Continuity of Darboux Functions

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Abstract

In 1965, Whyburn proved that if X is locally connected and first countable, Y is Hausdorff, and f: X → Y a function, then: f is continuous iff f preserves compactness and connectedness. Definition: A function f: X → Y is preserving path connectedness (a Darboux function) if the image of any path-connected subset of X is path connected.

As an example, the derivative f’ of a real differentiable function f defined on an interval is path preserving although f’ is not always continuous. In the paper we prove the following theorem

Theorem: Suppose X is Hausdorff, locally path-connected and 1-countable, Y is Hausdorff, and f: X → Y a function. Then f is continuous iff f preserves compactness and f preserves path connectedness.

By elementary theorems if f : X → Y is a continuous function then the image of compact subspace of X is compact set, and the image of connected set is connected. In 1970 McMillan proved that if X s Hausdorff, locally connected and Frechet, Y is Hausdorff, then the converse is also true. In fact McMillan generalized Whyburn’s theorem from 1965. The last advance in this direction is made in [2] by the following result of Whyburn [5]: Suppose X is locally connected and 1-countable, Y is Hausdorff, and f : X → Y a
function. The following conditions are equivalent: 1) \( f \) is continuous 2) any image of a compact set is compact and any image of a connected set is connected.

**Definition:** A function \( f : X \to Y \) preserves path connectedness (has a property of Darboux, is a Darboux function) if the image of any path-connected subset of \( X \) is path connected.

**Remark:** The derivative \( f' \) of a real function \( f \) defined on an interval need not to be continuous. However, the derivative \( f' \) has the property of Darboux i.e. if \( d \) is a real number such that \( f(a) < d < f(b) \), then there exists a real number \( c \) between \( a \) and \( b \) such that \( f(c) = d \).

A function having a Darboux property is not always continuous. An example of such function is the function of Cesaro \( \omega : [0, 1] \to [0, 1] \) defined by

\[
\omega(x) = \limsup_n \frac{a_1 + \ldots + a_n}{n},
\]

for \( 0 \leq x \leq 1 \), and \( x = 0, a_1 \ldots a_n \) is a dyadic expression of \( x \) (if \( x \) has two expressions we take the finite expression i.e with infinite number of 0’s).

Some of the properties of Darboux functions are presented in [4].

Now, let \( X \) be locally path-connected and 1-countable. We want to show that the following conditions 1) and 2’) are also equivalent:

1) \( f \) is continuous.

2’) Image of a compact set is compact and \( f \) preserves path connectedness.

It is clear that from 2) it follows 2’). We need to show that from 2’) it follows 2). First we will show the following:

**Theorem 1.** Let \( f : X \to Y \) be a mapping and \( X \) be locally path-connected metric space. If \( f \) maps compact to compact and preserves path connectedness, then the image of any subset of \( X \) that is both compact and connected, is connected (and compact).

To prove Theorem 1, we will need the following

**Proposition 1:** Let \( (C_n) \) be a sequence of compact, path-connected sets, such that each is a subset of the previous one. Then

\[
C = \bigcap_{n=1}^{\infty} C_n.
\]

is also compact and connected

(Note: Proposition 1 also holds for connected spaces instead of path-connected spaces.)

**Example:** As an example we consider the space consisting of two parallel lines in the plane, and of a spiral \( a(t) \) starting from a point \( a(0) \) between the two lines and \( a(t) \) approaching to both lines as \( t \to \infty \). Then
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\( a([0, \infty)) \supset a([1, \infty)) \supset a([2, \infty)) \supset \ldots \) is a decreasing sequence of connected sets while their interection \( \bigcap_{n=0}^{\infty} a([n, \infty)) \) is a union of two parallel lines i.e. their intersection isn’t connected.

**Proof of Proposition 1:** \( C \) is compact as an intersection of compacta. Let us assume that \( C = A \cup B \), where \( A \) and \( B \) are separated. For two points \( a \in A \) and \( b \in B \) there exists a path \( k_n : I \to C_n \), such that \( k_n(0) = a, k_n(1) = b \). If we put

\[
Z = \bigcup_{n=1}^{\infty} k_n(I)
\]

then \( Z' \subseteq C = A \cup B \).

We define a mapping for each \( n \), \( h_n : k_n(I) \to \mathbb{R} \) in the following way:

\[
h_n(x) = d(x, A) - d(x, B)
\]

Since for each \( n \), \( h_n(a) < 0, h_n(b) > 0 \), there exists \( z_n \in k_n(I) \), so that \( h_n(z_n) = 0 \), i.e. \( d(z_n, A) = d(z_n, B) \), where \( z_n = k_n(t_n) \), \( 0 < t_n < 1 \).

There exists a convergent subsequence \( (z_{n_k}) \), such that \( z_{n_k} \to z \in Z' \subseteq A \cup B \). Because of \( d(z, A) = d(z, B) \), it follows \( z \notin A \) and \( z \notin B \), which is a contradiction.

**Proof of Theorem 1:** Let \( f : X \to Y \) be a mapping and let \( C \) be both connected and compact in \( X \). For each \( x \in C \), there exists a sequence of path-connected open neighbourhoods \( V^n_x \), each a subset of the previous one, so that

\[
\bigcap_{n=1}^{\infty} V^n_x = \{x\}
\]

Since

\[
C \subseteq \bigcup_{x \in C} V^n_x
\]

is compact, it follows that there exists a finite union

\[
\bigcup_{i=1, \ldots, p} V^n_{x_i}
\]

that covers \( C \). Let
There exists a subsequence \((U^n)\) of \((W^n)\), so that

\[ U^n \supset U^{n+1} \supset \ldots \]

It follows that

\[ \bigcap_{n=1}^{\infty} U^n = C \]

and

\[ f(C) = f\left( \bigcap_{n=1}^{\infty} U^n \right) = \bigcap_{n=1}^{\infty} f(U^n) \]

Then, \(f(C)\) is compact and also connected as an intersection of compact path-connected sets, each a subset of the previous one.

**Proposition 2.** Let \(X\) be a Hausdorff space and \( f : [0,1] \to X \) preserves path connectedness and compactness. Then \( f \) is continuous.

**Proof.** Let \(C\) be connected in \([0,1]\). It follows that \(C\) is an interval, which means \(C\) is path-connected. \(f\) is preserves path connectedness, so it follows that \(f(C)\) is path-connected, so \(f(C)\) is connected. It follows \(f\) is continuous.

**Proposition 3.** Let \( h : [0,1] \to X \) be an embedding (i.e. \( h([0,1]) \) is an arc) and let \( f : X \to Y \) be a function that preserves path connectedness and compactness. Then, the restriction \( f \mid_{h(I)} \) is continuous.

**Proof.** Let \(C \subseteq h(I)\) be connected. Then \(C\) is homeomorphic image of an interval of \(I\), and because of Proposition 2, it follows the conclusion in Proposition 3.

**Theorem 2.** Let \(X\) be Hausdorff, locally path-connected and 1-countable, \(Y\) is Hausdorff, and let \( f : X \to Y \) be a function preserving path connectedness and compactness. Then \( f \) is continuous.

**Proof.** First, by \([1]\) since \(X\) is locally path connected, it is arcwise connected. Let \(z \in X\). We will show that \( f \) is continuous by showing that the sequence of images of the members of an arbitrary sequence that converges to \(z\), would converge to \( f(z) \in Y \).

Let \((a_n)\) be a sequence, such that \(a_n \to z\) and let \(\{V_k : k \in \mathbb{N}\}\) be a family of open arcwise connected neighbourhoods of \(z\), each a subset of the previous one. We may choose an increasing sequence \((n_k)\) of natural numbers (except in the trivial case), and a subsequence \((a_{n_k})\) of \((a_n)\), so that \(a_{n_k} \in V_{n_k}\) while...
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\[ a_{n_k-1} \notin V_{n_k}. \] The set \( \{a_{n_k} : k \in \mathbb{N}\} \cup \{z\} \) is compact, and since \( f : X \to Y \) maps compact to compact, it follows that

\[ \{f(a_{n_k}) : k \in \mathbb{N}\} \cup \{f(z)\} \]

is compact.

By taking subsequences we may assume that \( f(a_{n_k}) \) converges. If \( f(a_{n_k}) \to f(z) \) then the proof is completed. Let us assume that \( f(a_{n_k}) \to w \) where \( w \neq f(z) \). Then because of the compactness of the set \( \{f(a_{n_k}) : k \in \mathbb{N}\} \cup \{f(z)\} \), it follows that \( w \in \{f(a_{n_k}) : k \in \mathbb{N}\} \) i.e. \( w = f(a_j) \).

Now, because of the properties of the neighbourhoods \( V_{n_k} \), there exists an arc \( A_k \) from \( a_{n_k} \) to \( z \). We can make the construction of the arcs \( A_k \) by induction, in the following way (Figure 1): If \( A_1, A_2, \ldots, A_{k-1} \) are constructed, we put \( A = A_1 \cup A_2 \cup \ldots \cup A_{k-1} \). If \( A^*_k \) is an arbitrary arc in \( V_{n_k} \) from \( a_{n_k} \) to \( z \), we define \( t^* = \inf\{t|h_k(t) \in A\} \) where \( h_k^* : I \to A^*_k \) is an isomorphism and we put \( b_{n_k} = h_k^*(t^*) \). Then we define the arc \( A_k \) from \( a_{n_k} \) to \( b_{n_k} \) to be the same as the arc \( A_k^* \), and from \( b_{n_k} \) to \( z \) to be identical with some of the previously constructed arcs. We define a set

\[ Q = \bigcup_{k=1}^{\infty} A_k \]

which is compact because of the compactness of the arcs and the way how neighbourhoods \( V_k \) are chosen.

Now, let \( C \) be an arbitrary connected set in \( Q \).
1) If \( C \) intersects only one arc, then it follows that \( C = C \cap h_j(I) \) is homeomorphic to an interval, so \( f(C) \) is path connected also.
2) If \( x, y \in C \) then there is a path in \( Q \) contained in \( C \) connecting \( x \) and \( y \) i.e. \( C \) is path connected. It follows that \( f(C) \) is path connected.

It follows the restriction \( f|_Q: Q \to Y \) is continuous. So, if \( W \) is a neighbourhood of \( f(z) \), such that \( f(a) \notin W \), it follows that there exists a neighbourhood \( U \) of \( z \), such that \( f|_Q(U \cap Q) \subseteq W \).

There exists \( k' \), such that for each \( k > k' \), \( a_{nk} \in U \cap Q \), and so \( f(a_{nk}) \in W \), which is a contradiction to \( f(a_{nk}) \to f(a_j) \).

**Definition:** The map \( f: X \to Y \) has a property (RD) at \( x_0 \) if there exists an \( y_0 \in Y \) such that for any neighbourhood \( V \) of \( y_0 \) there exists an neighbourhood \( U \) of \( x_0 \) such that \( f(U \setminus \{x_0\}) \subseteq V \).

If \( f: X \to Y \) has the property (RD) at \( x_0 \) and is not continuous at \( x_0 \), we say that \( f: X \to Y \) has a removable discontinuity at \( x_0 \).

**Theorem 3.** Let \( X \) be locally path-connected and 1-countable and let \( f: X \to Y \) be a function preserving path connectedness and compactness. If \( f: X \to Y \) has the property (RD) at \( x_0 \), than it is continuous at \( x_0 \).

**REFERENCES**


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