Left $R$-prime $(R, S)$-submodules

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Abstract

We introduced $(R, S)$-modules as a generalization of bimodule structures. Moreover, we presented the ways to study primalities of $(R, S)$-submodules, which we called fully prime and jointly prime $(R, S)$-submodules. In this paper, we define the third way to study primality of $(R, S)$-submodules, which we call left $R$-prime $(R, S)$-submodules. Characterizations and some properties of left $R$-prime submodules are also investigated.

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1 Preliminary

Throughout this paper, let $R$ and $S$ be rings and $M$ an abelian group.

Definition 1.1. [1] Let $R$ and $S$ be rings and $M$ an abelian group under addition. We say that $M$ is an $(R, S)$-module if there is a function $+: R \times M \times S \rightarrow M$ satisfying the following properties: for all $r, r_1, r_2 \in R$, $m, n \in M$ and $s, s_1, s_2 \in S$,

(i) $r \cdot (m + n) \cdot s = r \cdot m \cdot s + r \cdot n \cdot s$

(ii) $(r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s$

(iii) $r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2$
(iv) \( r_1 \cdot (r_2 \cdot m \cdot s_1) \cdot s_2 = (r_1 r_2) \cdot m \cdot (s_1 s_2) \).

We usually abbreviate \( r \cdot m \cdot s \) by \( rns \). We may also say that \( M \) is an \((R, S)\)-module under + and \( \cdot \).

An \((R, S)\)-submodule of an \((R, S)\)-module \( M \) is a subgroup \( N \) of \( M \) such that \( rns \in N \) for all \( r \in R \), \( n \in N \) and \( s \in S \).

One can see that \((R, S)\)-modules are a generalization of modules. We introduced this and extended the notion of prime submodules to prime \((R, S)\)-submodules. However, there are many choices to study the concept of prime \((R, S)\)-submodules. In [1], we studied two possible ways to define prime \((R, S)\)-submodules, namely, fully prime \((R, S)\)-submodules and jointly prime \((R, S)\)-submodules. In the studying of primality for \((R, S)\)-submodules, we obtain various properties in the same way of prime ideal for a ring. These results inspire us to looking for other \((R, S)\)-submodules which more general. That is the origin of the third way to study the concept of primality for \((R, S)\)-submodules which we call left \( R \)-prime.

**Definition 1.2.** [1] Let \( M \) be an \((R, S)\)-module. A proper \((R, S)\)-submodule \( P \) of \( M \) is called **fully prime** if for each left ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \((R, S)\)-submodule \( N \) of \( M \),

\[
\text{INJ} \subseteq P \text{ implies } \text{IMS} \subseteq P \text{ or } N \subseteq P \text{ or } \text{RMJ} \subseteq P.
\]

A proper \((R, S)\)-submodule \( P \) of \( M \) is called **jointly prime** if for each left ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \((R, S)\)-submodule \( N \) of \( M \),

\[
\text{INJ} \subseteq P \text{ implies } \text{IMJ} \subseteq P \text{ or } N \subseteq P.
\]

A proper \((R, S)\)-submodule \( P \) of \( M \) is called **left \( R \)-prime** if for all ideals \( I \) and \( J \) of \( R \),

\[
(IJ)\text{MSS} \subseteq P \text{ implies } \text{IMS} \subseteq P \text{ or } \text{JMS} \subseteq P.
\]

Note that right \( S \)-prime \((R, S)\)-submodules can be defined and studied analogously. It is clear that all fully and jointly prime \((R, S)\)-submodules are left \( R \)-prime. The converse does not hold in general.

**Example 1.3.** For each \( r, s \in \mathbb{Z}^+ \), all jointly prime \((r\mathbb{Z}, s\mathbb{Z})\)-submodules of \( \mathbb{Z} \) are of the form \( \{0\} \), \( p\mathbb{Z} \) where \( p \) is a prime integer or \( k\mathbb{Z} \) where \( k \mid rs \). It is clear that \( \{0\} \), \( p\mathbb{Z} \) where \( p \) is a prime integer and \( k\mathbb{Z} \) where \( k \mid rs \) are left \( r\mathbb{Z} \)-prime \((r\mathbb{Z}, s\mathbb{Z})\)-submodules of \( \mathbb{Z} \). Moreover, we obtain other three forms of left \( r\mathbb{Z} \)-prime \((r\mathbb{Z}, s\mathbb{Z})\)-submodules of \( \mathbb{Z} \) which are \( rp\mathbb{Z} \) where \( p \) is a prime integer, \( p \mid r^2 \) and \( p \mid s^2 \) or \( rsp\mathbb{Z} \) where \( p \) is a prime integer, \( p \mid rs \). In particular, \( \mathbb{Z} \) is a \((2\mathbb{Z}, 2\mathbb{Z})\)-module and \( 6\mathbb{Z} \) is a proper \((2\mathbb{Z}, 2\mathbb{Z})\)-submodule of \( \mathbb{Z} \). It is easy to verify that \( 6\mathbb{Z} \) is a left \( 2\mathbb{Z} \)-prime but is not a jointly prime \((2\mathbb{Z}, 2\mathbb{Z})\)-submodule.
Example 1.4. Let $R$ be the ring of all $n \times n$ matrices over a division ring. Then $R$ has no proper ideals. Moreover, $R$ is an $(R, R)$-module and $0$ is a left $R$-prime submodule of $R$.

Definition 1.5. [1] Let $R$ and $S$ be rings and $M$ an $(R, S)$-module. Then $M$ is called a left multiplication $(R, S)$-module provided that for each $(R, S)$-submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IMS$.

For each $(R, S)$-submodule $N$ of an $(R, S)$-module $M$, the set $(N : M)_R = \{ r \in R \mid rMS \subseteq N \}$ is an ideal of $R$ if $RMS = M$. First of all, we improve the characterization of left multiplication $(R, S)$-modules as follows (compare to Proposition 3.2 (iv) in [1]). We prove this without assuming that $S^2 = S$.

Proposition 1.6. Let $M$ be an $(R, S)$-module. Then $M$ is a left multiplication $(R, S)$-module if and only if $N = (N : M)_RMS$ for all $(R, S)$-submodule $N$ of $M$.

Proof. ($\Rightarrow$) Assume that $M$ is a left multiplication $(R, S)$-module and let $N$ be an $(R, S)$-submodule of $M$. Then $N = IMS$ for some ideal $I$ of $R$. Clearly, $I \subseteq (N : M)_R$. This implies that $N = IMS \subseteq (N : M)_RMS \subseteq N$. Hence $N = (N : M)_RMS$.

($\Leftarrow$) Assume that $N = (N : M)_RMS$ for all $(R, S)$-submodule $N$ of $M$. This implies that $RMS = M$. By Proposition 3.2 (ii) in [1], $(N : M)_R$ is an ideal of $R$. Hence $M$ is a left multiplication $(R, S)$-module. 

Proposition 1.6 yields that if $M$ is a left multiplication $(R, S)$-module, then $M = RMS$. The contrapositive of this fact helps us to study when a $(bZ, cZ)$-module $aZ$ is a left multiplication $(bZ, cZ)$-module. The result is stated as follows.

Proposition 1.7. Let $a, b, c \in Z^+_0$. Then $aZ$ is a left multiplication $(bZ, cZ)$-module if and only if $b = c = 1$.

Definition 1.8. [1] Let $N$ and $K$ be $(R, S)$-submodules of a left multiplication $(R, S)$-module of $M$. The product of $N$ and $K$, denoted by $NK$, is defined by

$$(N : M)_R(K : M)_R MSS.$$

Let $a \in Z^+_0$. Then $aZ$ is a left multiplication $(Z, Z)$-module and all $(Z, Z)$-submodules of $aZ$ are of the form $akZ$ for some integer $k$. It can be shown that $(akZ : aZ)_Z = kZ$ for all $k \in Z$. Moreover, for each $k_1, k_2 \in Z$, the product of $(Z, Z)$-submodules $ak_1Z$ and $ak_2Z$ of $aZ$ is

$$(ak_1Z)(ak_2Z) = (ak_1Z : aZ)_Z(ak_2Z : aZ)_Z(aZ)ZZ = (k_1Z)(k_2Z)(aZ) = ak_1k_2Z.$$
2 Left $R$-prime $(R, S)$-submodules

The third way to study prime $(R, S)$-submodules is to give an extreme importance to one of the rings $R$ and $S$. Without loss of generality, we focus on the ring “$R$”. Recall that left $R$-prime $(R, S)$-submodules are considered as a generalization of both fully and jointly prime $(R, S)$-submodules.

The following Lemma is a major tool in order to characterize left $R$-prime $(R, S)$-submodules. Note that $(X)_l$, $(X)_t$ and $(X)_r$ is the two-sided ideal generated by $X$, the left ideal generated by $X$ and the right ideal generated by $X$, respectively, for any subset $X$ of a ring $R$.

**Lemma 2.1.** Let $P$ be a proper $(R, S)$-submodule of $M$.

(i) Let $A$ and $B$ be left (right) ideals of $R$. Then $[(A)_t(B)_t]_{MSS} \subseteq P$ if $(AB)_{MS} \subseteq P$. Moreover, if $S^2 = S$, then $(AB)_{MS} \subseteq P$ if and only if $[(A)_t(B)_t]_{MS} \subseteq P$.

(ii) Let $A$ and $B$ be right ideals of $R$. Then $[(A)_t(B)_l]_{MSS} \subseteq P$ if $(AB)_{MS} \subseteq P$. Moreover, if $S^2 = S$, then $(AB)_{MS} \subseteq P$ if and only if $[(A)_l(B)_l]_{MS} \subseteq P$.

(iii) Let $A$ be a right ideal of $R$ and $B$ a left ideal of $R$. Then $[A(B)_r]_{MSS} \subseteq P$ if $(AB)_{MS} \subseteq P$. Moreover, if $S^2 = S$, then $(AB)_{MS} \subseteq P$ if and only if $[A(B)_r]_{MS} \subseteq P$.

**Proof.** (i) Let $A$ and $B$ be left ideals of $R$. Assume that $(AB)_{MS} \subseteq P$. Note that $(A)_t = A + AR$ and $(B)_t = B + BR$. Then

$$[(A)_t(B)_t]_{MSS} = (A + AR)(B + BR)_{MSS}$$

$$\subseteq (AB + ABR)_{MSS}$$

$$\subseteq (AB)_{MSS} + (ABR)_{MSS}$$

$$\subseteq (AB)_{MS}$$

$$\subseteq P.$$

Next, if $S^2 = S$, then $(AB)_{MS} \subseteq [(A)_t(B)_t]_{MS}$.

In the rest case, let $A$ and $B$ be right ideals of $R$. Assume that $(AB)_{MS} \subseteq P$. Note that $(A)_r = A + RA$ and $(B)_r = B + RB$. Then

$$[(A)_t(B)_l]_{MSS} = (A + RA)(B + RB)_{MSS}$$

$$\subseteq (AB + RAB)_{MSS}$$

$$\subseteq (AB)_{MSS} + (RAB)_{MSS}$$

$$\subseteq (AB)_{MS} + R(AB_{MS})S$$

$$\subseteq P + RPS$$

$$\subseteq P.$$
Left $R$-prime $(R, S)$-submodules

Next, if $S^2 = S$, then $(AB)MS \subseteq [(A)_t(B)_s]MS$.

(ii) Let $A$ and $B$ be right ideals of $R$. Assume that $(AB)MS \subseteq P$. Then $(A)_t = (A)_t$ and $(B)_t = (B)_t$, so this part follows from (i).

(iii) Assume that $(AB)MS \subseteq P$. Note that $(B)_r = B + BR$. Then

$$[A(B)_r]MSS = A(B + BR)MSS$$
$$\subseteq (AB + ABR)MSS$$
$$\subseteq P.$$

Next, if $S^2 = S$, then $(AB)MS \subseteq [A(B)_r]MS$. $\Box$

Applying Lemma 2.1 yields the characterization of left $R$-prime $(R, S)$-submodules.

**Theorem 2.2.** Let $M$ be an $(R, S)$-module such that $S^2 = S$ and $P$ a proper $(R, S)$-submodule of $M$. The following statements are equivalent:

1. $P$ is left $R$-prime.
2. For all left ideals $I$ and $J$ of $R$,
   
   $$(IJ)MS \subseteq P \text{ implies } IMS \subseteq P \text{ or } JMS \subseteq P.$$

3. For all right ideals $I$ and $J$ of $R$,
   
   $$(IJ)MS \subseteq P \text{ implies } IMS \subseteq P \text{ or } JMS \subseteq P.$$

4. For all right ideal $I$ and left ideal $J$ of $R$,
   
   $$(IJ)MS \subseteq P \text{ implies } IMS \subseteq P \text{ or } JMS \subseteq P.$$

**Proof.** This follows from Lemma 2.1. $\Box$

We seem to have forgotten one equivalent statement in the above theorem: the case where $I$ is a left ideal and $J$ is a right ideal. In fact we have not forgotten anything. As the following example shows, this case is not equivalent to the others. As an aside we give an example providing that $0$ in Example 1.4 is left $R$-prime but there are a left ideal $I$ of $R$ and a right ideal $J$ of $R$ such that $(IJ)MR = 0$ but $IMR \neq 0$ and $JMR \neq 0$.

**Example 2.3.** Let $I = \left(\begin{array}{cc}1 & 0 \\ 0 & 0\end{array}\right)_l$ and $J = \left(\begin{array}{c}0 \\ 1 & 0 \end{array}\right)_r$.

It is easy to show that $(IJ)MR = 0$ but $IMR \neq 0$ and $JMR \neq 0$.  

**Corollary 2.4.** Let $M$ be an $(R,S)$-module such that $S^2 = S$ and $P$ a proper $(R,S)$-submodule of $M$. The following statements are equivalent:

(i) $P$ is left $R$-prime.

(ii) For all $a, b \in R$, $(a)_l(b)_l MS \subseteq P$ implies $aMS \subseteq P$ or $bMS \subseteq P$.

(iii) For all $a, b \in R$, $(a)_r(b)_r MS \subseteq P$ implies $aMS \subseteq P$ or $bMS \subseteq P$.

(iv) For all $a, b \in R$, $(a)_r(b)_l MS \subseteq P$ implies $aMS \subseteq P$ or $bMS \subseteq P$.

**Proof.** This follows from Theorem 2.2

**Theorem 2.5.** Let $M$ be an $(R,S)$-module such that $a \in RaS$ for all $a \in M$ and $P$ a proper $(R,S)$-submodule of $M$. The following statements are equivalent:

(i) $P$ is left $R$-prime.

(ii) For all left ideal $I$ and right ideal $J$ of $R$,

\[(IRJ)MS \subseteq P \implies IMS \subseteq P \text{ or } JMS \subseteq P.\]

(iii) For all $a, b \in R$,

\[(a)_r R(b)_l MS \subseteq P \implies aMS \subseteq P \text{ or } bMS \subseteq P.\]

**Proof.** (i) $\rightarrow$ (ii) Assume (i). Let $I$ and $J$ be a left ideal of $R$ and a right ideal of $R$, respectively, such that $(IRJ)MS \subseteq P$. Then $(IR)(RJ)MS \subseteq P$. Since $IR$ and $RJ$ are ideals of $R$ and $P$ is left $R$-prime, $(IR)MS \subseteq P$ or $(RJ)MS \subseteq P$. By Proposition 2.1 in [1], $IMS \subseteq P$ or $JMS \subseteq P$.

(ii) $\rightarrow$ (iii) This is obvious.

(iii) $\rightarrow$ (i) Assume (iii). Let $I$ and $J$ be ideals of $R$ such that $IJMSS \subseteq P$. Assume that $JMS \notin P$. Let $b \in J$ be such that $bMS \notin P$. Then $(b)_r MS \notin P$. Let $a \in I$. Then $(a)_l R(b)_r MSS \subseteq (IR)MSS \subseteq (IJ)MSS \subseteq P$. By (iii), $aMS \subseteq P$. This shows that $aMS \subseteq P$ for all $a \in R$. Hence $IMS \subseteq P$. Therefore $P$ is a left $R$-prime $(R,S)$-submodule of $M$. □

Characterizations of left $R$-prime $(R,S)$-submodules of left multiplication $(R,S)$-modules are also given. Compare the following results to [3] and [4].

**Theorem 2.6.** Let $P$ be a proper $(R,S)$-submodule of a left multiplication $(R,S)$-module $M$. If $P$ is a left $R$-prime $(R,S)$-submodule of $M$, then for each $(R,S)$-submodules $U$ and $V$ of $M$,

\[UV \subseteq P \implies U \subseteq P \text{ or } V \subseteq P.\] (1)

Furthermore, if $R$ is commutative, then the converse is also true: If the condition (1) holds, then $P$ is left $R$-prime.
Proof. Straightforward. □

Let \( M \) be an \((R, S)\)-module and \( a \in M \). Then \( \langle a \rangle \) is an \((R, S)\)-submodule of \( M \) generated by \( a \).

**Corollary 2.7.** Let \( P \) be a proper \((R, S)\)-submodule of a left multiplication \((R, S)\)-module \( M \). If \( P \) is a left \( R \)-prime \((R, S)\)-submodule of \( M \), then for each \( a, b \in M \),

\[
\langle a \rangle \langle b \rangle \subseteq P \text{ implies } a \in P \text{ or } b \in P.
\]

Furthermore, if \( R \) is commutative, then the converse is true as well: If the condition (2) holds, then \( P \) is left \( R \)-prime.

**Corollary 2.8.** Let \( P \) be a proper \((R, S)\)-submodule of a left multiplication \((R, S)\)-module \( M \) and \( S^2 = S \). If \( R \) is commutative, then the following statements are equivalent:

(i) \( P \) is a fully prime \((R, S)\)-submodule.

(ii) \( P \) is a jointly prime \((R, S)\)-submodule.

(iii) \( P \) is a left \( R \)-prime \((R, S)\)-submodule.

(iv) \( (P : M)_R \) is a prime ideal of \( R \).

The result of Proposition 1.7, Theorem 2.6 and the method of products of \((R, S)\)-submodule give a characterization of left \( \mathbb{Z} \)-prime \((\mathbb{Z}, \mathbb{Z})\)-submodule of \( \mathbb{Z} \) where \( a \in \mathbb{Z}^+ \) as follow.

**Proposition 2.9.** Let \( a \in \mathbb{Z}^+ \). For each \( p \in \mathbb{Z}_0^+ \setminus \{1\} \), \( ap\mathbb{Z} \) is a left \( \mathbb{Z} \)-prime \((\mathbb{Z}, \mathbb{Z})\)-submodule of \( a\mathbb{Z} \) if and only if \( p = 0 \) or \( p \) is a prime integer.

In a studying of ring theory, a subset of a ring is called **multiplicatively closed** if it is closed under multiplication. For commutative rings, the complement of a prime ideal is an especially important example of a multiplicatively closed set. In [2], an ideal \( P \) of a commutative ring \( R \) is prime if and only if the complement \( R \setminus P \) is multiplicatively closed. In this paper, we introduce the concept of closed sets of a left multiplication \((R, S)\)-module and give a characterization of left \( R \)-prime \((R, S)\)-submodules in the term of closed set.

For each nonempty subset \( C \) of a left multiplication \((R, S)\)-module, we call \( C \) a **closed set** if \( \langle a \rangle \langle b \rangle \cap C \neq \emptyset \) for all \( a, b \in C \).

**Theorem 2.10.** Let \( P \) be a proper \((R, S)\)-submodule of a left multiplication \((R, S)\)-module \( M \). If \( P \) is left \( R \)-prime, then \( M \setminus P \) is a closed set. Moreover, if \( R \) is commutative, then \( P \) is left \( R \)-prime if and only if \( M \setminus P \) is a closed set.
Proof. Firstly, assume that \( P \) is left \( R \)-prime and let \( a, b \in M \setminus P \). Then \( \langle a \rangle \langle b \rangle \not\subseteq P \). Hence \( \langle a \rangle \langle b \rangle \cap (M \setminus P) \neq \emptyset \).
Secondly, assume that \( R \) is commutative and \( M \setminus P \) is a closed set. We prove that \( P \) is left \( R \)-prime by showing that the condition (2) holds. Let \( a, b \in M \) be such that \( a \notin P \) and \( b \notin P \). Since \( M \setminus P \) is a closed set, \( \langle a \rangle \langle b \rangle \cap (M \setminus P) \neq \emptyset \).
Hence \( \langle a \rangle \langle b \rangle \not\subseteq P \).

**Theorem 2.11.** Let \( A \) be an \((R, S)\)-submodule of a left multiplication \((R, S)\)-module \( M \) and let \( C \) be a closed set in \( M \) such that \( A \cap C = \emptyset \). Then there exists an \((R, S)\)-submodule \( K \) of \( M \) which is maximal with respect to the property that \( A \subseteq K \) and \( K \cap C = \emptyset \).
Furthermore, if \( R \) is commutative, then \( K \) is a left \( R \)-prime \((R, S)\)-submodule of \( M \).

Proof. Let \( \Omega \) be the set of all \((R, S)\)-submodules \( B \) of \( M \) such that \( A \subseteq B \) and \( B \cap C = \emptyset \). We can see that \( A \in \Omega \). The Zorn’s Lemma provides that \( \Omega \) has a maximal element, say \( K \). Note that \( K \neq M \).
Assume that \( R \) is commutative. Suppose for contradiction that \( K \) is not left \( R \)-prime. Then \( M \setminus K \) is not closed set. Let \( a, b \in M \setminus K \) be such that \( \langle a \rangle \langle b \rangle \cap (M \setminus K) = \emptyset \). Then \( \langle a \rangle \langle b \rangle \subseteq K \). Since \( K \) is maximal in \( \Omega \), \( K + \langle a \rangle \) and \( K + \langle b \rangle \) are not in \( \Omega \). There are \( s, t \in C \) such that \( s \in K + \langle a \rangle \) and \( t \in K + \langle b \rangle \).
Since \( C \) is a closed set, \( \langle s \rangle \langle t \rangle \cap C \neq \emptyset \). Hence \( \langle s \rangle \langle t \rangle \subseteq (K + \langle a \rangle) (K + \langle b \rangle) \subseteq K \).
Therefore \( K \cap C \neq \emptyset \) which is a contradiction.

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**References**


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