Domination Number of Jump Graph

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Abstract

A set $D \subseteq V[J(G)]$ is dominating set of jump graph, if every vertex not in $D$ is adjacent to a vertex in $D$. The domination number of the jump graph is the minimum cardinality of dominating set of jump graph $J(G)$. We also study the graph theoretic properties of $\gamma[J(G)]$ and its exact values for some standard graphs. The relation between $\gamma[J(G)]$ with other parameters are also investigated.

Mathematics Subject Classification: 05C69, 05C70, 05C76

Keywords: Diameter, domination number, jump graph
1 Introduction

Let $G(p, q)$ be a graph with $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph $G$, respectively. All the graphs considered here are finite, non-trivial, undirected and connected without loops or multiple edges.

In general, the degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and it is denoted by $deg v$. The maximum (minimum) degree among the vertices of $G$ is denoted by $\Delta(G)$ $(\delta(G))$. We denote the minimum number of edges in edge cover of $G$ (i.e., edge cover number) by $\alpha_1(G)$ and the minimum number of edges in independent set of edges of $G$ (i.e., edge independent set) by $\beta_1(G)$. The subgraph induced by $X \subseteq V$ is denoted by $\langle X \rangle$.

A vertex of degree one is called a pendant vertex. A vertex adjacent to pendant vertex is called the support vertex. The maximum $d(u, v)$ for all $u$ in $G$ is eccentricity of $v$ and the maximum eccentricity is the diameter $\text{diam}(G)$. As usual, $P_n$, $C_n$, and $K_n$, are respectively, the path, cycle and complete graph of order $n$. $K_{r,s}$ is the complete bipartite graph with two partite sets containing $r$ and $s$ vertices. Any undefined term or notation in this paper can be found in Harary [2].

2 Preliminary Notes

The line graph $L(G)$ of $G$ has the edges of $G$ as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in $G$. We call the complement of line graph $L(G)$ as the jump graph $J(G)$ of $G$, found in [1]. The jump graph $J(G)$ of a graph $G$ is the graph defined on $E(G)$ and in which two vertices are adjacent if and only if they are not adjacent in $G$. Since both $L(G)$ and $J(G)$ are defined on the edge set of a graph $G$, it follows that isolated vertices of $G$ (if $G$ has ) play no role in line graph and jump graph transformation. We assume that the graph $G$ under consideration is nonempty and has no isolated vertices [1].

**Definition 2.1** We now define the domination number of jump graph. Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is said to be a dominating set, if every vertex not in $D$ is adjacent to a vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. Analogously, a set $D \subseteq V[J(G)]$ is said to be dominating set of $J(G)$, if every vertex not in $D$ is adjacent to a vertex in $D$. The domination number of Jump graph, denoted by $\gamma[J(G)]$, is the minimum cardinality of a dominating set in $J(G)$. For any graph $G$, with $p \leq 4$, the jump graph $J(G)$ of $G$, is disconnected. Since
we study only the connected jump graph, we choose \( p > 4 \) [3].

We recall following classical theorems to prove our results.

**Theorem 2.2** [4]: Let \( G \) be a graph without isolated vertices. If \( D \) is a minimal dominating set, then \( V - D \) is a dominating set.

**Theorem 2.3** [4]: If \( G \) is a graph with no isolated vertices, then
\[
\gamma(G) \leq \frac{p}{2}
\]

**Theorem 2.4** [4]: If \( G(V, E) \) is a simple graph then
\[
2 | q | \leq | p^2 | - | p |
\]

3 Main Results

We state some preliminary result in the following theorem for some standard graphs.

**Theorem 3.1**
1. For any path \( P_p \), with \( p \geq 5 \),
\[
\gamma[J(P_p)] = 2.
\]
2. For any cycle \( C_p \), with \( p \geq 5 \),
\[
\gamma(J[C_p]) = 2.
\]
3. For any Complete graph \( K_p \), with \( p \geq 5 \),
\[
\gamma[J(K_p)] = 3.
\]
4. For any Spider, \( \gamma[J(G)] = 2 \).
5. For any Complete bipartite graph \( K_{m,n} \),
\[
\gamma[J(K_{m,n})] = \begin{cases} 2 & \text{for } K_{2,n} \text{ where } n > 2 \\ 3 & \text{for } K_{m,n} \text{ where } m,n \geq 3. \end{cases}
\]
6. For any Wheel \( W_p \),
\[
\gamma[J(W_p)] = \begin{cases} 3 & \text{for } p = 5,6 \\ 2 & \text{for } p \geq 7. \end{cases}
\]

**Theorem 3.2** For any connected graph \( G \), \( \gamma[J(G)] \geq 2 \).

Proof of the theorem is obvious.
Theorem 3.3 If $D$ is any dominating set of $J(G)$ such that $|D| = \gamma[J(G)]$, then $V[J(G)] - D \leq \sum \deg v_i$.

Proof. Since every vertex in $V[J(G)] - D$ is adjacent to at least one vertex in $D$, there will be a contribution from each vertex of $V[J(G)] - D$ by one to the sum of degrees of vertices of $D$. Hence the proof of the theorem.

Theorem 3.4 For any connected $(p, q)$ graph $G$, $\gamma[J(G)] \leq q - \beta_1(G) + 1$.

Proof. Let $V_1 = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices in $J(G)$ corresponding to the set of independent edges $\{e_1, e_2, e_3, \ldots, e_n\}$ of $G$. By the definition of $J(G)$, the elements of $V_1$ form an induced subgraph $\langle K_n \rangle$ in $J(G)$. Further, let $S \cup \{v_1\}$, where $S \subset [V[J(G)] - V_1]$, be a dominating set in $J(G)$. It follows that $|S \cup \{v_1\}| \leq |V[J(G)] - V_1| + |v_1|$. Therefore $\gamma[J(G)] \leq q - \beta_1(G) + 1$.

The following theorem gives the relationship between domination number of a graph with its jump domination number of a graph.

Theorem 3.5 For any connected $(p, q)$ graph $G$, $\gamma(G) + \gamma[J(G)] < \left(\frac{p+1}{2}\right)^2$.

Proof. For any $(p, q)$ graph $G$, we have $\gamma(G) \leq \min\{|D|, |V - D|\}$, $\leq \frac{p}{2}$, by virtue of theorem B. Further, by the definition of Jump graph, we have $V(J(G)) = q$. Hence, $\gamma[J(G)] \leq \frac{q}{2}$. But for any simple graph $G$, $q \leq \frac{p(p-1)}{2}$. and therefore, we get $\gamma[(J(G)] \leq \frac{p(p-1)}{4}$. From the above equations, we get

$$\gamma(G) + \gamma[J(G)] \leq \frac{p}{2} + \frac{p(p-1)}{4} \leq \frac{p(p+1)}{4} < \left(\frac{p+1}{2}\right)^2.$$

Theorem 3.6 For any connected graph $G$ with diameter, $\text{diam}(G) \geq 2$, $\gamma[J(G)] \geq 2$.

Proof. Let $uv$ be a path of maximum distance in $G$. Then $d(u, v) = \text{diam}(G)$. We can prove the theorem with the following cases.

Case 1. For $\text{diam}(G) = 2$, choose a vertex $v_1$ of eccentricity 2 with maximum
degree among others. Let $V_1 = \{v_1^1, v_1^2, \ldots\}$ corresponding to the elements of $\{e_1, e_2, \ldots\}$ forming a dominating set in Jump graph $J(G)$. Every vertex $u \notin V_1$ is adjacent to a vertex in $V_1$. Hence $V_1$ is a minimum dominating set. So domination number of the jump graph will be equal to the degree of $v_1$, also by theorem (3.2), we say $\gamma[J(G)] > 2$.

**Case 2.** For $diam(G) > 2$, let $e_1$ be any edge adjacent to $u$ and $e_2$ be any edge adjacent to $v$. Let $\{e_1, e_2\} \subseteq E(G)$ form a corresponding vertex set $\{v_1^1, v_1^2\} \subseteq V(J(G))$. These two vertices form a dominating set in jump graph. Since these vertices $\{v_1^1, v_1^2\}$ are adjacent to all other vertices of $V(J(G))$, it follows that $\{v_1^1, v_1^2\}$ becomes a minimum dominating set. Hence $\gamma[J(G)] = 2$.

In view of above cases, we can conclude that for any connected graph $G$, $\gamma[J(G)] \geq 2$.

Similar type of result can be proved for any tree $T$.

**Theorem 3.7** For any tree $T$ with diameter greater than 3, $\gamma[J(T)] = 2$.

**Proof.** If the diameter is less than or equal to 3, then the jump graph will be disconnected. Let $uv$ be a path of maximum length in a tree $T$ where diameter is greater than 3. Let $e_i$ be the pendant edge adjacent to $u$ and $e_k$ be the pendant edge adjacent to $v$. The vertex set $\{v_i^1, v_i^1\}$ of $J(T)$ corresponding to the edges of $\{e_i, e_k\}$ in $T$ will form the dominating set in $J(T)$. Since all the other vertices of $V[J(T)]$ are adjacent with $\{v_i^1, v_i^1\}$, it forms a minimum dominating set. Hence $\gamma[J(T)] = 2$.

**Theorem 3.8** For any connected $(p, q)$ graph $G$, $\gamma[J(G)] \leq q - \Delta(G)$ where $\Delta(G)$ is the maximum degree of $G$.

**Proof.** Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices in $G$ and let $V_1 = V - v_1$ where $v_1$ is one of the vertices with maximum degree. By definition of jump graph, $E(G) = V[J(G)]$. Consider $I = \{e_1, e_2, e_3, \ldots e_k\}$ as the set of edges adjacent to $v_1$ in $G$. Let $H \subseteq V(J(G))$ be the set of vertices of $J(G)$ such that, $H \subseteq E - I$. Then $H$ itself form a minimally dominating set. Therefore $\gamma[J(G)] \leq |E| - |I|$. Hence $\gamma[J(G)] \leq q - \Delta(G)$.

**Theorem 3.9** For any connected $(p, q)$ graph $G$, $2 \leq \gamma[J(G)] \leq \left\lceil \frac{q}{2} \right\rceil$.

**Proof.** An edge $\{e_i\}$ in any connected graph $G$ is adjacent to at least one more edge in $G$. In Jump graph, the vertex $\{v_i^1\}$ corresponding to $\{e_i\}$ is
non adjacent to \( \{v^k_i, v^l_i\} \) of \( e_k, e_i \) in \( J(G) \). Therefore by definition of domination number of graph \( \gamma(G) \), the dominating set contains at least two elements. Hence \( \gamma[J(G)] \geq 2 \). 

Let \( E \) be the set of edges in \( G \). Then \( E = V[J(G)] \). Suppose \( D = \{v_1, v_2, v_3, \ldots, v_k\} \) be the dominating set. Then \( V - D \) is also a dominating set. One among these two sets will form a minimal dominating set. So by the definition of domination number of graph, we can say domination number, \( \gamma[J(G)] \) of jump graph is given by

\[
\gamma[J(G)] \leq \min\{|D|, |V - D|\}
\]

\[
\leq \left\lfloor \frac{q}{2} \right\rfloor \tag{2}
\]

From (1) and (2), we get \( 2 \leq \gamma[J(G)] \leq \left\lfloor \frac{q}{2} \right\rfloor \).

**Theorem 3.10** For any connected graph \( G \) without pendent vertex, \( \gamma[J(G)] \leq \delta(G) \).

**Proof.** Let \( V = \{v_1, v_2, v_3, \ldots, v_n\} \) be the set of vertices in \( G \) and \( v_1 \) be one among the vertices with minimum degree. Let \( \{e_1, e_2, e_3, \ldots, e_k\} \) be the set of edges adjacent to \( v_1 \) in \( G \). Then \( E_1 \subseteq V[J(G)] \) will form the dominating set in \( J(G) \). So \( |E_1| = \delta(G) \). Obviously it becomes the minimum dominating set. Therefore \( \gamma[J(G)] \leq \delta(G) \).

**References**


**Received: January 16, 2013**