Notes on Operator Classes \((A,k)_N\) and \((A,k)_N^*\)

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Abstract

In this article we have proved that every operator that belongs to the \((A,k)_N\) class, respectively \((A,k)_N^*\) class, then it is a \(\sqrt[N]{N-k}\)–paranormal operator, respectively a \(\sqrt[N]{N-k}\)–*paranormal operator, for \(k \geq 2\). We showed that if \(T\) is a \(N-k\)–*paranormal operator such that \(T^n\) is compact, for some \(n \geq k\) then \(T\) is also compact. We also give some properties about these classes.

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1. Introduction

Let us denote by \(H\) the complex Hilbert space and \(B(H)\) the space of all bounded linear operators defined in Hilbert space \(H\). In the following we will mention some known classes of operators defined in Hilbert space \(H\). Let \(T\) be an element in the algebra of bounded operators \(B(H)\). The operator \(T\) is called quasi-normal if \(T(T^*T) = (T^*T)T\), it is \(N\)-hyponormal if \(N\|Tx\| \geq \|T^*x\|\), for all \(x\) in \(H\) and for a fixed constant \(N > 0\). We say that an operator \(T\) is \(N\)–quasi-hyponormal if it satisfies the following condition \(N\|r^2x\| \geq \|T^*Tx\|\), for all \(x\) in \(H\) and for a fixed constant \(N > 0\). We say that an operator \(T\) is of \((M,k)_N\) class if \(NT^{\ast k}T^k \geq (T^*T)^k\), for \(k \geq 2\), and for a fixed constant \(N > 0\) (see [7]). We say that an operator \(T\) is of \((M,k)_N^*\) class if \(NT^{\ast k}T^k \geq (TT^*)^k\), for \(k \geq 1\) and for a fixed constant \(N > 0\) (see [7]). The operator \(T\) is called
$N - k - *$-paranormal if it satisfies the following condition $N\|T^k x\| \geq \|T^* x\|^k$, for all unit vectors $x$ in $H$, $k \geq 2$ and for a fixed constant $N > 0$. The operator $T$ is called $N - k$-paranormal if it satisfies the following condition $N\|T^k x\| \geq \|T x\|^k$, for all unit vectors $x$ in $H$, $k \geq 2$ and for a fixed constant $N > 0$. The operator $T$ is of $(A,k)_N$ class if $N\|T^k x\|^2 \geq \|T^* T x\|^k$, for all unit vectors $x$ in $H$, $k \geq 2$ and for a fixed constant $N > 0$. We see that the $(A,2)_N$ class coincides with the $(M,2)_N$ class. The operator $T$ is of $(N k A)_N$ class if $N \|T^k x\|^2 \geq \|T T^* x\|^k$, for all unit vectors $x$ in $H$, $k \geq 1$ and for a fixed constant $N > 0$ ($k$ - an integer). We see that the $(A,2)^*_N$ class coincides with the $(M,2)^*_N$ class. We will prove that every operator that belongs to the $(A,k)_N$ class, respectively $(A,k)^*_N$ class, then it is a $\sqrt{N} - k$-paranormal operator, respectively a $\sqrt{N} - k - *$-paranormal operator, for $k \geq 2$. We showed that if $T$ is a $N - k - *$-paranormal operator such that $T^n$ is compact, for some $n \geq k$ then $T$ is also compact. We also give some properties about these classes.

**Theorem A. (Hölder-McCarthy inequality [1]).** Let $A$ be a positive operator. Then the following inequalities hold for all $x$ in $H$

i) $\langle A^r x, x \rangle \leq \langle A x, x \rangle^r \|x\|^{2(1-r)}$, for $0 < r \leq 1$,

ii) $\langle A^r x, x \rangle \geq \langle A x, x \rangle^r \|x\|^{2(1-r)}$, for $r \geq 1$.

**2. Operator classes $(A,k)_N$ and $(A,k)^*_N$ in Hilbert space**

In this section we will show some properties of $(A,k)_N$ and $(A,k)^*_N$ classes in Hilbert space.

**Proposition 2.1.** If the operator $T$ belongs to the $(A,k)_N$ class, for $k \geq 2$, then $T$ is $\sqrt{N} - k -$ paranormal.

**Proof.** Let $T \in (A,k)_N$, then for every unit vectors $x \in H$ we have

$$N^2 \|T^k x\|^4 \geq \|T^* T x\|^{2k} = \langle T^* T x, T^* T x \rangle^k$$

$$= \langle (T^* T)^k x, x \rangle^k \geq \langle T^* T x, x \rangle^{2k}$$ (Hölder-McCarthy ineq.)
Thus $\sqrt{N}\|T^k x\| \geq \|Tx\|^k$, for every unit vector $x \in H$, respectively $T$ is a $\sqrt{N} - k$-paranormal operator. Therefore the proof is completed.

Corollary 2.1. If the operator $T$ belongs to the $(A,k)$ class, for $k \geq 2$, then $T$ is $k$-paranormal.

Proposition 2.2. If the operator $T$ belongs to the $(M,k)_N$ class, for $k \geq 2$, then $T$ belongs to the $(A,k)_N$ class.

Proof. The proof of the Proposition is similar to that of Proposition 2.2. in [8].

Corollary 2.2. If the operator $T$ belongs to the $(M,k)_N$ class, for $k \geq 2$, then $T$ is $\sqrt{N} - k$-paranormal.

Proposition 2.3. If the operator $T$ belongs to the $(A,k)_N^*$ class, for $k \geq 2$, then $T$ is $\sqrt{N} - k$-paranormal.

Proof. The proof of the Proposition is similar to that of Proposition 2.1.

Corollary 2.3. If the operator $T$ belongs to the $(A,k)_N^*$ class, for $k \geq 2$, then $T$ is $k$-paranormal.

Theorem 2.1. Let $T$ be a $N - k - *$paranormal operator such that $T^n$ is compact for some $n \geq k$, then $T$ is compact, too.

Proof. It suffices to prove that the compactness of $T^n$ for some $n \geq k$ implies that of $T^{n-1}$ is compact, too. Let $T$ be a $N - k - *$paranormal operator for $k \geq 2$ and let $\frac{T^{n-k}x}{\|T^{n-k}x\|}$ be a unit vector in Hilbert space $H$. Then we have

$$\|T^* T^{n-k} x\|^k \leq N \|T^n x\| \cdot \|T^{n-k} x\|^{k-1}. \quad (1)$$

Let $\{x_n\} \in H, \|x_n\| = 1$ be weakly convergent sequence with limit 0 in $H$. From compactness of $T^n$ and inequality (1) we get the following relation
\[ \left\| T^* T^{n-k} x_m \right\|^2 \leq N \left\| T^n x_m \right\| \cdot \left\| T^{n-k} x_m \right\|^{k-1} \to 0, \ m \to \infty. \]

If \( n = k \), this implies the compactness of \( T^* \), and hence that of \( T \). If \( n > k \), this implies the compactness of \( T^* T^{n-k} \) from which follows that \( T^{n+1} T^{n-1} \) is also compact, respectively \( T^{n-1} \) is a compact operator (see Theorem 5.2.4 in [3]).

**Corollary 2.4.** Let \( T \) be a \( k - \ast \)-paranormal operator such that \( T^n \) is compact for some \( n \geq k \), then \( T \) is compact, too.

**Corollary 2.5.** Let \( T \) be a \( \ast \)-paranormal operator such that \( T^n \) is compact for some \( n \geq 2 \), then \( T \) is compact, too.

**Remark 2.1.** For \( k = 2 \) we obtain theorem 1.4 in [4].

**Lemma 2.1.** If \( T \) is a bilateral weighted shift operator, with weighted sequence \( \{ \omega_n \} \), \( (Te_n = \omega_n e_{n+1}) \), then \( T \) is in \((A,k)_N\) class if and only if

\[ \sqrt{N} |\omega_n| |\omega_{n+1}| \cdots |\omega_{n+k-1}| \geq |\omega_n|^k, \ \ n \in \mathbb{Z} \ \text{and} \ k \geq 2. \]

**Proof.** The proof follows immediately from the definition of \((A,k)_N\) class. ■

**Lemma 2.2.** If \( T \) is a bilateral weighted shift operator, with weighted sequence \( \{ \omega_n \} \), \( (Te_n = \omega_n e_{n+1}) \), then it is \((A,k)_N^\ast\) class if and only if

\[ \sqrt{N} |\omega_n| |\omega_{n+1}| \cdots |\omega_{n+k-1}| \geq |\omega_{n-1}|^k, \ \ n \in \mathbb{Z} \ \text{and} \ k \geq 1. \]

**Proof.** The proof follows immediately from the definition of \((A,k)_N^\ast\) class. ■

**Example 2.1.** Let \( \{e_n\}_{n=1}^\infty \) be an orthonormal basis of the Hilbert space \( H \). Define a bilateral weighted shift operator \( T \) on \( H \), with weighted sequence \( \{\omega_n\} \) given by
For \( k = 2 \) and \( N = 4 \) by Lemma 2.1, we have \( 2|\omega_{n+1}| \geq |\omega_n| \), for \( n \in \mathbb{Z} \). Therefore, \( T \) is in \((A,2)_4\) class. Now for \( k = 2 \) and \( N = 4 \) by Lemma 2.2, we have \( 2|\omega_n| \cdot |\omega_{n+1}| \geq |\omega_{n-1}|^2 \), for \( n \in \mathbb{Z} \). Therefore, \( T \) isn’t in \((A,2)_4^*\) class, because for \( n = 1 \) we have \( 2|\omega_1| \cdot |\omega_2| = 2 \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{8} < |\omega_0|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \).

**Example 2.2.** Let \( T \) is a bilateral weighted shift operator, with weighted sequence \( \{\omega_n\} \) given as follows

\[
\omega_n = \begin{cases} 
\frac{1}{4}, & \text{for } n \leq -1 \\
1, & \text{for } n = 0 \\
\frac{1}{2}, & \text{for } n = 1 \\
\frac{1}{4}, & \text{for } n \geq 2 
\end{cases}
\]

From lemma 2.1 and lemma 2.2, for \( k = 2 \) and \( N = 4 \) we have that \( T \) belongs to the \((A,2)_4\) class if \( 2|\omega_{n+1}| \geq |\omega_n| \), for \( n \in \mathbb{Z} \) and \( T \) belongs to the \((A,2)_4^*\) class if \( 2|\omega_n| \cdot |\omega_{n+1}| \geq |\omega_{n-1}|^2 \), for \( n \in \mathbb{Z} \). Therefore, \( T \) belongs to the \((A,2)_4^*\) class, but \( T \) does not belong to the \((A,2)_4\) class, because for \( n = 0 \) we have \( 2|\omega_1| = 2 \cdot \frac{1}{4} = \frac{1}{2} < |\omega_0| = 1 \).

**Lemma 2.3.** If \( T \) is a regular bilateral weighted shift operator, with weighted sequence \( \{\omega_n\} \neq 0 \), \( (Te_n = \omega_n e_{n+1}) \), then \( T^{-1} \in (A,k)_N^* \) if and only if

\[
\sqrt{N} |\omega_k|^k \geq |\omega_{n-1}| \cdot |\omega_{n-2}| \cdot \ldots \cdot |\omega_{n-k}|, \quad n \in \mathbb{Z}, |\omega_n| \neq 0 \text{ and } k \geq 1.
\]

**Proof.** This follows immediately from the definition of \((A,k)_N^*\) class. \( \blacksquare \)

By the following example we show that there exists an operator \( T \) which belongs to the \((A,2)_4^*\) class but its inverse \( T^{-1} \) is not element of \((A,2)_4\).
Example 2.3. Let \( T \) is a bilateral weighted shift operator, with weighted sequence \( \{\omega_n\} \) given as follows

\[
\omega_n = \begin{cases} 
\frac{1}{4}, & \text{for } n \leq -1 \\
1, & \text{for } n = 0 \\
\frac{1}{4}, & \text{for } n = 1 \\
1, & \text{for } n \geq 2 
\end{cases}
\]

From example 2.2. follows that \( T \) belongs to the \((A,2)_4^*\) class. Now from lemma 2.3 for \( k = 2 \) and \( N = 4 \) we have \( T^{-1} \) belongs to the \((A,2)_4^*\) class if

\[
2 \cdot |\omega_n|^2 \geq |\omega_{n-1}| \cdot |\omega_{n-2}|, \quad \text{for } n \in \mathbb{Z}.
\]

Therefore \( T^{-1} \) does not belong to the \((A,2)_4^*\) class, because for \( n = 1 \) we have

\[
2 |\omega_0|^2 = 2 \cdot \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \frac{1}{8} < |\omega_0| \cdot |\omega_{-1}| = 1 \cdot \frac{1}{4} = \frac{1}{4}.
\]

Theorem 2.2. If a quasi-normal operator \( T \) belongs to the \((A,1)_N^*\) class, then it belongs to the \((M,k)_N^*\) class too, for \( k \geq 1 \).

Proof. Suppose \( T \in (A,1)_N^* \) and \( x \in H \|x\|=1 \). Then we have

\[
N\|Tx\|^2 \geq \|TT^* x\|
\]

\[
N \langle Tx, Tx \rangle \geq \langle TT^* x, TT^* x \rangle^{\frac{1}{2}}
\]

\[
N \langle T^* Tx, x \rangle \geq \langle (TT^*)^2 x, x \rangle^{\frac{1}{2}}
\]

\[
N \langle T^* Tx, x \rangle \geq \langle TT^* x, x \rangle^{\frac{1}{2}} \text{ (Hölder-McCarthy ineq.)}
\]

\[
N \langle T^* Tx, x \rangle \geq \langle TT^* x, x \rangle
\]

\[
\langle NT^* T - TT^* x, x \rangle \geq 0.
\]
Notes on operator classes

Thus $NT'T \geq TT'$, respectively $T$ belongs to the $(M,1)_N^*$ class. Therefore the result is true for $k = 1$. Now suppose that the result is true for $k = n$ so that $NT'^nT' \geq (TT')^n$, respectively $T$ belongs to the $(M,n)_N^*$ class. Since $T'T$ operator commutes with $NT'^nT'$ and $(TT')^n$ (because $T$ is a quasi-normal), then we get $N(T'^nT')(T'T) \geq (TT')^n(T'T)$.

Now we have

$$N(T'^nT')(T'T) = \frac{NT'^nTT...T(T'T)}{n}$$

Thus

$$= NT'^nTT...T$$

Respectively

$$N(T'^nT')(T'T) = NT'^{n+1}T^{n+1}.$$ (2)

Also

$$(TT')^n(T'T) = (TT')^{n-1}(TT')(T'T) = (TT')^{n-1}(TT'TTT)$$

$$= (TT')^{n-1}(TT'TTT) = (TT')^{n-1}(TT')^2 = (TT')^{n+1}.$$ (3)

Therefore by (2) and (3) follows that $NT'^{n+1}T^{n+1} \geq (TT')^{n+1}$, respectively $T$ belongs to the $(M,n+1)_N^*$ class. Now by induction, operator $T$ belongs to the $(M,k)_N^*$ class, for every $k \geq 1$.

**Corollary 2.6.** If a quasi-normal operator $T$ belongs to the $(A,1)_N^*$ class, then it is $\sqrt[N]{N} - (k+1)$ - paranormal, for $k \geq 1$.

**Proof.** This proof follows from Theorem 2.2, Proposition 4.10 in [7] and Corollary 2.2.

**Proposition 2.4.** If $T \in B(H)$, then the following assertions hold:

1. Operator $T$ belongs to the $(M,l)_N^*$ class if and only if $T$ is $\sqrt[N]{N}$-hyponormal.
2. If $T \in (A,l)_N^*$, then $T$ is a $\sqrt[N]{N}$-paranormal operator.
Proof. i). Suppose $T \in (M,l)_N^*$, then $NT^*T \geq TT^*$. Now for every $x \in H$ we have

$$
\langle (NT^*T - TT^*)x, x \rangle \geq 0, \ (x \in H)
$$

$$
\iff \langle (NT^*Tx, x) - \langle TT^*x, x \rangle \geq 0, \ (x \in H)
$$

$$
\iff N\langle Tx, Tx \rangle - \langle T^*x, T^*x \rangle \geq 0, \ (x \in H)
$$

$$
\iff N\|Tx\|^2 \geq \|T^*x\|^2, \ (x \in H)
$$

Respectively $T$ is a $\sqrt{N}$-hyponormal operator.

ii). Since $T \in (A,l)_N^*$ and $x \in H, \|x\| = 1$, then $T$ is a $\sqrt{N}$-hypernormal operator. Now by (i), Proposition 4.10 in [7] and Corollary 2.2 follows that $T$ is a $\sqrt{N}$-paranormal operator. The proof is completed.

REFERENCES


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