(\alpha\) and \(\beta\) are regular normal fuzzy equivalences on \(FE(S/L^*_f)\) and \(FE(S/R^*_f)\), respectively. Thus, \(A\) is a regular normal fuzzy subset on \(S\).

We note that \(\bar{\alpha} = \rho \lor L^*_f\) and \(\bar{\beta} = \rho \lor R^*_f\). Since \(R^*_f \leq \bar{\beta}\), it follows that \((\bar{\alpha} \cap \bar{\beta})^0 \lor R^*_f \leq \bar{\beta}\). Further, since \(\rho \leq (\bar{\alpha} \cap \bar{\beta})^0\), we have \(\bar{\beta} = \rho \lor R^*_f \leq (\bar{\alpha} \cap \bar{\beta})^0 \lor R^*_f\). Therefore \(\bar{\beta} = (\bar{\alpha} \cap \bar{\beta})^0 \lor R^*_f\) and Dually, \(\bar{\alpha} = (\bar{\alpha} \cap \bar{\beta})^0 \lor L^*_f\).
\[ K_\rho = A \] gives \( \rho \leq \xi_A \) whence \( \bar{\beta} = \rho \vee R_f^* \leq \xi_A \vee R_f^* \) and dually, \( \bar{\alpha} = \rho \vee L_f^* \leq \xi_A \vee L_f^* \). We have proved that \((\alpha, A, \beta)\) is a fuzzy strong regular congruence triple. From the first part of the proof we may now conclude that \( \rho = \rho_{(\alpha,A,\beta)} \).

By lemma 3.6, we can easily obtain the following corollary.

**Corollary 3.10** Let \( FRCT(S) \) be the poset of all fuzzy strong regular congruence triples for an \( E \)-inversive semigroup \( S \) under the partial order given by
\[
(\alpha, A, \beta) \leq (\alpha', A', \beta') \iff \alpha \leq \alpha', A \leq A', \beta \leq \beta'.
\]
Then the mappings
\[
\rho \rightarrow ((\rho \vee L_f^*)/L_f^*, K_\rho, (\rho \vee R_f^*)/R_f^*), (\alpha, A, \beta) \rightarrow \rho_{(\alpha,A,\beta)}
\]
are mutually inverse isomorphisms of \( FRC(S) \) and \( FRCT(S) \).

**References**


Received: January 20, 2013