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Fuzzy Strong Regular Congruence Triples for an E-inversive Semigroup†

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Abstract

In this paper, we introduce the concept of fuzzy strong regular congruence triple of a fuzzy strong regular congruence on an E-inversive semigroup, and then we show that each fuzzy strong regular congruence on an E-inversive semigroup is uniquely determined by its fuzzy strong regular congruence triple. Finally, it is obtained that there exists a one-one correspondence between fuzzy strong regular congruence triples and fuzzy strong regular congruences on an E-inversive semigroup.

Keywords: Fuzzy strong regular congruence; Fuzzy strong regular congruence triple; $\mathcal{L}^*(\mathcal{R}^*)$-regular part; Green* equivalence relations; E-inversive semigroup

1 Introduction and preliminaries

The concepts of fuzzy sets and fuzzy congruences on inverse semigroups are introduced by Al-Thukair in [1] and he sets up a one-one correspondence between the set of fuzzy congruences and the set of fuzzy congruence pairs on an inverse

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semigroup.Wang [8] and Tian [6] studied the relationships between fuzzy congruences and fuzzy congruence triples on completely 0-simple semigroup and completely simple semigroup respectively. Francis Pastijn and Mario Petrich [2] studied the congruence triples by using Green's relations on regular semigroup and concluded that congruence relation is uniquely determined by its associated triple. Li [3] introduced the concepts of fuzzy congruence triples and fuzzy congruences on a regular semigroup and he obtained that there exists a one-one correspondence between the fuzzy congruence and fuzzy congruence triples on a regular semigroup. Luo [5] first generalized Green' relations on arbitrary semigroups and introduced the strong regular congruence pairs for an E-inversive semigroup. It is proved that each strong regular congruence ρ on an E-inversive semigroup S is uniquely determined by the strong regular congruence pair. In this paper, we introduce the concept of fuzzy strong regular congruence triple of a fuzzy strong regular congruence on an E-inversive semigroup, and then we show that each fuzzy strong regular congruence on an E-inversive semigroup is uniquely determined by its fuzzy strong regular congruence triple. Finally, it is obtained that there exists a one-one correspondence between fuzzy strong regular congruence triples and fuzzy strong regular congruences on an E-inversive semigroup.

2 Definition and basic results

An element a of a semigroup S is called regular if there exists x such that $axa = a$. A semigroup S is called regular semigroup if its all elements are regular. A semigroup S is called E-inversive semigroup if there exists $x \in S$ such that $ax \in E(S)$. As usual, $E(S)$ is the set of all idempotents and $Reg(S)$ is the set of regular elements of S. If a is a regular element of S, $V(a) = \{x \in S | axa = a, xax = x\}$ is the set of all inverses of a. An element $x \in S$ is called weak inverse of $a \in S$ if $xax = x$. Devote by $W(a)$ the set of all weak inverses of $a \in S$.

Let S be a semigroup and $e, f \in E(S)$, and let

$$M(e, f) = \{g \in E(S) |\ ge = g = fg\}$$

and

$$S(e, f) = \{g \in E(S)) |\ ge = g = fg, egf = ef\}.$$

$S(e, f)$ is called the *sandwith set* of e and f. Clearly, if $e, f \in E(S)$ and $e\mathcal{R}f(e\mathcal{L}f)$, then $S(e, f) = \{e\}(S(e, f) = \{f\})$. If ρ is a congruence on S and $h \in S(e, f)$, then $h\rho \in S(e\rho, f\rho)$. Note that in an E-inversive semigroup S, $M(e, f)$ is not empty for any $e, f \in E(S)$.

A congruence on an E-inversive semigroup S is called to be *strong regular* if for any $a \in S$, there exists $a' \in W(a)$ such that $a\rho aa'a$. Clearly, in this case, S/ρ is a regular semigroup.

As the generalization of Green' relations on regular semigroup, Green* relations on E-inversive semigroup S are defined by:

Definition 2.1[5] Let S be a semigroup and τ be an equivalence relation on $E(S)$. Define the following binary relations on S by, for $a, b \in S$,

$$a\mathcal{L}^*_\tau b \Leftrightarrow (\forall a' \in W(a))(\exists b' \in W(b))(a'a\tau b'b) \\ \&(\forall b' \in W(b))(\exists a' \in W(a))(a'a\tau b'b);$$

$$a\mathcal{R}^*_\tau b \Leftrightarrow (\forall a' \in W(a))(\exists b' \in W(b))(aa'\tau bb') \\ \&(\forall b' \in W(b))(\exists a' \in W(a))(aa'\tau bb');$$

$$a\mathcal{H}_\tau b \Leftrightarrow \begin{cases} (\forall a' \in W(a))(\exists b' \in W(b))(aa'\tau bb', a'a\tau b'b) \\ (\forall b' \in W(b))(\exists a' \in W(a))(aa'\tau bb', a'a\tau b'b). \end{cases}$$

If τ is the identity relation on $E(S)$, we use the notation $\mathcal{L}^*$, $\mathcal{R}^*$ in place of $\mathcal{L}^*_\tau$, $\mathcal{R}^*_\tau$, respectively. It is clear that $\mathcal{L}^*_\tau$, $\mathcal{R}^*_\tau$, $\mathcal{L}^*_\tau \cap \mathcal{R}^*_\tau$ and $\mathcal{H}_\tau$ are equivalence relations on S and $\mathcal{H}_\tau \subseteq \mathcal{L}^*_\tau \cap \mathcal{R}^*_\tau$. Recall from Theorem 2.3 [4] that if a, b are regular elements of a semigroup S, then $a\mathcal{L}^*b(a\mathcal{R}^*b,\ a(\mathcal{L}^* \cap \mathcal{R}^*)b)$ if and only if $a\mathcal{L}b(a\mathcal{R}b, a\mathcal{H}b)$.

Let ρ be a congruence on a semigroup S. The restriction of ρ to $E(S)$ is called the trace of ρ and is denoted by $\mathrm{tr}\rho$. The subset $\{a \in S | a\rho a^2\}$ of S is called the kernel of ρ and is denoted by $\ker\rho$. From lemma 1.1 of [6], if ρ is a strong regular congruence on an E-inversive semigroup S, then

$$\ker\rho = \{a \in S | \exists e \in E(S), a\rho e\}.$$

If γ is an equivalence on a semigroup S, the greatest congruence on S contained in γ is denoted by γ^0. If Γ is a family of equivalences on S, then $\cap_{\gamma\in\Gamma}\gamma^0 = (\cap_{\gamma\in\Gamma}\gamma)^0$(see[2]).

Lemma 2.2[5] Let ρ be a strong regular congruence on an E-inversive semigroup S with $\tau = \mathrm{tr}\rho$. Then

(1) $(e\rho)\mathcal{R}^*(f\rho) \Leftrightarrow e(\tau\mathcal{R}^*\tau)f \Leftrightarrow e\mathcal{R}^*_\tau f\ (e, f \in E(S))$;

(2) $(a\rho)\mathcal{R}^*(b\rho) \Leftrightarrow a\mathcal{R}^*_\tau b$;

(3) $\mathcal{R}^*_\tau = \rho\mathcal{R}^*\rho = \rho \vee \mathcal{R}^*$;

(4) $\mathcal{R}^*_\tau|_{E(S)} = \tau\mathcal{R}^*\tau = (\rho \vee \mathcal{R}^*)|_{E(S)} = \tau \vee (\mathcal{R}^*|_{E(S)})$;

(5) $\tau = tr(\mathcal{L}^*_\tau \cap \mathcal{R}^*_\tau)^0 = \tau\mathcal{L}^*\tau \cap \tau\mathcal{L}^*\tau$.

By lemma 2.7 in [2] and lemma 2.2, we can easily obtain the following lemma.

Lemma 2.3 Let ρ be a strong regular congruence on an E-inversive semigroup S with $\tau = \mathrm{tr}\rho$. Then

(1) $e\tau\mathcal{R}^*\tau f$;

(2) $\varnothing \neq (e\tau)(f\tau) \cap E(S) \subseteq f\tau, \varnothing \neq (f\tau)(e\tau) \cap E(S) \subseteq e\tau$;

(3) $S(e, f) \subseteq e\tau, S(f, e) \subseteq f\tau$,

for any $e, f \in E(S)$.

Lemma 2.4 Let Γ be a family of strong regular congruences on an E-inversive semigroup S. Then

$$((\cap_{\rho\in\Gamma}\rho)\vee\mathcal{R}^*)|_{E(S)}=(\cap_{\rho\in\Gamma}(\rho\vee\mathcal{R}^*))|_{E(S)}.$$

Proof. Let $e,f\in E(S)$ be such that $e(\cap_{\rho\in\Gamma}(\rho\vee\mathcal{R}^*))f$. Then for every strong regular congruence $\rho\in\Gamma$, we have $e(\rho\vee\mathcal{R}^*)f$. Thus, if $g\in S(e,f)$ we have from lemma 2.2 and lemma 2.3 that $g\rho e$ for all $\rho\in\Gamma$. Further, $(gf)\rho=(g\rho)(f\rho)=(e\rho)(f\rho)=f\rho$ for all $\rho\in\Gamma$ since $(e\rho)\mathcal{R}^*(f\rho)$ by lemma 2.2(1),(3). Consequently

$$e(\cap_{\rho\in\Gamma}\rho)g\mathcal{R}^*gf(\cap_{\rho\in\Gamma}\rho)f,$$

that is,

$$e((\cap_{\rho\in\Gamma}\rho)\vee\mathcal{R}^*)f.$$

Hence

$$((\cap_{\rho\in\Gamma}\rho)\vee\mathcal{R}^*)|_{E(S)}\supseteq(\cap_{\rho\in\Gamma}(\rho\vee\mathcal{R}^*))|_{E(S)}$$

holds and the reverse inclusion obviously holds. □

Lemma 2.5[5] For any strong regular congruence ρ on an E-inversive semigroup S, $K=\ker\rho$, τ=trρ. $a,b\in S$, we have

$$a\rho b\Leftrightarrow a(\mathcal{L}^*_\tau\cap\mathcal{R}^*_\tau)b,\ ab'\in K\ for\ all\ b'\in W(b).$$

Let X be a non-empty, a map $A:X\to[0,1]$ is called a fuzzy subset of X. Devote the set of all fuzzy subsets on X by $F(X)$. Let $A\in F(X)$, $t\in[0,1]$, define $A^t=\{a\in X|A(a)\geq t\}$, we call A^t a t-cut set. Let $t\in[0,1)$, define $A^\ell=\{a\in X|A(a)>t\}$, we call A^ℓ a power t-cut set. The mapping $\rho:X\times X\to[0,1]$ is called a fuzzy relation on X, let ρ and σ be fuzzy relations on X, $\rho\leq\sigma$ means that $\rho(a,b)\leq\sigma(a,b)$ for all $a,b\in X$. Their composition denoted by $\rho\circ\sigma$ and defined as:

$$(\rho\circ\sigma)(a,c)=\sup\{\min\{\rho(a,b),\sigma(b,c)\},b\in X\},$$

for any $a,c\in X$. We denote $\rho\circ\sigma$ by $\rho\sigma$ for the sake of simplicity.

Let $t\in[0,1]$, define

$$\rho^t=\{(a,b)\in X\times X|\ \rho(a,b)\geq t\}.$$

Let $t\in[0,1)$, define

$$\rho^\ell=\{(a,b)\in X\times X|\ \rho(a,b)>t\}.$$

Let $\Gamma\subseteq F(X)$, define $\cap_{A\in\Gamma}A$ as

$$(\cap_{A\in\Gamma}A)(x)=\inf\{A(x)|\ A\in\Gamma\}$$

for all $x \in X$.

A fuzzy relation ρ is called a fuzzy equivalence relation on a semigroup S if

(1) $\rho(a,a) = 1$ for all $a \in S$;

(2) $\rho(a,b) = \rho(b,a)$ for all $a, b \in S$;

(3) $\rho(a,c) \geq \min\{\rho(a,b), \rho(b,c)\}$ for all $a, b, c \in S$.

A fuzzy equivalence relation ρ on a semigroup S is called a fuzzy congruence if $\rho(ab, cd) \geq \min\{\rho(a,c), \rho(b,d)\}$ or ($\rho(ac, bc) \geq \rho(a,b)$ and $\rho(ca, cb) \geq \rho(a,b)$) for all $a, b, c, d \in S$.

Let S be an E-inversive semigroup. A fuzzy congruence ρ is called a fuzzy strong regular congruence on S if there exists $a' \in W(a)$ such that $\rho(a, aa'a) = 1$ for any $a \in S$. Denote by $FE(S)$ the set of all fuzzy equivalences of a semigroup S, $FC(S)$ the set of all fuzzy congruences and $FRC(S)$ the set of all fuzzy strong regular congruences. If $\rho \in FRC(S)$, then $K_\rho(a) = \sup\{\rho(a,e) | \ e \in E(S)\}$ is called the fuzzy kernel of ρ for all $a \in S$. If $\Gamma \subseteq FE(S)(\Gamma \subseteq FC(S))$, then $\vee_{\rho\in\Gamma}\rho$ is the least fuzzy equivalence(fuzzy congruence) relation containing all $\rho \in \Gamma$, $\cap_{\rho\in\Gamma}\rho$ is the greatest fuzzy equivalence(fuzzy congruence) relation contained in each $\rho \in \Gamma$. A fuzzy relation A is called regular normal on S if there exists $\rho \in FRC(S)$ such that $A = K_\rho$. For $A \in FRC(S)$, we define ξ_A as:

$$\xi_A(a,b) = \sup\{\rho(a,b) | \ K_\rho = A, \rho \in FRC(S)\}.$$

It is clear that ξ_A is the greatest fuzzy strong regular congruence in $\{\rho \in FRC(S) | \ K_\rho = A\}$.

Lemma 2.6[3] Let ρ be a fuzzy relation on a semigroup S, then ρ is a fuzzy equivalence relation if and only if ρ^ℓ is an equivalence relation on S for all $t \in [0, 1)$.

Lemma 2.7[3] Let μ be a fuzzy equivalence relation on a semigroup S. μ^0 is defined as $\mu^0(a,b) = \inf\{\mu(xay, xby) | \ x, y \in S^1\}$, if $\varnothing \neq \Gamma \subseteq FE(S)$, then

(1) μ^0 is the greatest fuzzy congruence contained in μ;

(2) $\cap_{\mu\in\Gamma} \mu^0 = (\cap_{\mu\in\Gamma}\mu)^0$.

Let S be a semigroup and τ be an equivalence relation on $E(S)$. Define the following fuzzy binary relations on S by, for $a, b \in S$,

$$\mathcal{L}^*_f(a,b) = \begin{cases} 1, & (a,b) \in \mathcal{L}^*_\tau. \\ 0, & (a,b) \notin \mathcal{L}^*_\tau. \end{cases}$$

$$\mathcal{R}^*_f(a,b) = \begin{cases} 1, & (a,b) \in \mathcal{R}^*_\tau. \\ 0, & (a,b) \notin \mathcal{R}^*_\tau. \end{cases}$$

It is clear that both of $\mathcal{L}^*_f$ and $\mathcal{R}^*_f$ are fuzzy equivalences.

3 Fuzzy strong regular congruence triples

Let $\rho \in FE(S)$. Define fuzzy equivalence ρ_a as $\rho_a(x) = \rho(a, x)$.

Lemma 3.1[3] Let S be a semigroup.

(1) If $\rho, \sigma \in FE(S)$ and $\nu_0 = \rho, \nu_1 = \rho\sigma, \nu_2 = \rho\sigma\rho, \nu_3 = \rho\sigma\rho\sigma, \cdots$, then

$$\rho \vee \sigma = \sup_{n=0,1,2,\cdots,\infty} \nu_n;$$

(2) If $\rho \in FE(S)(FC(S))$, then $\rho_a = \rho_b$ if and only if $\rho(a, b) = 1$ for $a, b \in S$.

Let S be an E-inversive semigroup. Let $\rho, \sigma \in FE(S)$ be such that $\rho \le \sigma$. The fuzzy relation on $S/\rho = \{\rho_a | \ a \in S\}$, devote by σ/ρ, is defined as for all $a, b \in S$, $(\sigma/\rho)(\rho_a, \rho_b) = \sigma(a, b)$. where $\rho_a, \rho_b \in S/\rho$.

Lemma 3.2 Let S be an E-inversive semigroup. Let $\rho, \sigma \in FE(S)$ be such that $\rho \le \sigma$. Then σ/ρ is a fuzzy equivalence relation on S/ρ.

Proof. Let $\rho, \sigma \in EF(S)$. Suppose that (a, b) and (c, d) in $S \times S$ such that $\sigma(a, b) \ne \sigma(c, d)$, then $(\rho_a, \rho_b) \ne (\rho_c, \rho_d)$. Otherwise, we have $\rho_a = \rho_c$ and $\rho_b = \rho_d$, which leads to $\rho(a, c) = 1$ and $\rho(b, d) = 1$. Since $\rho \le \sigma$, we have $\rho(a, c) \le \sigma(a, c) = 1$ and $\rho(b, d) \le \sigma(b, d) = 1$. That is $\sigma(a, b) = \sigma(c, d)$, which contradicts the assumption. Thus, σ/ρ is a mapping. It is a routine matter to show that σ/ρ is a a fuzzy equivalence relation on S/ρ, that is, for all $\rho_a, \rho_b, \rho_c \in S/\rho$, it satisfies the following conditions:

(1) $(\sigma/\rho)(\rho_a, \rho_a) = \sigma(a, a) = 1$;

(2) $(\sigma/\rho)(\rho_a, \rho_b) = \sigma(a, b) = \sigma(b, a) = (\sigma/\rho)(\rho_b, \rho_a)$;

(3) $(\sigma/\rho)(\rho_a, \rho_c) = \sigma(a, c) \ge \min\{\sigma(a, b), \sigma(b, c)\}$
$= \min\{(\sigma/\rho)(\rho_a, \rho_b), (\sigma/\rho)(\rho_b, \rho_c)\}$. □

Let S be an E-inversive semigroup and let $\alpha \in FE(S/\mathcal{L}^*_f)$, $\beta \in FE(S/\mathcal{R}^*_f)$. Define $\bar{\alpha}$, $\bar{\beta}$ as follow:

$$\bar{\alpha}(a, b) = \alpha((\mathcal{L}^*_f)_a, (\mathcal{L}^*_f)_b), \ \bar{\beta}(a, b) = \beta((\mathcal{R}^*_f)_a, (\mathcal{R}^*_f)_b)$$

for all $a, b \in S$. It is easy to verify that $\bar{\alpha}, \bar{\beta} \in FE(S)$, $\bar{\alpha}^0, \bar{\beta}^0 \in FRC(S)$, and $\mathcal{L}^*_f \le \bar{\alpha}, \mathcal{R}^*_f \le \bar{\beta}$. A fuzzy equivalence relation $\alpha \in FE(S/\mathcal{L}^*_f)$ ($\beta \in FE(S/\mathcal{R}^*_f)$) is called regular normal, if $\alpha = (\bar{\alpha}^0 \vee \mathcal{L}^*_f)/\mathcal{L}^*_f$ ($\beta = (\bar{\beta}^0 \vee \mathcal{R}^*_f)/\mathcal{R}^*_f$). Let ρ be a strong regular congruence on S. $(\rho \vee \mathcal{L}^*_f)/\mathcal{L}^*_f$ is called an $\mathcal{L}^*$-regular part of ρ, $(\rho \vee \mathcal{R}^*_f)/\mathcal{R}^*_f$ is called a $\mathcal{R}^*$-regular part of ρ.

Lemma 3.3 Let S be an E-inversive semigroup and $\beta \in FE(S/\mathcal{R}^*_f)$. Then β is regular normal if and only if β is $\mathcal{R}^*$-regular part of some fuzzy regular congruences on S.

Proof. If β is regular normal, choose $\bar{\beta}^0$ in $FRC(S)$ such that β is $\mathcal{R}^*$-regular part of $\bar{\beta}^0$.

Conversely, let β be a $\mathcal{R}^*$-regular part of $\rho \in FRC(S)$, that is $\beta = (\rho \vee \mathcal{R}^*_f)/\mathcal{R}^*_f$. By lemma 3.2, we have $\bar{\beta} = \rho \vee \mathcal{R}^*_f$. Clearly, $\rho \leq \bar{\beta}^0$, thus, $\rho \vee \mathcal{R}^*_f \leq \bar{\beta}^0 \vee \mathcal{R}^*_f$. On the other hand, from lemma 3.1 know that $\bar{\beta}^0 \leq \rho \vee \mathcal{R}^*_f$. Further, $\bar{\beta}^0 \vee \mathcal{R}^*_f \leq \rho \vee \mathcal{R}^*_f$. Therefore, $\beta = (\rho \vee \mathcal{R}^*_f)/\mathcal{R}^*_f = (\bar{\beta}^0 \vee \mathcal{R}^*_f)/\mathcal{R}^*_f$, and so β is regular normal. □

Lemma 3.4 Let S be an E-inversive semigroup. If $\rho \in FRC(S)$, then $\rho \vee \mathcal{L}^*_f = \rho\mathcal{L}^*_f\rho$, $\rho \vee \mathcal{R}^*_f = \rho\mathcal{R}^*_f\rho$.

Proof. It is easy to verify that $\rho\mathcal{L}^*_f\rho \leq \rho \vee \mathcal{L}^*_f$ and $\rho\mathcal{L}^*_f\rho$ is a fuzzy equivalence relation. Since $\rho \vee \mathcal{L}^*_f$ is the least fuzzy equivalence relation containing ρ and $\mathcal{L}^*_f$, thus, $\rho\mathcal{L}^*_f\rho \geq \rho \vee \mathcal{L}^*_f$, that is $\rho \vee \mathcal{L}^*_f = \rho\mathcal{L}^*_f\rho$. Similarly, $\rho \vee \mathcal{R}^*_f = \rho\mathcal{R}^*_f\rho$. □

By lemma 3.1 and lemma 2.4, we can easily obtain the following lemma.

Lemma 3.5 Let S be an E-inversive semigroup.

(1) If $\mathcal{R}^*_f(a,c) = 1$, $\mathcal{R}^*_f(b,d) = 1$ and $\rho \in FRC(S)$, then $(\rho \vee \mathcal{R}^*_f)(a,b) = (\rho \vee \mathcal{R}^*_f)(c,d)$ for any $a,b,c,d \in S$;

(2) If $\varnothing \neq \Gamma \subseteq FRC(S)$, then $((\cap_{\rho\in\Gamma}\rho) \vee \mathcal{R}^*_f)(e,f) = (\cap_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f))(e,f)$ for any $e,f \in E(S)$.

Lemma 3.6 Let S be an E-inversive semigroup. Then the mapping

$$\theta : \rho \to \rho \vee \mathcal{R}^*_f \ (\rho \in FRC(S))$$

is a complete lattice homomorphism $FRC(S)$ into $FE(S)$.

Proof. Let $\Gamma \subseteq FRC(S)$ and $a,b \in S$. It clear that $(\vee_{\rho\in\Gamma}\rho) \vee \mathcal{R}^*_f \geq \vee_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f)$.

On the other hand, since $\vee_{\rho\in\Gamma}\rho \leq \vee_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f)$, we have

$$(\vee_{\rho\in\Gamma}\rho) \vee \mathcal{R}^*_f \leq (\vee_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f)) \vee \mathcal{R}^*_f = \vee_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f).$$

Therefore

$$(\vee_{\rho\in\Gamma}\rho) \vee \mathcal{R}^*_f = \vee_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f).$$

It follows that properties of Green* relations that there exist $e,f \in E(S)$ such that $e\mathcal{R}^*a$, and $f\mathcal{R}^*b$, by lemma 3.5, we have

$$\begin{aligned}(\cap_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f))(a,b) &= (\cap_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f))(e,f)\\ &= ((\cap_{\rho\in\Gamma}\rho) \vee \mathcal{R}^*_f)(e,f)\\ &= ((\cap_{\rho\in\Gamma}\rho) \vee \mathcal{R}^*_f)(a,b),\end{aligned}$$

that is

$$(\cap_{\rho\in\Gamma}\rho) \vee \mathcal{R}^*_f = \cap_{\rho\in\Gamma}(\rho \vee \mathcal{R}^*_f).$$

We have proved that the mapping $\theta : \rho \to \rho \vee \mathcal{R}_f^*$ is a complete lattice homomorphism. □

Definition 3.7 A triple (α, A, β) consisting of regular normal fuzzy equivalence α in $FE(S/\mathcal{L}_f^*)$ and β in $FE(S/\mathcal{R}_f^*)$ and a regular normal fuzzy subset A on S, is a fuzzy strong regular congruence triple if

(1) $\bar{\alpha} = (\bar{\alpha} \cap \bar{\beta})^0 \vee \mathcal{L}_f^*, \bar{\beta} = (\bar{\alpha} \cap \bar{\beta})^0 \vee \mathcal{R}_f^*$;

(2) $A \leq K_{\alpha^0}, \bar{\alpha} \leq \xi_A \vee \mathcal{L}_f^*$;

(3) $A \leq K_{\beta^0}, \bar{\beta} \leq \xi_A \vee \mathcal{R}_f^*$.

If this is the case, we define

$$\rho_{(\alpha,A,\beta)} = \xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0.$$

Lemma 3.8 Let S be an E-inversive semigroup. If $\rho \in FRC(S)$, then

$$\rho = \xi_{K_\rho} \cap (\rho \vee \mathcal{L}_f^*)^0 \cap (\rho \vee \mathcal{R}_f^*)^0.$$

Proof. It is clear that

$$\begin{aligned}\rho(a,b) &\leq (\xi_{K_\rho} \cap (\rho \vee \mathcal{L}_f^*)^0 \cap (\rho \vee \mathcal{R}_f^*)^0)(a,b) \\ &= \min\{\xi_{K_\rho}(a,b), (\rho \vee \mathcal{L}_f^*)^0(a,b), (\rho \vee \mathcal{R}_f^*)^0(a,b)\}\end{aligned}$$

for any $a, b \in S$, $\rho \in FRC(S)$.

On the other hand, let $r = \min\{(\rho \vee \mathcal{L}_f^*)^0(a,b), (\rho \vee \mathcal{R}_f^*)^0(a,b), \xi_{K_\rho}(a,b)\}$. If $r = 0$, then $\rho(a,b) = 0$. Suppose that $r > 0$, from lemma 3.4, for any $\varepsilon > 0 (\varepsilon < r)$, if $t = r - \varepsilon$, then there exist $g, h \in S$ such that $\min\{\rho(a,g), \mathcal{R}_f^*(g,h), \rho(h,b)\} > t$, thus, $(a\rho^\ell)\mathcal{R}_\tau^*(b\rho^\ell)$. Similarly, we have $(a\rho^\ell)\mathcal{L}_\tau^*(b\rho^\ell)$. Therefore, $(a\rho^\ell)(\mathcal{L}_\tau^* \cap \mathcal{R}_\tau^*)(b\rho^\ell)$. Since $K_\rho(ab') \geq \xi_{K_\rho}(ab', bb') \geq \xi_{K_\rho}(a,b) > t$, where $b' \in W(b)$, then there exists $e = bb' \in E(S)$ such that $\rho(ab', e) > t$. Thus, $ab' \in ker\rho^\ell$. It follows from lemma 2.5 that $\rho(a,b) > t = r - \varepsilon$. We obtain $\rho(a,b) \geq r$, by making use of arbitrariness of ε. Thus, $\rho(a,b) = r$. It means

$$\rho = \xi_{K_\rho} \cap (\rho \vee \mathcal{L}_f^*)^0 \cap (\rho \vee \mathcal{R}_f^*)^0.$$

□

Theorem 3.9 Let S be an E-inversive semigroup. If (α, A, β) is a fuzzy strong regular congruence triple, then $\rho_{(\alpha,A,\beta)}$ is a unique fuzzy strong regular congruence ρ on S such that α is the $\mathcal{L}^*$-regular part of ρ, $A = K_{\rho_{(\alpha,A,\beta)}}$ and β is the $\mathcal{R}^*$-regular part of ρ.

Conversely, if ρ is a fuzzy strong regular congruence on S, then $(\alpha, A, \beta) = ((\rho \vee \mathcal{L}_f^*)/\mathcal{L}_f^*, K_\rho, (\rho \vee \mathcal{R}_f^*)/\mathcal{R}_f^*)$ is a fuzzy strong regular congruence triple for S and $\rho = \rho_{(\alpha,A,\beta)}$.

Proof. Let (α, A, β) be a fuzzy strong regular congruence triple. Then

$$\begin{aligned}\rho_{(\alpha,A,\beta)}(a, aa'a) &= (\xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0)(a, aa'a)\\ &= (\xi_A \cap \bar{\alpha}^0 \cap \bar{\beta}^0)(a, aa'a)\\ &= \inf\{\xi_A(a, aa'a), \bar{\alpha}^0(a, aa'a), \bar{\beta}^0(a, aa'a)\}\\ &= 1\end{aligned}$$

for any $a \in S$, $a' \in W(a)$. This shows that $\rho_{(\alpha,A,\beta)}$ is a fuzzy strong regular congruence.

By lemma 2.7 and definition 3.7, for all $x \in S$, we have

$$\begin{aligned}K_{\rho_{(\alpha,A,\beta)}}(x) &= K_{(\xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0)}(x)\\ &= \sup\{(\xi_A \cap \bar{\alpha}^0 \cap \bar{\beta}^0)(x, e) | e \in E(S)\}\\ &= K_{\xi_A}(x) \cap K_{\bar{\alpha}^0}(x) \cap K_{\bar{\beta}^0}(x)\\ &= A(x) \cap K_{\bar{\alpha}^0}(x) \cap K_{\bar{\beta}^0}(x)\\ &= A(x).\end{aligned}$$

Further

$$\begin{aligned}\rho_{(\alpha,A,\beta)} \vee \mathcal{R}_f^* &= (\xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0) \vee \mathcal{R}_f^*\\ &= (\xi_A \vee \mathcal{R}_f^*) \cap ((\bar{\alpha} \cap \bar{\beta})^0 \vee \mathcal{R}_f^*)\\ &= (\xi_A \vee \mathcal{R}_f^*) \cap \bar{\beta}\\ &= \bar{\beta},\end{aligned}$$

and thus

$$(\rho_{(\alpha,A,\beta)} \vee \mathcal{R}_f^*)/\mathcal{R}_f^* = \beta$$

is the $\mathcal{R}^*$-regular part of $\rho_{(\alpha,A,\beta)}$. Dually, α is the $\mathcal{L}^*$-regular part of $\rho_{(\alpha,A,\beta)}$.

Let $\rho \in FRC(S)$ be such that α is the $\mathcal{L}^*$-regular part of ρ, $A = K_\rho$ and β is the $\mathcal{R}^*$-regular part of ρ. Then $K_\rho = A = K_{\rho_{(\alpha,A,\beta)}}$. By lemma 3.8, we have

$$\begin{aligned}\rho &= \xi_{K_\rho} \cap (\rho \vee \mathcal{L}_f^*)^0 \cap (\rho \vee \mathcal{R}_f^*)^0\\ &= \xi_A \cap \bar{\alpha}^0 \cap \bar{\beta}^0\\ &= \xi_A \cap (\bar{\alpha} \cap \bar{\beta})^0\\ &= \rho_{(\alpha,A,\beta)}.\end{aligned}$$

Conversely, let ρ in $FRC(S)$ and let $\alpha = (\rho \vee \mathcal{L}_f^*)/\mathcal{L}_f^*$, $A = K_\rho$ and $\beta = (\rho \vee \mathcal{R}_f^*)/\mathcal{R}_f^*$. By lemma 3.3 and its dual α and β are regular normal fuzzy equivalences on $FE(S/\mathcal{L}_f^*)$ and $FE(S/\mathcal{R}_f^*)$, respectively. Thus, A is a regular normal fuzzy subset on S.

We note that $\bar{\alpha} = \rho \vee \mathcal{L}_f^*$ and $\bar{\beta} = \rho \vee \mathcal{R}_f^*$. Since $\mathcal{R}_f^* \leq \bar{\beta}$, it follows that $(\bar{\alpha} \cap \bar{\beta})^0 \vee \mathcal{R}_f^* \leq \bar{\beta}$. Further, since $\rho \leq (\bar{\alpha} \cap \bar{\beta})^0$, we have $\bar{\beta} = \rho \vee \mathcal{R}_f^* \leq (\bar{\alpha} \cap \bar{\beta})^0 \vee \mathcal{R}_f^*$. Therefore $\bar{\beta} = (\bar{\alpha} \cap \bar{\beta})^0 \vee \mathcal{R}_f^*$ and Dually, $\bar{\alpha} = (\bar{\alpha} \cap \bar{\beta})^0 \vee \mathcal{L}_f^*$.

$K_\rho = A$ gives $\rho \leq \xi_A$ whence $\bar{\beta} = \rho \vee \mathcal{R}_f^* \leq \xi_A \vee \mathcal{R}_f^*$ and dually, $\bar{\alpha} = \rho \vee \mathcal{L}_f^* \leq \xi_A \vee \mathcal{L}_f^*$. We have proved that (α, A, β) is a fuzzy strong regular congruence triple. From the first part of the proof we may now conclude that $\rho = \rho_{(\alpha,A,\beta)}$. □

By lemma 3.6, we can easily obtain the following corollary.

Corollary 3.10 Let $FRCT(S)$ be the poset of all fuzzy strong regular congruence triples for an E-inversive semigroup S under the partial order given by

$$(\alpha, A, \beta) \leq (\alpha', A', \beta') \Leftrightarrow \alpha \leq \alpha', A \leq A', \beta \leq \beta'.$$

Then the mappings

$$\rho \to ((\rho \vee \mathcal{L}_f^*)/\mathcal{L}_f^*, K_\rho, (\rho \vee \mathcal{R}_f^*)/\mathcal{R}_f^*), (\alpha, A, \beta) \to \rho_{(\alpha,A,\beta)}$$

are mutually inverse isomorphisms of $FRC(S)$ and $FRCT(S)$.

References

[1] A.Fawzi, Al-Thukair, Fuzzy Congruence Pairs of Inverse Semigroup, Fuzzy Sets and Systems, 56 (1993), 117-122.

[2] F.Pastijn, M.Petrich, Congruences on Regular Semigroups. Trans Amer-Math. Soc, 295 (1986), 607-633.

[3] Li Yonghua, Xu Chengxian, Fuzzy Congruence Triples for a Regular Semigroup. Journal of xi'an jiaotong university, 36 (2002), 418-421.

[4] Luo Yanfeng, Li Xiaoling, The Maximum Idempotent Separating Congruence on Eventually Regular Semigroups. Semigroup Forum, 74(2) (2007), 306-318.

[5] Luo Yanfeng, Fan Xingkui, Li Xiaoling, Regular Congruences on E-inversive Semigroup. Semigroup Forum, 76 (2008), 107-123.

[6] Ma Yunfeng, Tian Zhenji, Fuzzy Congruence on Completely Simple Semigroups. International Mathematical Forum, 64 (2011), 3199-3204.

[7] V.Murali, Fuzzy Equaivalence Relations, Fuzzy Sets and Systems, 87 (1989), 155-163.

[8] Wang Xijian, Fuzzy Congruence on Completely 0-simple Semigroups. Fuzzy Systems and Mathematics, 21(2) (2007), 65-71.

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