Asymptotic Distribution of Sample Variance in Skew Normal-Uniform Distribution

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Abstract

Sample variance is one of the most applicable measures in statistics discussions, among which it can be pointed out to its skewed and kurtosis measures. In this article, limited distribution of sample variance is presented in the parametric skew normal-uniform distribution.

Keywords: sample variance, skew normal-uniform distribution, U-statistics.

1 Introduction

Being a function of random variables, the sample variance is itself a random variable, and it is natural to study its distribution. In the case that random sample are from normal distribution, Cochran's theorem shows that sample variance follow a scaled chi-squared distribution [see: Knight 2000, proposition 2.11].

Skew normal distribution has attracted the attention of many researchers. The emergence of this new trend resulted in the applied research done by Arnold (1983) and Hill and Dixon (1982). Research has also been done on the basis of robust and Bayes estimation along with skew normal distribution, O'Hagan and Leonard (1976). Gupta, et al (1976) discussed and found the models which include normal, t, Cauchy, Laplace, and logistic distribution. In all of the above findings, a new parameter such as λ played a key role for the control of the extent of kurtosis and skew.

Let the density uniform function has been defined on the interval of $[-h, h]$, then the skew normal-uniform density function will be as follows:

$$f_X(x) = \begin{cases} 0 & \lambda x \leq -h \\ \frac{\lambda x + h}{\sqrt{2\pi} \sigma h} e^{\left(-\frac{x^2}{2\sigma^2}\right)} & -h \leq \lambda x \leq h \\ \frac{2}{\sqrt{2\pi} \sigma} e^{\left(-\frac{x^2}{2\sigma^2}\right)} & h \leq \lambda x \end{cases} \quad \sigma > 0, h > 0, \lambda \in R. \quad (1)$$

The parameter of σ is from normal distribution. The distribution in (1) includes the normal distribution family and its moments with the use of Gamma functions and incomplete Gamma moments which yielded the following formulas.

For even n s

$$E(X^n) = \frac{2^{n/2} \sigma^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right).$$

For odd n s

$$E(X^n) = \frac{2^{(n+1)/2} \sigma^{n+1} \lambda}{\sqrt{\pi} h} \gamma\left(\frac{n}{2} + 1, \frac{h^2}{2\sigma^2 \lambda^2}\right) + \frac{2^{n/2} \sigma^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}, \frac{h^2}{2\sigma^2 \lambda^2}\right),$$

where $\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$, $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$, and $\phi(\cdot)$ is density of normal distribution. Nang-Cheng Su (2012) showed; with the help of altered form Azalini distribution that explicit forms of its cumulative distribution function, and moments was derived. Then the corresponding method of moments estimation and the maximum likelihood estimation of location-scale skew-normal-uniform distribution are discussed. They also obtained the distribution of the sample mean and then studied control charts for the skew-normal-uniform distribution.

In the next part, we obtain the limiting distribution of sample variance from the distribution (1). Since the results show some complicated formulas, we present special cases that result in simpler forms along with the numerical results.

2 Results

In the two measures of skew and kurtosis, the role of sample variance estimator, s^2 , is very effective. First we deal with the determination of the limited distribution of this statistic. The asymptotic distribution for sample variances is

$$\sqrt{n}(S^2 - \mu_2) \sim AN(0, \mu_4 - \mu_2^2), \quad (2)$$

where μ_2 and μ_4 are the second and fourth central moment of skew normal-uniform distribution respectively. The variance of S^2 equals to

$$\begin{aligned} \text{var}[(X - E(X))^2] &= \mu_4 - \mu_2^2 \\ &= \frac{12\sigma^8 \sigma \lambda}{\sqrt{\pi} h} \left\{ 2 \Phi\left(\frac{h}{\sigma \lambda}\right) - 1 \right\} \left(\sqrt{2} e^{-\frac{1}{2\sigma^2 \lambda^2}} - 3\sqrt{\pi} \left\{ 2 \Phi\left(\frac{h}{\sigma \lambda}\right) - 1 \right\} \right. \\ &\quad \left. + 6\sigma^4 \left(\frac{\sigma \lambda}{h}\right)^2 \left\{ 2 \Phi\left(\frac{h}{\sigma \lambda}\right) - 1 \right\}^2 - 3\sigma^4 \left(\frac{\sigma \lambda}{h}\right)^4 \left\{ 2 \Phi\left(\frac{h}{\sigma \lambda}\right) - 1 \right\}^4 \right), \quad (3) \end{aligned}$$

However, in (3) by considering $\frac{h}{\sigma\lambda}$ as a new parameter such as α , the number of parameters will be reduced, but the analysis of results are done according to the varies values of skew parameter, λ , and uniform distribution parameter, h , we do not do it. Therefore, we use the $var[(X - E(X))^2] = k(h, \lambda, \sigma)$. Now, with changing of the distribution parameters of (1), we show the results which are obtained from their relationships.

- 1- $k(\sigma, \lambda, \lambda, \sigma) = 0.873487723 \sigma^4$,
 - 2- $k(\lambda, \lambda, \sigma) = \frac{\sigma^4}{\sqrt{\pi}} \left(3\sqrt{\pi} + 4\sqrt{2} \sigma e^{-\frac{1}{2\sigma^2}} M - 6\sigma^2 \sqrt{\pi} M^2 - 3\sqrt{\pi} \sigma^4 M^4 \right)$,
 - 3- $\lim_{\lambda \rightarrow \infty} k(h, \lambda, \sigma) = \sigma^4 \left(3 + \frac{4}{\pi} - \frac{12}{\pi^2} \right)$,
 - 4- $\lim_{h \rightarrow \infty} k(h, \lambda, \sigma) = 3\sigma^4$,
- where $M = erf\left(\frac{1}{2} \frac{\sqrt{2}}{\sigma}\right)$.
- 5- By applying $\lambda = \frac{i}{\sigma}$; $i = 1, 2, \dots$, the phrase $\frac{k(h, \lambda, \sigma)}{\sigma^4}$ will be depended on the h and i . The results will be shown in table (1).

Table 1: The values of $\frac{k(h, \lambda, \sigma)}{\sigma^4}$ under varies of h and i

$i \downarrow h \rightarrow$	1	3	5	7	9
1	1.68389	2.61359	2.83101	2.90551	2.93970
3	0.67761	1.12447	1.68389	2.14552	2.44040
5	0.57202	0.74810	1.01890	1.34793	1.68389
7	0.54225	0.63429	0.78163	0.97559	1.20291
9	0.52991	0.58617	0.67761	0.80107	0.95210

In order to show the validation of above results, we have made a simulation according to distribution (1). Table (2) shows the differences in the amount statistics of Shapiro-Wilk test (SW). The behavior of asymptotic distribution of sample variance by different amounts of the parameters and the size of sample are shown. Skew parameter can change the judgment about limited distribution with respect to the parameter gained from the uniform distribution.

Table 2: The values of SW test of asymptotic distribution of S^2

Size sample	Parameters	Sample Variance
$n = 50$	$\lambda = 8$	sample mean = 37.060
	$\sigma = 10$	sd = 8.774
	$h = 5$	SW = 0.9293
		pvalue = 0.2637
$n = 25$	$\lambda = 8$	sample mean = 44.433
	$\sigma = 10$	sd = 15.07
	$h = 5$	SW = 0.9105
		pvalue = 0.1385
$n = 25$	$\lambda = 1$	sample mean = 3.761
	$\sigma = 2$	sd = 1.1034
	$h = 10$	SW = 0.99
		pvalue = 0.997

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