Further Results on Square Sum Graph

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Abstract

A (p, q) graph G is said to be square sum, if there exists a bijection $f : V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$ such that the induced function $f^* : E(G) \rightarrow N$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. In this paper we establish that if G is a square sum graph then $G \cup P_m$ is square sum, $(K_{m,n})^2$ is square sum if and only if $m+n \leq 5$ and W_n^2 is square sum if and only if $n \leq 5$. We proved that shadow graph and split graph of P_n and $K_{1,n}$ are square sum for every $n \in N$. Also union of paths, the sequential join of some classes of square sum graph, $mK_{1,n}$ for $m, n \in N$ and $P_n \odot K_2$ (attaching K_2 to each vertex of P_n) are some classes of square sum graphs.

Keywords: Square sum graphs

1 Introduction

Graph labeling, where the vertices and edges are assigned real values or subset of a set are subject to certain conditions, have often been motivated by their utility to various applied fields. Several practical problems in real life situations have motivated the study of labelings of graphs which are required to obey a variety of conditions depending on the structure of graphs. Graph labeling has a strong communication between number theory [3] and structure of graphs [4] and [7]. Here we are interested in the study of vertex functions $f : V(G) \rightarrow$

¹Second author is indebted to the University Grants Commission(UGC) for granting her Teacher Fellowship under UGC's Faculty Development Programme during XI plan.

A , $A \subseteq N$ for which the induced edge function $f^*(uv) = (f(u))^2 + (f(v))^2$, for all $uv \in E(G)$ is injective. The wide-angular history of sum of squares of numbers motivated the authors to study the particular graphs named square sum graphs. Unless mentioned otherwise, by a *graph* we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [7]. Square sum graphs are vertex labeled graphs with the labels from the set $\{0, 1, 2, \dots, p-1\}$ such that the induced edge labels as the sum of the squares of the labels of the end vertices are all distinct. Not every graph is square sum. For example, any complete graph K_n , where $n \geq 6$ is not square sum [2]. We are interested to study different classes of graphs, which are square sum. In this paper we establish that if G is a square sum graph then $G \cup P_m$ is square sum, $(K_{m,n})^2$ is square sum if and only if $m+n \leq 5$ and W_n^2 is square sum if and only if $n \leq 5$. Also we proved that shadow graph of P_n and $K_{1,n}$ are square sum for every arbitrary n . Split graph of P_n and $K_{1,n}$ are square sum for every arbitrary n . We also establish that union of paths is square sum, the sequential join of some classes of square sum graph is square sum and proved that for the integers m and n , $mK_{1,n}$ is square sum and $P_n \odot K_2$ (attaching K_2 to each vertex of P_n) is square sum.

2 Square sum graphs

Acharya and Germina [1] defined a square sum labeling of a (p, q) -graph G [2] as follows.

Definition 2.1. A (p, q) graph G is said to be square sum, if there exists a bijection $f : V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$ such that induced function $f^* : E(G) \rightarrow N$ is given by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective.

Square of a graph G is denoted by G^2 , has $V(G^2) = V(G)$ with u, v adjacent in G^2 if and only if $d(u, v) \leq 2$ in G . The shadow graph $S(G)$ of a graph G is obtained from G by adding for each vertex v of G , a new vertex v' , called the shadow vertex of v , and joining v' to the neighbours of v in G . Split graph of a graph G is obtained by taking two copies of G say, G' and G'' and join each vertex u' in G' to the neighbourhood of corresponding vertex v' in G'' .

Theorem 2.2. [5] For every positive integer n , P_n is square sum.

Theorem 2.3. For every positive integer n , P_n^2 is square sum.

Proof. Let $V(P_n^2) = \{u_i; 1 \leq i \leq n\}$. Edge set of P_n^2 can be partition into two classes as follows. $E(P_n^2) = \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i u_{i+2}; 1 \leq i \leq n-2\}$. Define $f : V(P_n^2) \rightarrow \{0, 1, 2, \dots, n-1\}$ by $f(u_i) = i-1; 1 \leq i \leq n$. Clearly f is injective. In P_n^2 , edge labels of one class is strictly increasing sequence of odd numbers and the other class forms strictly increasing sequence of even numbers. Hence the induced edge labels are distinct and P_n^2 is square sum. $\square$

Observation 2.4. In square sum graph P_n^2 , the difference between consecutive edge labels in the form $f^*(u_i u_{i+1}); 1 \leq i \leq n-1$ form a sequence of even positive integers which form an A.P. Similarly in P_n^2 , the difference between consecutive edge labels in the form $f^*(u_i u_{i+2}); 1 \leq i \leq n-2$ form a sequence of even positive integers which form an A.P.

Theorem 2.5. [2] Complete graph K_n is square sum if and only if $n \leq 5$.

Corollary 2.6. Since $K_n^2 \cong K_n$, K_n^2 is square sum if and only if $n \leq 5$.

Theorem 2.7. [2] The complete bipartite graph $K_{m,n}$ is square sum if $m \leq 4$ and for all n .

Theorem 2.8. For every positive integer m, n , $(K_{m,n})^2$ is square sum if and only if $m+n \leq 5$.

Proof. The square sum labeling of $(K_{m,n})^2$ for $m+n \leq 5$ is shown in figure below.

Figure 1: All square sum $(K_{m,n})^2$

Assume that $m+n \geq 6$ and let $f : V(K_{m,n})^2 \rightarrow \{0, 1, 2, \dots, m+n-1\}$ be a square sum labeling such that $f^*(uv) = (f(u))^2 + (f(v))^2$, for every

$uv \in E(K_{m,n})^2$. For $(m+n) \geq 6$, the graph $(K_{m,n})^2$ is a complete graph and we always have edges e_1 and e_2 such that $f^*(e_1) = 0^2 + 5^2 = 25$ and $f^*(e_2) = 3^2 + 4^2 = 25$. Hence $f^*(e_1) = f^*(e_2)$. This is a contradiction to the fact that f is a square sum labeling. Hence $(K_{m,n})^2$ is square sum if and only if $m+n \leq 5$. $\square$

Remark 2.9. *The graph $(K_{1,n})^2$ is square sum if and only if $n \leq 4$.*

Square of a wheel W_n is isomorphic to complete graph K_n , hence by applying theorem 2.5 we get the following theorem.

Theorem 2.10. W_n^2 is square sum if and only if $n \leq 5$.

Theorem 2.11. *For any square sum graph G , $GU P_m$ is square sum for all m .*

Proof. Let G be a square sum graph of order n . Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(P_m) = \{u_{n+1}, u_{n+2}, \dots, u_{n+m}\}$. Label the vertices of G by $0, 1, 2, \dots, n-1$ and the vertices of P_m by $\{n, n+1, n+2, \dots, n+m-1\}$. Define $f : V(P_m) \rightarrow \{n, n+1, n+2, \dots, n+m-1\}$ by $f(u_i) = i-1$; $n+1 \leq i \leq n+m$. The edge labels of P_m is strictly increasing with increasing value of i and the edge labels of the square sum graph G is strictly less than that of P_m . Hence the induced edge labels of $G \cup P_m$ is distinct and is square sum. $\square$

Theorem 2.12. $K_{1,n} \cup K_{1,n+1}$ where $n \geq 1$ is square sum.

Proof. Let $V(K_{1,n} \cup K_{1,n+1}) = \{u, u_i; 1 \leq i \leq n\} \cup \{v, v_i; 1 \leq i \leq n+1\}$, where u and v are apex vertex of $K_{1,n}$ and $K_{1,n+1}$ respectively. Edge set of $K_{1,n} \cup K_{1,n+1}$ can be partition into two classes as follows.

$E(K_{1,n} \cup K_{1,n+1}) = \{uu_i; 1 \leq i \leq n\} \cup \{vv_i; 1 \leq i \leq n+1\}$. Define $f : V(K_{1,n} \cup K_{1,n+1}) \rightarrow \{0, 1, 2, \dots, 2n+2\}$ by

$$f(u) = 0;$$

$$f(u_i) = i; 1 \leq i \leq n;$$

$$f(v) = 2n+2;$$

$$f(v_i) = f(u_n) + i; 1 \leq i \leq n+1.$$

Clearly f is injective. Since f is strictly increasing with increasing value of i , the edge labels of both classes, $f^*(uu_i); 1 \leq i \leq n$ and $f^*(vv_i); 1 \leq i \leq n+1$ are distinct. Hence $f^*(uu_i) \neq f^*(vv_i)$ for all i . Also the maximum value of $f^*(uu_i); 1 \leq i \leq n$ is n^2 and the minimum value of $f^*(vv_i); 1 \leq i \leq n+1$ is $(2n+2)^2 + (n+1)^2$. Hence $f^*(uu_i) \neq f^*(vv_j)$ for $i \neq j$. Hence all the induced edge labels are distinct. $\square$

Theorem 2.13. *For integers m and n , $mK_{1,n}$ is square sum.*

Proof. Let $V(mK_{1,n}) = \{u_i; 1 \leq i \leq m\} \cup \{v_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\}$, where $u_i; 1 \leq i \leq m$ be the apex vertices of $mK_{1,n}$.

$$E(mK_{1,n}) = \{u_i v_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\}$$

Define $f : V(mK_{1,n}) \rightarrow \{0, 1, 2, \dots, m(n+1) - 1\}$ by

$$f(u_i) = i - 1; 1 \leq i \leq m$$

$$f(v_{i1}) = m + (i - 1)n; 1 \leq i \leq m \text{ and}$$

$$f(v_{ij}) = f(v_{i1}) + (j - 1); 1 \leq i \leq m; 2 \leq j \leq n$$

clearly f is injective. The induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ for every $uv \in E(mK_{1,n})$ is distinct, as the edge labels are strictly increasing with increasing value of i . Hence $mK_{1,n}$ is square sum. $\square$

However we strongly believe that if G_1 and G_2 are square sum, then $G_1 \cup G_2$ is square sum. So we pose the following conjecture.

Conjecture 2.14. *If G_1 and G_2 are square sum, then $G_1 \cup G_2$ is square sum.*

Lemma 2.15. *For every positive integer n , the graph $P_2 + \overline{K_n}$ is square sum.*

Proof. Let $V(P_2 + \overline{K_n}) = \{u_1, u_2, u_3, \dots, u_{n+2}\}$ where $V(P_2) = \{u_1, u_2\}$ and $V(\overline{K_n}) = \{u_3, u_4, \dots, u_{n+2}\}$. The edge set of $P_2 + \overline{K_n}$ can be partition into three classes as follows. $E(P_2 + \overline{K_n}) = \{u_1 u_2\} \cup \{u_1 u_i; 3 \leq i \leq n+2\} \cup \{u_2 u_i; 3 \leq i \leq n+2\}$.

Define $f : V(P_2 + \overline{K_n}) \rightarrow \{0, 1, 2, \dots, n+1\}$ by

$$f(u_i) = i - 1; 1 \leq i \leq n+2.$$

Clearly f is injective. Since f is strictly increasing with increasing value of i , the edge labels of each of the above classes are distinct. The label of the edge $u_1 u_2$ is clearly distinct from all other edge labels. Now we have to show that $f^*(u_1 u_i) \neq f^*(u_2 u_j)$ for $i \neq j$. If possible let $f^*(u_1 u_i) = f^*(u_2 u_j)$, for $i \neq j$. Then $(f(u_1))^2 + (f(u_i))^2 = (f(u_2))^2 + (f(u_j))^2$. Put $f(u_i) = a$ and $f(u_j) = b$ we get an equation of the form $a^2 - b^2 = 1$, so that either $f(u_i) = 0$ or $f(u_j) = 0$, which is not possible since f is a bijection. Hence all the edge labels are distinct and $P_2 + \overline{K_n}$ is square sum. $\square$

Lemma 2.16. *$P_3 + \overline{K_n}$ is square sum for every positive integer n .*

Proof. Let $V(P_3 + \overline{K_n}) = \{u_1, u_2, \dots, u_{n+3}\}$, where $V(P_3) = \{u_1, u_2, u_3\}$ and $V(\overline{K_n}) = \{u_4, u_5, \dots, u_{n+3}\}$. The edge set of $P_3 + \overline{K_n}$ can be partition into four classes as follows. $E(P_3 + \overline{K_n}) = \{u_i u_{i+1}; 1 \leq i \leq 2\} \cup \{u_1 u_i; 4 \leq i \leq n+3\} \cup \{u_2 u_i; 4 \leq i \leq n+3\} \cup \{u_3 u_i; 4 \leq i \leq n+3\}$.

Define $f : V(P_3 + \overline{K_n}) \rightarrow \{0, 1, 2, \dots, n+2\}$ by

$$f(u_i) = i; 1 \leq i \leq n+2; f(u_{n+3}) = 0;$$

Clearly f is injective. The edge labels of each of the above classes are distinct as $u_i; 4 \leq i \leq n+3$ are distinct. Now $f^*(u_2 u_i) = f^*(u_1 u_i) + 3; 4 \leq i \leq n+3$ and $f^*(u_3 u_i) = f^*(u_2 u_i) + 5; 4 \leq i \leq n+3$. Hence $f^*(u_1 u_i) \neq f^*(u_2 u_i)$, $f^*(u_1 u_i) \neq f^*(u_3 u_i)$ and $f^*(u_2 u_i) \neq f^*(u_3 u_i)$, for $4 \leq i \leq n+3$. Now we have to show that

$f^*(u_1u_i) \neq f^*(u_2u_j)$ for $i \neq j$, $f^*(u_1u_i) \neq f^*(u_3u_j)$ for $i \neq j$ and $f^*(u_2u_i) \neq f^*(u_3u_j)$ for $i \neq j$.

If possible assume that $f^*(u_1u_i) = f^*(u_2u_j)$. Put $f(u_i) = a$ and $f(u_j) = b$, then we get an equation $a^2 - b^2 = 3$; for $a, b \neq 1, 2$. This is not possible as f is injective. Hence $f^*(u_1u_i) \neq f^*(u_2u_j)$, for $i \neq j$.

If possible assume that $f^*(u_1u_i) = f^*(u_3u_j)$ for $i \neq j$. Then we get $a^2 - b^2 = 8$, for $a, b \neq 1, 3$. This is not possible as f is injective. Also if possible assume that $f^*(u_2u_i) = f^*(u_3u_j)$, for $i \neq j$. Put $f(u_i) = a$ and $f(u_j) = b$. Then we get an equation $a^2 - b^2 = 5$ for $a, b \neq 2, 3$. This is not possible, as f is injective. Also $f^*(u_iu_{i+1})$; $1 \leq i \leq 2$ are clearly distinct and distinct from all other edge labels. Hence all the edge labels are distinct. $\square$

However we strongly believe that for $m \geq 4$, $P_m + \overline{K_n}$ is not square sum. So we pose the following conjecture.

Conjecture 2.17. $P_m + \overline{K_n}$, $m, n \in \mathbb{N}$ is square sum if and only if $m \leq 3$.

Theorem 2.18. The complete tripartite graph $K_{1,1,n}$ is square sum for $n \in \mathbb{N}$.

Proof. Let $X = \{u\}$, $Y = \{x_1\}$, and $Z = \{y_1, y_2, \dots, y_n\}$ be three sets define by $f(u) = 0$; $f(x_1) = 1$; and $f(y_i) = i + 1$; $1 \leq i \leq n$. Clearly f is injective. Edge set of $K_{1,1,n}$ can be classified into three classes as follows.

$$E(K_{1,1,n}) = \{ux_1\} \cup \{uy_i; 1 \leq i \leq n\} \cup \{x_1y_i; 1 \leq i \leq n\}.$$

Since $f^*(ux_1) = 1$;

$$f^*(uy_i) = (f(u))^2 + (f(y_i))^2 = (i + 1)^2; 1 \leq i \leq n \text{ and}$$

$f^*(x_1y_i) = 1 + (i + 1)^2; 1 \leq i \leq n$, the edge labels of each class is distinct and hence $f^*(uy_i) \neq f^*(x_1y_i); 1 \leq i \leq n$. The edge labels of $f^*(uy_i); 1 \leq i \leq n$ are all perfect squares. But edge labels of $f^*(x_1y_j); 1 \leq j \leq n$ are not perfect squares and hence $f^*(uy_i) \neq f^*(x_1y_j)$ for $i \neq j$. Hence all the edge labels are distinct and $K_{1,1,n}$ is square sum for all n . $\square$

Theorem 2.19. Shadow graph of $K_{1,n}$ is square sum.

Proof. Let $V(K_{1,n}) = \{u_1, u_2, \dots, u_{n+1}\}$ where u_1 is the apex vertex of $K_{1,n}$ and let $v_1, v_2, \dots, v_{n+1}$ be the corresponding shadow vertices.

Define $f : V(S(K_{1,n})) \rightarrow \{0, 1, 2, \dots, 2n + 1\}$ by

$$f(u_1) = 0;$$

$$f(v_1) = 2n + 1;$$

$$f(u_i) = i - 1; 2 \leq i \leq n + 1;$$

$$f(v_i) = f(u_i) + (2n + 2) - (2i - 1); 2 \leq i \leq n + 1$$

clearly f is injective.

The edge set of $S(K_{1,n})$ can be partition into three classes as follows. $E(S(K_{1,n})) = \{u_1u_i; 2 \leq i \leq n + 1\} \cup \{u_1v_i; 2 \leq i \leq n + 1\} \cup \{v_1u_i; 2 \leq i \leq n + 1\}$.

Since f is strictly increasing with increasing value of i , clearly the edge labels of each of the above classes are distinct. Since the maximum value of u_1u_i

for $2 \leq i \leq n+1$ is n^2 and the minimum value of u_1v_i for $2 \leq i \leq n+1$ is $(n+1)^2$, $f^*(u_1v_i) \neq f^*(u_1u_i)$ for all i . Edge labels of u_1u_i is strictly less than the edge labels of v_1u_i for every $2 \leq i \leq n+1$ and hence $f^*(u_1u_i) \neq f^*(v_1u_j)$ for $i \neq j$. Also the edge labels of u_1v_i is strictly less than the edge labels of v_1u_i for $2 \leq i \leq n+1$. Hence $f^*(u_1v_i) \neq f^*(v_1u_j)$ for $i \neq j$. Hence all the induced edge labels are distinct and $S(K_{1,n})$ is square sum. $\square$

Theorem 2.20. *The shadow graph of P_n is square sum.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of P_n and $u_1, u_2, \dots, u_n$ be the corresponding shadow vertices. Define $f : V(S(P_n)) \rightarrow \{0, 1, 2, \dots, 2n-1\}$ by

$$f(v_i) = 2i - 2; \quad 1 \leq i \leq n;$$

$$f(u_i) = 2i - 1; \quad 1 \leq i \leq n;$$

Edge set of $S(P_n)$ is classified into three classes as follows. $E(S(P_n)) = \{v_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{v_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i v_{i+1}; 1 \leq i \leq n-1\}$. Clearly f is injective. Since f is strictly increasing with increasing value of i , the edge labels of each of the above classes are distinct. The induced edge labels of $f^*(v_i v_{i+1}); 1 \leq i \leq n-1$ are strictly increasing sequence of even numbers and the edge labels of $f^*(v_i u_{i+1})$ and $f^*(u_i v_{i+1})$ are strictly increasing sequence of odd numbers and hence $f^*(v_i v_{i+1}) \neq f^*(v_i u_{i+1})$ and $f^*(v_i v_{i+1}) \neq f^*(u_i v_{i+1})$ for $1 \leq i \leq n-1$.

Now it is enough to prove that $f^*(u_i v_{i+1}) \neq f^*(v_j u_{j+1})$, for $i \neq j$. $f^*(v_i u_{i+1}) = f^*(u_i v_{i+1}) + 4$ for every i . Because of symmetry it is enough to verify the case for the edges connecting the vertices with labels 0, 3 and 1, 2. The corresponding edge induced labels will be 9 and 5, which are clearly different. Hence one may conclude that $f^*(u_i v_{i+1}) \neq f^*(v_j u_{j+1})$ for $i \neq j$. Hence $S(P_n)$ is square sum. $\square$

Theorem 2.21. *Split graph of P_n is square sum.*

Proof. Let $v_i; 1 \leq i \leq n$ be the vertices of first copy of P_n and $u_i; 1 \leq i \leq n$ be the corresponding vertices of second copy of P_n . Let G be the graph $spl(P_n)$. Define $f : V(G) \rightarrow \{0, 1, \dots, 2n-1\}$ by

$$f(v_i) = 2i - 2; \quad 1 \leq i \leq n$$

$f(u_i) = 2i - 1; \quad 1 \leq i \leq n$. Clearly f is injective. The edge set of G can be partitioned into four classes as follows. $E(G) = \{v_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{v_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i v_{i+1}; 1 \leq i \leq n-1\}$. Since f is strictly increasing with increasing value of i , induced edge labels of each of the above classes are distinct. $v_i v_{i+1}; 1 \leq i \leq n-1$ are edges connecting consecutive even numbers and $u_i u_{i+1}; 1 \leq i \leq n-1$ are edges connecting consecutive odd numbers. Labels of the edges connecting consecutive even

numbers and labels of the edges connecting consecutive odd numbers are distinct and hence $f^*(u_i u_{i+1}) \neq f^*(v_i v_{i+1})$ for all i . Also $f^*(v_i u_{i+1})$; $1 \leq i \leq n-1$ and $f^*(u_i v_{i+1})$; $1 \leq i \leq n-1$ are strictly increasing sequence of odd numbers and $f^*(v_i u_{i+1}) = f^*(u_i v_{i+1}) + 4$; $1 \leq i \leq n-1$. Hence one may conclude that $f^*(v_i u_{i+1}) \neq f^*(u_i v_{i+1})$; $1 \leq i \leq n-1$ and $f^*(v_i u_{i+1}) \neq f^*(u_j v_{j+1})$; $i \neq j$. Hence all the induced edge labels of G are distinct. $\square$

Theorem 2.22. *Split graph of $K_{1,n}$ is square sum.*

Proof. Let G be the graph $spl(K_{1,n})$. Let u_i ; $1 \leq i \leq n+1$ be the vertices of first copy of $K_{1,n}$ where u_1 is the apex vertex of $K_{1,n}$ and v_i ; $1 \leq i \leq n+1$ be the corresponding vertices of second copy of $K_{1,n}$, with v_1 as the apex vertex. Define $f : V(G) \rightarrow \{0, 1, \dots, 2n+1\}$ by

$$f(u_1) = 0;$$

$$f(v_1) = 2n+1;$$

$$f(u_i) = i-1; \quad 2 \leq i \leq n+1; \text{ and}$$

$$f(v_i) = f(u_i) + (2n+2) - (2i-1); \quad 2 \leq i \leq n+1.$$

Clearly f is injective. The edge set of G can be classified into four classes as follows.

$$E(G) = \{u_1 u_i; \quad 2 \leq i \leq n+1\} \cup \{u_1 v_i; \quad 2 \leq i \leq n+1\} \cup \{v_1 u_i; \quad 2 \leq i \leq n+1\} \cup \{v_1 v_i; \quad 2 \leq i \leq n+1\}.$$

Since f is strictly increasing with increasing value of i , clearly edge labels of each of the above class are distinct. Labels of the edges $(u_1 u_i)$; $2 \leq i \leq n+1$ are strictly less than labels of the edges $(u_1 v_i)$; $2 \leq i \leq n+1$, so that $f^*(u_1 u_i) \neq f^*(u_1 v_i)$ for all i . The maximum value of $f^*(u_1 u_i)$ is n^2 and the minimum value of $f^*(u_1 v_i)$ is $(n+1)^2$, hence $f^*(u_1 u_i) \neq f^*(u_1 v_j)$ for $i \neq j$. The minimum value of $f^*(v_1 u_i) = (2n+1)^2 + 1$, hence $f^*(u_1 u_i) \neq f^*(v_1 u_j)$; for $i \neq j$. The minimum value of $f^*(v_1 v_i) = (2n+1)^2 + (2n)^2$, hence $f^*(u_1 u_i) \neq f^*(v_1 v_j)$, for $i \neq j$. Hence all the induced edge labels of G are distinct. $\square$

Theorem 2.23. 1) mP_n , for $m \geq 1$ is square sum.

2) $K_3 \cup mK_4$, for $m \geq 1$ is square sum.

Proof. 1) It follows from the fact that P_n is square sum for all n .

2) Let $G = K_3 \cup mK_4$. Let $V(G) = \{v_i; \quad 1 \leq i \leq 3; \quad v_{ij}; \quad 1 \leq i \leq m; \quad 1 \leq j \leq 4\}$, where $V(K_3) = v_i; \quad 1 \leq i \leq 3$ and $V(mK_4) = v_{ij}; \quad i \leq i \leq m; \quad 1 \leq j \leq 4$.

$$E(G) = \{E(K_3) \cup E(mK_4)\}$$

Define $f : V(G) \rightarrow \{0, 1, 2, \dots, 4m+2\}$ by $f(v_i) = i-1$; $1 \leq i \leq 3$ and label the vertices of first K_4 by $v_{1j} = v_3 + j$; $1 \leq j \leq 4$ in cyclic order. Clearly the edge labels of K_3 and K_4 are distinct and hence $K_3 \cup K_4$ is square sum. Label the corresponding vertices of second K_4 by $v_{2j} = v_{1j} + 4$; $1 \leq j \leq 4$. The edge labels of $K_3 \cup 2K_4$ is in strictly increasing order and hence square sum. Label the vertices of third K_4 by $v_{3j} = v_{2j} + 4$; $1 \leq j \leq 4$. Continue in this way, label the vertices of m^{th} K_4 by $v_{mj} = v_{(m-1)j} + 4$. Clearly the edge labels of K_3 are

strictly less than the edge labels of mK_4 ; $1 \leq i \leq m$ and the edge labels of mK_4 are in strictly increasing order and hence $K_3 \cup mK_4$ is square sum. $\square$

In general the class $K_3 \cup mK_4$ are special cases of sequential join of graphs.

Definition 2.24. For three or more disjoint graphs $G_1, G_2, \dots, G_n$, the sequential join $G_1 + G_2 + G_3 + \dots + G_n$ is the graph $\{G_1 + G_2\} \cup \{G_2 + G_3\} \cup \dots \cup \{G_{n-1} + G_n\}$.

Corollary 2.25. If $G_1 \cong K_1$ and $G_i \cong K_2$; $2 \leq i \leq n$, then the sequential join of G_i is square sum.

Corollary 2.26. If $G_1 \cong K_2$ and $G_i \cong K_2$; $2 \leq i \leq n$, then the sequential join of G_i is square sum.

Theorem 2.27. $\bigcup_{i=1}^k P_{n_i}$ is square sum.

Proof. $\bigcup_{i=1}^k P_{n_i}$ is of order $p = \sum_{i=1}^k n_i$.

Let $V(\bigcup_{i=1}^k P_{n_i}) = \{v_1, v_2, \dots, v_{n_1}, v_{n_1+1}, \dots, v_{n_1+n_2-1}, \dots, v_{\sum n_i}\}$. Without loss of generality assume that $|V(P_{n_i})| \leq |V(P_{n_j})|$ for $i < j$, $j \leq k$. Define $f :$

$V(\bigcup_{i=1}^k P_{n_i}) \rightarrow \{0, 1, 2, \dots, p-1\}$, by $f(v_j) = j-1$; $1 \leq j \leq \sum n_i$. Then clearly the edge labels form a strictly increasing sequence of odd integers and are distinct. $\square$

Theorem 2.28. $K_{1,n} + K_1$ is square sum for all n .

Proof. Let $V(K_{1,n}) = \{u_1, u_2, \dots, u_{n+1}\}$, where u_1 is the apex vertex of $K_{1,n}$ and let $V(K_1) = v_1$. Define $f : V(K_{1,n} + K_1) \rightarrow \{0, 1, 2, \dots, n+1\}$ by

$$f(u_1) = 0;$$

$$f(v_1) = n+1;$$

and $f(u_i) = i-1$; $2 \leq i \leq n+1$. Clearly f is injective. $E(K_{1,n} + K_1) = \{u_1 u_i; 2 \leq i \leq n+1\} \cup \{v_1 u_i; 2 \leq i \leq n+1\} \cup \{u_1 v_1\}$

The induced edge labels of $u_1 u_i$; $2 \leq i \leq n+1$ is strictly less than the induced edge labels of $v_1 u_i$; $2 \leq i \leq n+1$. Hence $f^*(u_1 u_i) \neq f^*(v_1 u_i)$ for all i . $f^*(u_1 v_1) = (n+1)^2$ is clearly distinct from all other induced edge labels. Hence all the induced edge labels are distinct. $\square$

Theorem 2.29. The graph $P_n \odot K_2$ (attaching K_2 to each vertex of P_n) is square sum for all n .

Proof. $P_n \odot K_2$ is isomorphic to the path join of n copies of K_3 by path P_2 . Let $v_i; 1 \leq i \leq n$ be the vertices of P_n in $P_n \odot K_2$. Let $a_i, b_i; i = 1, 2, \dots, n$ be the vertices adjacent to $v_i; 1 \leq i \leq n$. Let $K_{3,i}; 1 \leq i \leq n$ be the n copies of K_3 . Let G be the graph $P_n \odot K_2$. $|V(G)| = 3n$. Let $K_{3,1} : v_1 a_1 b_1 v_1, K_{3,2} : v_2 a_2 b_2 v_2, \dots, K_{3,n} : v_n a_n b_n v_n$ be the n copies of K_3 , where $v_1 v_2$ be the path joining $K_{3,1}$ and $K_{3,2}$, $v_2 v_3$ be the path joining $K_{3,2}$ and $K_{3,3}$, and $v_{n-1} v_n$ be the path joining $K_{3,n-1}$ and $K_{3,n}$ respectively.

Define $f : V(G) \rightarrow 0, 1, 2, \dots, 3n - 1$ as follows.

$$f(v_i) = 3i - 3; 1 \leq i \leq n.$$

$$f(a_i) = 3i - 2; 1 \leq i \leq n.$$

$$f(b_i) = 3i - 1; 1 \leq i \leq n.$$

Clearly f is injective. The labels of edges $v_i v_{i+1}; 1 \leq i \leq n$ are in strictly increasing order and hence edge labels of P_n in $P_n \odot K_2$ are distinct. Also the edge labels of $K_{3,i}; 1 \leq i \leq n - 1$ are distinct as f is injective. The edge having maximum label in $K_{3,i}$ is $a_i b_i; 1 \leq i \leq n$, and $f^*(v_i v_{i+1}) = f^*(a_i b_i) + 4; 1 \leq i \leq n - 1$. Hence the edge label of $v_1 v_2$ is strictly greater than edge label of $K_{3,1}$ and edge label of $v_2 v_3$ is strictly greater than edge labels of $K_{3,2}$ and finally the edge label of $v_{n-1} v_n$ is strictly greater than the edge labels of $K_{3,n-1}$. Hence all the edge labels are distinct and which implies $P_n \odot K_2$ is square sum. $\square$

We propose the following problems for further investigation.

problem 2.30. *Characterize the square sum sequential join of square sum graphs.*

problem 2.31. *Given an injective edge labeled graph, find the corresponding vertex labeling so that the edge labels are represented by the sum of the squares respective end vertices.*

References

- [1] B.D.Acharya, *Personal Communication*, September 2011.
- [2] Ajitha V, *Studies in Graph Theory-Labeling of Graphs*, Ph D thesis (2007), Kannur Univeristy, Kannur.
- [3] D.M Burton, *Elementary Number theory*, Second Edition, Wm.C.Brown Company publishers, 1980.
- [4] J.A.Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics (DS6), 2005.

- [5] K.A Germina and Reena Sebastian, *On square sum Graphs*, communicated.
- [6] M.C.Golumbic, *Algorithmic graph theory and perfect graphs*, academic press, San Diego, 1980.
- [7] F. Harary, **Graph Theory**, Addison-Wesley Pub. Comp., Reading, Massachusetts, 1969.
- [8] Shelly, *Sum to perfect squares*, Internet YouTube:
<http://www.youtube.com/watch?v=YM2a7My1flo>.
- [9] S.A. Shirali, *To find four distinct positive integers such that the sum of any two is a quare*, math@rishivalley.org, 18 September 2009.

Received: September, 2012