Consistent FE-Analysis of Strang's Problem Based on Mixed Formulations

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Abstract

In this paper our studies on techniques for a posteriori error control and adaptive mesh design for finite element models in perfect plasticity are continued. The focus is on the numerical analysis of a low-order, dual-mixed discretisation. A posteriori error estimates are provided. Numerical tests confirm the theoretical results.

Keywords: adaptive finite elements, a posteriori error-bounds, mixed methods, variational inequality, anti-plane shear problem

1 Introduction

The work at hand is devoted to the numerical treatment of systems of partial differential equations, where the solution is subjected to inequality constraints. We employ the finite element Galerkin (FE) method to obtain approximate solutions of such systems.

The basis for applying an FE discretisation is a suitable mathematical setting, which in the topics under consideration takes the form of variational inequalities (VI). An introduction to mathematical issues of these topics can be found in Kinderlehrer & Stampacchia [9] and Duvaut & Lions [3]. A corresponding overview on numerical methods is given in Glowinski et. al. [5, 4].

Variational inequality settings typically arise in the field of continuum mechanics. Examples are contact problems, where the displacement is restricted by a rigid obstacle or plastic materials, where certain norms of the stresses are bounded.

From mathematical point of view elasto-plastic problems are optimisation problems, where the stress field is subjected to inequality constraints which have to be fulfilled in the pointwise sense. A first straight-forward approach for

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a corresponding FE-scheme starts from the *primal-mixed* formulation. Here, the stresses can be eliminated and then the problem can be written by means of a projection onto the yield surface as an nonlinear equality in terms of derivatives of the displacement field u .

The drawback of this formulation is, that even very simple examples in the elasto-plastic case show, that discontinuities in the displacement u may occur (see Suquet [12]) due to slip lines in the micro-structure, whereas the stresses σ , determined by linear combinations of derivatives of u , have a smoother behaviour. For example in Bensoussan & Frehse [2] and Seregin [10] there is proven, that the stresses belong to H^1_{loc} .

Due to these regularity results it seems to be more adequate to consider the *dual-mixed* approach since in this case derivatives of u are not entering this formulation. Furthermore, from a practical point of view, the stresses are required with high accuracy. Therefore dual-mixed methods treating the stresses directly are often more adequate for solving problems in continuum mechanics.

Model problem

In the following, we present our ideas at the anti-plane shear or Strang's example. (c.f. Strang [11]). The mathematical problem seeks for a scalar displacement u and a stress vector $\sigma = (\sigma_1, \sigma_2)$ as functions on a bounded domain $\Omega \subset \mathbb{R}^2$. Stresses and displacement are caused by an external load described by a scalar function f on Ω . A nonlinear relation between u and σ — the so called plastic behaviour of the material — is taken into account by the restriction $|\sigma| \leq 1$. The classical form of the whole system is given by

$$\begin{aligned} -\operatorname{div} \sigma &= f, \quad \sigma = \Pi \nabla u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Π denotes the pointwise projection onto the circle with radius 1.

In order to give a weak form for (1), we set

$$\begin{aligned} H &:= \{ \tau \in L^2(\Omega, \mathbb{R}^2) \mid \operatorname{div} \tau \in L^2(\Omega) \}, \\ \Pi H &:= \{ \tau \in H, |\tau| - 1 \leq 0 \}, \\ V &:= \{ \varphi \in L^2(\Omega) \}, \end{aligned}$$

where $|\tau|^2 = \tau_1^2 + \tau_2^2$. Now, similar to the approach in Johnson [7], the solution $\{\sigma, u\} \in \Pi H \times V$ is determined by the variational inequality

$$(\sigma, \tau - \sigma) + (u, \operatorname{div}(\tau - \sigma)) - (\operatorname{div} \sigma, \varphi) \geq (f, \varphi) \quad \forall \{\tau, \varphi\} \in \Pi H \times V \tag{2}$$

where $(\cdot, \cdot)$ denotes the L^2 -scalar product on H . Existence of the solution and uniqueness for the stresses σ have been proven, e.g., by Johnson [6]. From

regularity theory (see e.g. Seregin [10]) one knows (roughly spoken) the stresses to fulfill $\sigma \in (H^1(\Omega))^2$.

The discrete analogue seeks for a pair $\{\sigma_h, u_h\}$ fulfilling

$$\begin{aligned} &(\sigma_h, \tau - \sigma_h) + (u_h, \operatorname{div}(\tau - \sigma_h)) \\ &(-\operatorname{div} \sigma_h, \varphi) + \delta_h([u_h], [\varphi]) \geq (f, \varphi) \forall \{\tau, \varphi\} \in \Pi H_h \times V_h. \end{aligned} \quad (3)$$

The finite dimensional space $\Pi H_h \times V_h$ for a discretisation on triangulations $\mathbb{T}_h$ with mesh size parameter h only consisting of triangular elements T for this saddle point problem is determined by approximating each component of the stresses by the standard linear shape functions for ΠH_h . V_h is constructed by elementwise constant functions. Above the brackets $[\cdot]$ denote the jump across the element boundaries ∂T and δ_h is an appropriately chosen mesh-dependent parameter. For details to this notation we refer to Becker [1].

2 Error control

In order to provide an error estimate for $\|\sigma - \sigma_h\|$, we introduce - following Johnson e.a. [6, 8] - the regularised version with parameter $\mu > 0$ of (2) in the form

$$\begin{aligned} &(\sigma_\mu, \tau) + (\mu^{-1}(\sigma_\mu - \Pi\sigma_\mu), \tau) \\ &+ (u_\mu, \operatorname{div} \tau) - (\operatorname{div} \sigma_\mu, \varphi) = (f, \varphi) \quad \forall \{\tau, \varphi\} \in H \times V. \end{aligned} \quad (4)$$

In view of

$$\|\sigma - \sigma_h\| \leq \|\sigma - \sigma_\mu\| + \|\sigma_\mu - \sigma_h\|$$

it remains to provide separate estimates for $\|\sigma - \sigma_\mu\|$ and $\|\sigma_\mu - \sigma_h\|$, which will be done now.

2.1 A priori estimate

Here, we focus on the estimate for $\|\sigma - \sigma_\mu\|$. The computation is as follows. Exploiting monotonicity of the regularisation and recalling that by construction there holds $|\sigma| \leq 1$, we get

$$(\mu^{-1}(\sigma_\mu - \Pi\sigma_\mu) - \mu^{-1} \overbrace{(\sigma - \Pi\sigma)}^{=0}, \sigma_\mu - \sigma) \geq 0.$$

Additionally one employs $(-\operatorname{div} \sigma, \varphi) = (-\operatorname{div} \sigma_\mu, \varphi) = (f, \varphi)$, yielding

$$\begin{aligned} \|\sigma_\mu - \sigma\|^2 &\leq (-\sigma + \sigma_\mu, \sigma_\mu - \sigma) + (\mu^{-1}(\sigma_\mu - \Pi\sigma_\mu) - \mu^{-1}(\sigma - \Pi\sigma), \sigma_\mu - \sigma) \\ &= -(\sigma, \sigma_\mu - \sigma) + (\sigma_\mu, \sigma_\mu - \sigma) \\ &\quad + (\mu^{-1}(\sigma_\mu - \Pi\sigma_\mu), \sigma_\mu - \sigma) + (u_\mu, \operatorname{div}(\sigma_\mu - \sigma)). \end{aligned}$$

With (4) it remains

$$\|\sigma_\mu - \sigma\|^2 = -(\sigma, \sigma_\mu - \sigma) = 0 \quad \text{for } \mu \rightarrow 0,$$

where the limit process is ensured by the weak convergence of σ_μ (see Johnson [6]).

2.2 A posteriori estimate

Here, we focus on the estimate for $\|\sigma_\mu - \sigma_h\|$. Arguing as above the computation starts with

$$\begin{aligned} & (\sigma_\mu - \sigma_h, \sigma_\mu - \sigma_h) + \delta_h([u_\mu - u_h], [u_\mu - u_h]) \\ & \leq (\sigma_\mu, \sigma_\mu - \sigma_h) + (u_\mu, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) + \underbrace{(\mu^{-1}(\sigma_\mu - \Pi\sigma_\mu) - \mu^{-1}(\sigma_h - \Pi\sigma_h), \sigma_\mu - \sigma_h)}_{\substack{=0 \\ \geq 0, \text{monotone operator}}} \\ & \quad - (u_\mu - u_h, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) + \delta_h([u_\mu - u_h], [u_\mu - u_h]) \\ & \quad - (u_h, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) - (\sigma_h, \sigma_\mu - \sigma_h) \end{aligned}$$

Again using (4) the first terms vanish and it remains

$$(\sigma_\mu - \sigma_h, \sigma_\mu - \sigma_h) + \delta_h([u_\mu - u_h], [u_\mu - u_h]) \leq T1 + T2$$

with

$$\begin{aligned} T1 &= -(u_\mu - u_h, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) + \delta_h([u_\mu - u_h], [u_\mu - u_h]) \\ T2 &= -(u_h, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) - (\sigma_h, \sigma_\mu - \sigma_h). \end{aligned}$$

2.2.1 Estimation of T1

Below the standard interpolant of u_μ will be denoted by u_i . Next we observe that for arbitrary $\varphi_h \in V_h$ there holds

$$\begin{aligned} (\varphi_h, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) &= -(f, \varphi_h) + (f, \varphi_h) - \delta_h([u_h], [\varphi_h]) \\ &= -\delta_h([u_h], [\varphi_h]). \end{aligned}$$

Now we obtain the identity

$$\begin{aligned} & -(u_\mu - u_h, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) + \delta_h([u_h], [u_h]) \\ & \quad = -(u_\mu - u_i, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) + \delta_h([u_h], [u_i]). \end{aligned}$$

Here, using interpolation estimates found in Johnson & Hansbo [8], the first term on the right-hand-side can be estimated by

$$\begin{aligned} (u_\mu - u_i, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) &= \sum_T (u_\mu - u_i, f - f_i)_T \\ &\leq c \sum_T \|h_T(f - f_i)\|_{\infty, T} \|\nabla u_\mu\|_{L_1(T)}. \end{aligned}$$

Analogously the jumps can be controlled by

$$\begin{aligned}\delta_h([u_h], [u_i])_T &\leq \delta_h([u_h], u_i^l - u_\mu)_{\partial T} + \delta_h([u_h], u_\mu - u_i^r)_{\partial T} \\ &\leq c\delta_h \|h_T D_h u_h\|_{\infty, T} \|\nabla u_\mu\|_{L_1(T)},\end{aligned}$$

where u_i^l and u_i^r denote the values of u_i on the two triangles having an edge of ∂T in common and according to the notation introduced in Johnson & Hansbo [8] we set

$$D_h u_h = \max_{\partial T} \frac{|[u_h]|}{h_T}.$$

Under the assumption of the so-called *safe load hypothesis* (see e.g. Johnson [6]) $\|\nabla u_\mu\|_{L_1}$ is - uniformly with respect to μ - bounded. In this note in practice this norm is approximately evaluated by exploiting values of $D_h u_h$.

2.2.2 Estimation of T2

In order to tackle the estimation of T2 we introduce the auxiliary problem of finding u_L fulfilling

$$\begin{aligned}\left(\frac{1}{p(u_L)} \nabla u_L, \varphi_L\right) &= (\sigma_h, \nabla \varphi_L) \quad \forall \varphi_L \in V_L \\ p(u_L) &= \begin{cases} 1, & |\nabla u_L| \leq 1 \\ |\nabla u_L|, & \text{otherwise.} \end{cases}\end{aligned}$$

Here the discrete space $V_L \subset H_0^1(\Omega)$ is defined on $\mathbb{T}_h$ using the standard linear shape functions and H_0^1 is a standard Sobolev space including zero boundary conditions. In fact this is a discrete version of a primal formulation for our plasticity problem, where the external load is determined by information - namely σ_h - of the discrete dual-mixed setting.

Now we can estimate T2 as follows.

$$\begin{aligned}&-(u_h, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) - (\sigma_h, \sigma_\mu - \sigma_h) \\ &= -(u_h - u_L, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) - (u_L, \operatorname{div} \sigma_\mu - \operatorname{div} \sigma_h) - (\sigma_h, \sigma_\mu - \sigma_h) \\ &= (u_h - u_L, f + \operatorname{div} \sigma_h) + (\nabla u_L - p(u_L) \sigma_h, \sigma_\mu - \sigma_h) + (\lambda_L \sigma_h, \sigma_\mu - \sigma_h) \\ &\leq (u_h - u_L, f + \operatorname{div} \sigma_h) + \frac{1}{2} \|\nabla u_L - p(u_L) \sigma_h\|^2 \\ &\quad + \frac{1}{2} \|\sigma_\mu - \sigma_h\|^2 + \int_{\Omega} \lambda_L |\sigma_h| (1 + C\mu^{1/2} - |\sigma_h|) dx\end{aligned}$$

where λ_L is given by $\lambda_L = p(u_L) - 1$ and $\mu^{-1}|\sigma - \sigma_\mu|^2 \leq C$ (see Johnson [6]) is used.

2.3 Error bound

Eventually collecting terms from subsections above and passing to the limit $\mu \rightarrow 0$ we obtain

Theorem 2.1. *With notations introduced above there holds an estimate for the error between the solutions σ of (2) and σ_h of (3) in the form*

$$\|\sigma - \sigma_h\|^2 \leq \sum_T (\rho_u^2 + \rho_\sigma^2 + \rho_p^2 + \rho_j^2),$$

with the settings

$$\begin{aligned} \rho_u^2 &= (u_h - u_L, f + \operatorname{div} \sigma_h), & \rho_\sigma^2 &= \frac{1}{2} \|\nabla u_L - p(u_L) \sigma_h\|^2, \\ \rho_p^2 &= \int_\Omega \lambda_L |\sigma_h| (1 - |\sigma_h|) dx, & \rho_j^2 &= \delta_h([u_h], [u_h])_T. \end{aligned}$$

3 Numerical results

As a simple test we choose the second example proposed in Johnson & Hansbo [8]. The evaluation of the single components of our error estimate, determined on a sequence of regular refined meshes are shown in Table 1. Here one observes the jumps of u_h measured by ρ_j to be dominant.

Furthermore assuming the components to behave like $\mathcal{O}(h^\alpha)$ we determine an approximation $\alpha_{(\cdot)}$ on each level, by comparing the error bounds with the estimator on the previous level. The results are listed in Table 2. One observes $\alpha_{(\cdot)}$ tending towards 1, indicating the components of the estimate to behave like $\mathcal{O}(h)$, as one would anticipate. Again ρ_j turns out to be the critical term.

Eventually a sequence of adaptively refined grids, generated on the basis of the weighted estimate, is depicted in Figure 1. Especially the critical zone where plastification occurs is well resolved. The background colour is given by values of $p^{-1}(u_L)$.

Cells	ρ_σ	ρ_u	ρ_p	ρ_{jumps}
32	2.349925e-01	2.048310e-01	5.014007e-02	2.046361e-01
128	1.315747e-01	1.104788e-01	2.910051e-02	1.712313e-01
512	6.939553e-02	5.516487e-02	1.613156e-02	1.055047e-01
2048	3.467687e-02	2.495607e-02	9.385640e-03	5.952007e-02
8192	1.744209e-02	1.134331e-02	4.539062e-03	3.134596e-02
32768	8.751919e-03	5.082170e-03	2.304256e-03	1.620830e-02

Table 1: Computational results for the test example. Evaluation of the single components of our error estimate, determined on a sequence of regular refined meshes.

Cells	α_σ	α_u	α_p	α_{jumps}
32	0.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00
128	8.367326e-01	8.906645e-01	7.849196e-01	2.571142e-01
512	9.229675e-01	1.001948e+00	8.511585e-01	6.986392e-01
2048	1.000869e+00	1.144359e+00	7.813589e-01	8.258591e-01
8192	9.914008e-01	1.137549e+00	1.048061e+00	9.250967e-01
32768	9.949016e-01	1.158325e+00	9.780932e-01	9.515467e-01

Table 2: Computational results for the test example. Estimated exponents describing rates of convergence $\mathcal{O}(h^{\alpha(\cdot)})$ of the single components of the proposed error estimate.

Figure 1: Sequence of locally refined grids produced by our numerical simulation, showing that especially the critical zone where plastification occurs is well resolved. Background colour is given by values of $p^{-1}(u_L)$.

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