Further Results on Complementary Perfect Domination Number of a Graph

G. Mahadevan\textsuperscript{1} and B. Ayisha\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Gandhigram Rural Institute Deemed University, Gandhigram - 624 302, India

\textit{Current address:} Dept. of Mathematics, Anna University, Tirunelveli Region (Government Engineering college campus), Tirunelveli - 627 007, Tamilnadu, India

gmaha2003@yahoo.co.in

\textsuperscript{2}Department of Mathematics Thamirabharani Engineering College, Tirunelveli, India

ayisharahman3@gmail.com

Abstract

A subset \(S\) of \(V\) is called a dominating set in \(G\) if every vertex in \(V - S\) is adjacent to at least one vertex in \(S\). The minimum cardinality taken over all dominating sets in \(G\) is called the dominating number of \(G\) and is denoted by \(\gamma\). The concept of complementary perfect domination number of a graph was introduced by Paulraj Joseph, J, Mahadevan, G, Selvam. A in [9]. A subset \(S\) of \(V\) of a non trivial graph \(G\) is said to be complementary perfect dominating set, if \(S\) is a dominating set and \(<V-S>\) has a perfect matching. The minimum cardinality taken over all complementary perfect dominating sets is called complementary perfect domination number and is denoted by \(\gamma_{cp}\). The minimum number of colours required to colour all the vertices of \(G\) in such a way that adjacent vertices do not receive the same colour is called the chromatic number and is denoted by \(\chi\) of \(G\). In [6], the authors characterized the classes of graphs for which the sum of complementary perfect domination number and chromatic number of order upto \(2n-5\). In this paper we characterize the classes of graphs for which the sum of complementary perfect domination number and chromatic number = \(2n-6\), for any \(n>3\).

Mathematics Subject Classification: 05C

Keywords: Complementary Perfect domination number, chromatic number
1. Introduction

Let $G = (V,E)$ be a simple undirected graph. The degree of any vertex $u$ in $G$ is the number of edges incident with $u$ and is denoted by $d(u)$. The maximum degree of a vertex is denoted by $\Delta (G)$. The path of $n$ vertices is denoted by $P_n$. The vertex connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results a disconnected graph. Caterpillar is a tree in which removal of pendent edges results a path.

In [9], concept of complementary perfect dominating set was introduced by Paulraj Josep, J. Mahadevan, G. and Selvam, A.

A subset $S$ of $V$ of a non-trivial graph $G$ is said to be complementary perfect dominating set, if $S$ is a dominating set and $<V - S>$ has a perfect matching. The minimum cardinality taken over all complementary perfect dominating sets is called complimentary perfect domination number and is denoted by $\gamma_{cp}$.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameters and a graph theoretic parameter and characterized the corresponding external graphs. In [8], Paulraj Joseph and Arumugam proved that $\gamma + \kappa \leq P$. They also proved that $\gamma_c + \chi \leq P + 1$ and characterized the classes of graphs for which the upper bound is attained. They also proved similar results for $\gamma$ and $\gamma_c$. In [4], Paulraj Joseph J. and Mahadevan G. proved that $\gamma_{cc} + \chi \leq 2n - 1$ and characterized the corresponding external graphs. In [6], the authors characterized the classes of graphs for which the sum of complementary perfect domination number and chromatic number of order upto $2n - 5$. In this paper we characterize the classes of graphs for which the sum of complementary perfect domination number and chromatic number $= 2n - 6$ for any $n > 3$.

Previous Results
Theorem 1.1: [9] For any graph $G$, $\gamma_{cp}(G) \leq n - 2$.
Theorem 1.2: [9] For any graph $G$, $\gamma_{cp}(G) = n$ if and only if $G$ is a star.

2. Main Results

We use the following notation for the subsequent characterization:

Notations 2.1: $K_n(P_k)$ is the graph obtained from $K_n$ by attaching the end vertex of $P_k$ to any one vertices of $K_n$. $K_n(mP_k)$ is the graph obtained from $K_n$ by attaching the
end vertices of m copies of $P_k$ to any one vertices of $K_n$. $C_n(P_k)$ is the graph obtained from $C_n$ by attaching the end vertex of $P_k$ to any one vertices of $K_n$. The sub division of star $K_{1,n}$ with at most $n-1$ edges is called wounded spider and is denoted by $S^*(K_{1,n})$. $C_n(K_1P_{x1}K_2P_{x2},...,K_mP_{xn})$ is the graph obtained from $C_n$ by attaching the end vertices of $K_1$ copies of $P_{x1}$ to any one vertex of $C_n$, attaching the end vertices of $K_2$ copies of $P_{x2}$ to any one vertex of $C_n$ distinct from the previous vertex etc.

**Theorem 2.2:** For any connected graph $G$, $\gamma_{cp} + \chi = 2n-6$ if and only if $G$ is isomorphic to $K_{1,7}, P_6$, $K_7$, $K_8$, $K_4(P_3)$, $K_5(P_3)$, $C_3(2P_3)$, $C_4(P_3)$, $K_6(2,0,0,0,0,0)$, $K_6(1,1,0,0,0,0), K_6(2,0,0,0,0), K_6(1,1,0,0,0), K_4(4,0,0,0), K_4(3,1,0,0), K_4(2,2,0,0), K_4(2,1,1,0), P_4(0,1,0,0,0), P_4(0,0,1,0,0), P_4(0,2,0,0,0), P_4(0,1,1,0), C_4(2P_2,0,0,0), C_4(P_2P_2,0), C_4(4P_2,0,0), C_3(3P_2P_2,0), C_3(2P_2,2P_2,0), C_3(2P_2,2P_2), C_3(P_3,2P_2,0), C_3(P_3,2P_2,0), G_i, for $i = 1$ to 64, the graphs shown in the following Figure 2.2.
Complementary perfect domination number of a graph
Proof: If $G$ is any one of the graph in figure 2.2 then clearly $\gamma_{cp} + \chi = 2n-6$. Conversely, assume that $\gamma_{cp} + \chi = 2n-6$. This is possible only if (i) $\gamma_{cp} = n$ and $\chi = n-6$. (ii) $\gamma_{cp} = n-1$ and $\chi = n-5$. (iii) $\gamma_{cp} = n-2$ and $\chi = n-4$. (iv) $\gamma_{cp} = n-3$ and $\chi = n-3$. (v) $\gamma_{cp} = n-4$ and $\chi = n-2$. (vi) $\gamma_{cp} = n-5$ and $\chi = n-1$. (vii) $\gamma_{cp} = n-6$ and $\chi = n$. The cases for which (ii) $\gamma_{cp} = n-1$ and $\chi = n-5$, (iv) $\gamma_{cp} = n-3$ and $\chi = n-3$ and (vi) $\gamma_{cp} = n-5$ and $\chi = n-1$, are not possible. Hence we have to discuss only four cases.

Case 1: $\gamma_{cp} = n$ and $\chi = n-6$. 

Figure 2.2
Since $\gamma_{cp} = n$, by Theorem 1.2, G is a star. But for a star $\chi = 2$ and hence $n=8$. Hence $G \cong K_{1,7}$.

Case 2: $\gamma_{cp} = n-2$ and $\chi = n-4$

Since $\chi = n-4$, G contains a clique $K = K_{n-4}$ or does not contain a clique $K = K_{n-4}$.

Let G contains a clique $K = K_{n-4}$

Let $S = \{w, x, y, z\} = V(G)-V(K)$. Then $<S> = K_4$ or $K_4$ or $P_4$ or $C_4$ or $K_{3,1}$ or $K_{2,1,2}$ or $K_{2,1,2}$ or $P_3$ or $C_3(1,0,0)$ or $K_{1,3}$ or $K_{4-e}$.

Subcase 2(I): $<S> = K_4$

Let the vertices of $K_4$ be $w, x, y, z$. Since G is connected $w$ is adjacent to some $u_i$ in $K = K_{n-4}$.

(a) If $K = K_{n-4}$ has even number of vertices, then $\{x, u_i\}$ for $i \neq j$ in $K = K_{n-4}$ forms a $\gamma_{cp}$ set of G. Since, $\gamma_{cp} = n-2$, $n = 4$, $\chi = 0$, which is a contradiction. Hence no graph exists in this case.

(b) If $K = K_{n-4}$ has odd number of vertices, then $\{w, x, u_i\}$ forms a $\gamma_{cp}$ set of G. Since $\gamma_{cp} = n-2$, $n = 5$. But $\chi = 1$, which is a contradiction. Hence no graph exists in this case.

Sub case 2(II): $<S> = P_4$

Let the vertices of $P_4$ be $w, x, y, z$.

(i) If $K = K_{n-4}$ has even number of vertices. Since G is connected, at least one of the vertices say $u_i$ of $K = K_{n-4}$ is adjacent to $w$ (or equivalently $z$) or $x$ (or equivalently $y$).

(a) If $u_i$ is adjacent to $w$ then $\{y, z, u_i, u_j\}$ for $i \neq j$ forms a $\gamma_{cp}$ set of G. Since $\gamma_{cp} = n-2$, $n = 6$, $\chi = 2$. $K = K_{n-4} = K_2 = uv$. Let $w$ be adjacent to $u$. If $d(w) = d(x) = d(y) = 2$ and $d(z) = 1$, then $G \equiv P_5$. If $d(w) = 3$ then $\chi = 3$. Which is a contradiction. If $d(w) = d(x) = 2$, $d(y) = 3$, $d(z) = 1$ then $G \equiv C_4 (P_2P_2, 0, 0)$. If $d(w) = 2$, $d(x) = 3$, $d(y) = 2$, $d(z) = 1$ then $G \equiv G_1$. If $d(w) = d(x) = d(y) = d(z) = 2$, then $G \equiv C_6$.

(b) If $u_i$ is adjacent to $x$ then $\{w, x, u_i, u_j\}$ for $i \neq j$ form a $\gamma_{cp}$ set of G. Since $\gamma_{cp} = n-2$, $n = 6$, and $\chi = 2$, $\gamma_{cp} = 4$. Hence $K = K_{n-4} = K_2 = uv$. Let $u$ be adjacent to $x$.

If $d(w) = d(z) = 1$, $d(y) = 2$, $d(x) = 3$ then $G \equiv P_5 (0, 0, 1, 0, 0)$. If $d(w) = d(y) = d(z) = 2$, $d(x) = 3$ then $G \equiv G_1$. If $d(w) = d(z) = 1$, $d(y) = d(x) = 3$ then $G \equiv C_4 (P_2P_2, 0, 0)$. If $d(w) = 2$, $d(y) = 3$, $d(x) = 1$ and $d(x) = 3$ then $G \equiv G_2$. If $d(w) = 2$, $d(z) = 1$, $d(y) = 2$, $d(x) = 3$ then $G \equiv G_1$. If $d(w) = 1$, $d(z) = 2$, $d(y) = 2$, $d(x) = 3$ then $G \equiv C_4 (P_3)$.
Complementary perfect domination number of a graph

G = \tilde{C}_4(P_2, P_2, 0, 0). If d(w) = 1, d(z) = 2, d(y) = d(x) = 3, then G = G_2. In all other cases no new graphs exists.

(ii) If K = K_{n-4} has odd number of vertices. Since G is connected, at least one of the vertices say u_i in K = K_{n-4} is adjacent to w (or equivalently z) or x (or equivalently y). In both cases no graph exists.

Subcase 2(III): \langle S \rangle = C_4

Let w, x, y, z be the vertices of C_4. Since G is connected there exists a vertex u_i in K = K_{n-4} adjacent to any one of \{w, x, y, z\}.

(a) If K = K_{n-4} has even number of vertices, then \{w, x, u_i, u_j\} for i \neq j in K = K_{n-4} forms a \gamma_{cp} set of G. Since \gamma_{cp} = n-2, n = 6, \chi = 2, which is a contradiction. Hence no graph exists in this case.

(b) If K = K_{n-4} has odd number of vertices then \{w, x, u_i\} forms a \gamma_{cp} set of G. Since \gamma_{cp} = n-2, n = 5, \chi = 1, which is a contradiction. Hence no graph exists in this case.

Subcase 2(IV): \langle S \rangle = K_3UK_1

Let vertices of K_3 be w, x, y and the vertices of K_1 be z. Since G is connected,

(i) There exists a vertex u_i in K = K_{n-4} adjacent to w and z.

(a) Suppose K = K_{n-4} has even number of vertices, then \{w, x, u_i, u_j\} for i \neq j in K = K_{n-4} forms a \gamma_{cp} set of G. Since \gamma_{cp} = n-2, n = 6, \chi = 2, which is a contradiction. Hence no graph exists in this case.

(b) Suppose K = K_{n-4} has odd number of vertices, then \{w, z, u_i\} forms a \gamma_{cp} set of G. Since \gamma_{cp} = n-2, n = 5, \chi = 1, which is a contradiction. Hence no graph exists in this case.

(ii) There exists a vertex u_i and u_j in K = K_{n-4} such that u_i is adjacent to w and u_j is adjacent to z.

(a) Suppose K = K_{n-4} has even number of vertices, then \{w, z, u_i, u_j\} forms a \gamma_{cp} set of G. Since \gamma_{cp} = n-2, n = 6, \chi = 2, K = K_{n-4} = K_2 = uv and \gamma_{cp} = 4, so that \chi = 2, which is a contradiction. Hence no graph exists in this case.

(b) Suppose K = K_{n-4} has odd number of vertices, then \{x, u_i, z\} forms a \gamma_{cp} set of G. Since \gamma_{cp} = n-2, n = 5, \chi = 1, which is a contradiction. Hence no new graph exists.

Subcase 2(V): \langle S \rangle = K_2UK_2

Let the vertices of K_2UK_2 be w, x and y, z. Since G is connected,

(i) There exists a vertex u_i in K = K_{n-4} adjacent to any one of \{w, x\} say w and u_i is adjacent to any one of \{y, z\} say y.
(a) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{w,x,y,z,u_i\} \) forms a \( \gamma_{cp} \)-set of \( G \). Since \( \gamma_{cp} = n - 2 \), \( n = 7 \) and \( \chi = 3 \), \( K = K_{n-4} = K_3 \). Let the vertices of \( K_3 \) be \( u_1,u_2,u_3 \). Let \( u_i \) be adjacent to \( w,y \). If \( d(w) = 2, d(y) = 2, d(x) = d(z) = 1 \) then \( G \cong C_4(2P_3) \). In all other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{x,y,z,u_j\} \) forms a \( \gamma_{cp} \)-set of \( G \). Since \( \gamma_{cp} = n - 2 \), \( n = 6 \) and \( \chi = 2 \), \( K = K_{n-4} = K_2 = uv, \gamma_{cp} = 4 \). Let \( u_i \) be adjacent by \( w,x \). If \( d(w) = 4, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong P_4(0,2,0,0) \). If \( d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong G_3 \). In all other cases no new graph exists.

(ii) If \( u_i \) is adjacent to any one of \( \{x,y,z\} \) without loss of generality, Let \( u_i \) be adjacent to \( x \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{w, y, z, u_i\} \) for \( u_i \neq u_j \) forms a \( \gamma_{cp} \)-set of \( G \). Since \( \gamma_{cp} = n - 2 \), \( n = 6 \) and \( \chi = 2 \), \( K = K_{n-4} = K_2 = uv, \gamma_{cp} = 4 \). If \( d(w) = 4, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong P_4(0,2,0,0) \). If \( d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong G_3 \). In all other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{w,x,y,z,u_i\} \) forms a \( \gamma_{cp} \)-set of \( G \). Since \( \gamma_{cp} = n - 2 \), \( n = 7 \) and \( \chi = 3 \), Which is a contradiction, Hence no graph exists.

Subcase 2(VI): \( <S> = K_{1,3} \).

Let \( w \) be the root vertex and \( x,y,z \) are adjacent to \( w \). Since \( G \) is connected there exists a vertex \( u_i \) in \( K = K_{n-4} \) is adjacent to \( w \) or any one of \( \{x,y,z\} \).

(i) If \( u_i \) is adjacent to \( w \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{x,y,z,u_j\} \) for \( u_j \neq u_i \) forms a \( \gamma_{cp} \)-set of \( G \). Since \( \gamma_{cp} = n - 2 \), \( n = 6 \) and \( \chi = 2 \), \( K = K_{n-4} = K_2 = uv, \gamma_{cp} = 4 \). If \( d(w) = 4, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong P_4(0,2,0,0) \). If \( d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong G_3 \). In all other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{x,y,z,u_k\} \) forms a \( \gamma_{cp} \)-set of \( G \). Since \( \gamma_{cp} = n - 2 \), \( n = 7 \) and \( \chi = 1 \), Which is a contradiction, Hence no graph exists.

(ii) If \( u_i \) is adjacent to any one of \( \{x,y,z\} \) without loss of generality, Let \( u_i \) be adjacent to \( x \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{w,y,z,u_i\} \) for \( u_i \neq u_j \) forms a \( \gamma_{cp} \)-set of \( G \). Since \( \gamma_{cp} = n - 2 \), \( n = 6 \) and \( \chi = 2 \), \( K = K_{n-4} = K_2 = uv, \gamma_{cp} = 4 \). If \( d(w) = 4, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong P_4(0,2,0,0) \). If \( d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong G_3 \). In all other cases no new graph exists.
(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{y, z, u_i\} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2 \), \( n = 5 \) and \( \chi = 1 \), which is a contradiction. Hence no graph exists in this case.

**Subcase 2(VII):** \( \langle S \rangle = P_3 \cup K_1 \)

Let the vertices of \( P_3 \) be \( \{w, x, y\} \), and the vertex of \( K_1 \) is \( z \). Since \( G \) is connected, the following possibilities have been made.

i) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w, y\} \) and \( \{z\} \).

ii) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w, y\} \) and \( u_j \) \( i \neq j \) is adjacent to \( z \).

iii) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to both \( x \) & \( z \).

iv) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to \( x \) and \( u_j \) \( (i \neq j) \) is adjacent to \( z \).

i) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w, y\} \) say \( w \) and \( \{z\} \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{x, y, z, u_i\} \) \( u_i \neq u_j \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2 \), \( n = 6 \) and \( \chi = 2 \), \( \gamma_{cp} = 4 \). \( K = K_{n-4} = K_2 = uv \). If \( d(w) = 2 \), \( d(x) = 2 \), \( d(y) = 1 \), \( d(z) = 1 \) then \( G \cong \tilde{P}_5 \) \( (0, 1, 0, 0, 0) \). If \( d(w) = 2 \), \( d(x) = 2 \), \( d(y) = 2 \), \( d(z) = 1 \) then \( G \cong C_4(2P_2, 0, 0, 0) \). If \( d(w) = 2 \), \( d(x) = 3 \), \( d(y) = 1 \), \( d(z) = 1 \) then \( G \cong G_5 \). In all other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{y, z, u_i\} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2 \), \( n = 5 \) and \( \chi = 1 \), which is a contradiction. Hence no graph exists in this case.

(ii) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w, y\} \) say \( w \) and \( \{z\} \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{x, y, z, u_i\} \) \( u_i \neq u_j \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2 \), \( n = 6 \) and \( \chi = 2 \), \( \gamma_{cp} = 4 \). \( K = K_{n-4} = K_2 = uv \). If \( d(w) = d(x) = 2 \), and \( d(y) = d(z) = 1 \) then \( G \cong \tilde{P}_6 \). If \( d(w) = d(x) = 2 \), \( d(y) = 2 \), \( d(z) = 1 \) then \( G \cong C_4(2P_2, 0, 0, 0) \). If \( d(w) = 2 \), \( d(x) = 3 \), \( d(y) = 1 \), \( d(z) = 1 \) then \( G \cong G_5 \). In all the other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{y, z, u_i\} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2 \), \( n = 5 \) and \( \chi = 1 \), which is a contradiction. Hence no graph exists in this case.

iii) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to both \( x \) & \( z \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{w, y, z, u_i\} \) \( u_i \neq u_j \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2 \), \( n = 6 \) and \( \chi = 2 \), \( \gamma_{cp} = 4 \). \( K = K_{n-4} = K_2 = uv \).
If $d(w) = 1, d(x) = 3, d(y) = 1, d(z) = 1$ then $G \cong P_4(0,1,1,0)$. If $d(w) = 1, d(x) = 3, d(y) = 2, d(z) = 1$ then $G \cong C_4(P_2,P_2,0,0)$. If $d(w) = 2, d(x) = 3, d(y) = 2, d(z) = 1$ then $G \cong G_5$. In all the other cases no new graph exists.

(b) Suppose $K = K_{n-4}$ has odd number of vertices, then $\{w,x,y,z,u_i\}$ forms a $\gamma_{cp}$ set of $G$. Since $\gamma_{cp} = n-2, n = 7$ and $\chi = 3, \gamma_{cp} = 5$. Let the vertices of $K_3$ be $u_1, u_2, u_3$. If $d(w) = 1, d(x) = 3, d(y) = 1, d(z) = 1$ then $G \cong G_6$. If $d(w) = 1, d(x) = 3, d(y) = 1, d(z) = 2$ then $G \cong G_7$. If $d(w) = 1, d(x) = 4, d(y) = 1, d(z) = 1$ then $G \cong G_8$. If $d(w) = 1, d(x) = 4, d(y) = 1, d(z) = 2$ then $G \cong G_9$. In all the other cases no new graph exists.

iv) There exists a vertex $u_i$ in $K = K_{n-4}$ which is adjacent to $x$ and $u_j, i \neq j$ is adjacent to $z$.

Subcase 2(VIII): $<S> = C_3(P_2)$

Let the vertices of $C_3$ be $w,x,y$. Let the new vertex $z$ be adjacent to $w$. Since $G$ is connected, the following possibilities have been made.

i) There exists a vertex $u_i$ in $K = K_{n-4}$ which is adjacent to $w$.

ii) There exists a vertex $u_i$ in $K = K_{n-4}$ which is adjacent to one of $\{x,y\}$ say $x$.

iii) There exists a vertex $u_i$ in $K = K_{n-4}$ which is adjacent to the $z$.

In all the cases it can be verified that no graph exists.

Subcase 2(IX): $<S> = K_2 \cup K_2$

Let the vertices of $K_2$ be $w,x$ and the vertices of $K_2$ be $y,z$. Since $G$ is connected to the following possibilities have been made.

i) There exists a vertex $u_i$ in $K = K_{n-4}$ which is adjacent to any one of $\{w,x\}$ say $w,y$ and $z$.

ii) There exists a vertex $u_i$ in $K = K_{n-4}$ which is adjacent to any one of $\{w,x\}$ say $w$ and $u_j$ is adjacent to $y,z$, for $i \neq j$ in $K = K_{n-4}$.
There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w,x\} \) say \( w \),\( y \) and \( u_j \) is adjacent to \( z \) for \( i \neq j \) in \( K = K_{n-4} \).

iv) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w,x\} \) say \( w \) and \( u_j \) is adjacent to \( y \) and \( u_k \) is adjacent to \( z \) for \( i \neq j \neq k \) in \( K = K_{n-4} \).

i) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w,x\} \), say \( w,y,z \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{x,y,z,u_j\} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2, n = 6 \) and \( \chi = 2, \gamma_{cp} = 4 \). \( K = K_{n-4} = K_2 = uv \). If \( d(w) = 2, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong P_4(0,2,0,0) \). If \( d(w) = 2, d(x) = 2, d(y) = 1, d(z) = 1 \) then \( G \cong C_4 \). In all the other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{w,x,y,z,u_i\} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2, n = 7 \) and \( \chi = 3, \gamma_{cp} = 5 \). \( K = K_{n-4} = K_3 \). Let the vertices of \( K_3 \) be \( u_1,u_2,u_3 \). If \( d(w) = 2, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong C_3(P_3,2P_2,0,0) \) . If \( d(w) = 2, d(x) = 2, d(y) = 1, d(z) = 1 \) then \( G \cong C_4 \). In all the other cases no new graph exists.

ii) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w,x\} \) say \( w \),\( y \) and \( u_j \) is adjacent to \( z \) for \( i \neq j \) in \( K = K_{n-4} \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{x,y,z,u_j\} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2, n = 6 \) and \( \chi = 2, \gamma_{cp} = 4 \). \( K = K_{n-4} = K_2 = uv \). If \( d(w) = 2, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong P_4(0,2,0,0) \). If \( d(w) = 2, d(x) = 2, d(y) = 1, d(z) = 1 \) then \( G \cong C_4 \). In all the other cases no new graph exists.
b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{w,x,y,z,u_i\} \) forms a \( \gamma_{cp} \) set of \( G \). Since
\[
\gamma_{cp} = n-2, n = 7 \quad \text{and} \quad \chi = 3, \gamma_{cp} = 5. \quad K = K_{n-4} = K_3.
\]
Let the vertices of \( K_3 \) be \( u_1, u_2, u_3 \). If \( d(w) = 2, d(x) = d(y) = 1, d(z) = 2 \) then \( G \cong G_{14} \). If \( d(w) = 2, d(x) = 2, d(y) = 2, d(z) = 2 \) then \( G \cong G_{15} \). If \( d(w) = 2, d(x) = 1, d(y) = 2, d(z) = 2 \) then \( G \cong G_{16} \). If \( d(w) = 2, d(x) = 2, d(y) = 2, d(z) = 2 \) then \( G \cong G_{17} \). If \( d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong G_{18} \). If \( d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong G_{19} \). If \( d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong G_{20} \). If \( d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \) then \( G \cong G_{21} \). In all the other cases no new graph exists.

iv) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to any one of \( \{w,x\} \) say \( w \) and \( u_j \) is adjacent to \( y \) and \( u_k \) is adjacent to \( z \) for \( i \neq j \neq k \) in \( K = K_{n-4} \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{x,y,z,u_i\} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2, n = 6 \quad \text{and} \quad \chi = 2, \gamma_{cp} = 4. \quad K = K_{n-4} = K_2 = uv \), which is impossible. Hence no graph exists in this case.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{w,x,y,z,u_i\} \) forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2, n = 7 \quad \text{and} \quad \chi = 3, \gamma_{cp} = 5. \quad K = K_{n-4} = K_3 \). Let the vertices of \( K_3 \) be \( u_1, u_2, u_3 \). If
\[
d(w) = 2, d(x) = 1, d(y) = 1, d(z) = 1 \quad \text{then} \quad G \cong G_{14}. \]
\[
d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \quad \text{then} \quad G \cong G_{17}. \]
\[
d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \quad \text{then} \quad G \cong G_{20}. \]
\[
d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \quad \text{then} \quad G \cong G_{23}. \]
\[
d(w) = 3, d(x) = 1, d(y) = 1, d(z) = 1 \quad \text{then} \quad G \cong G_{25}. \]
In all the other cases no new graph exists.

Subcase 2(X): \( \langle S \rangle = K_4-e \) where \( e \) is any edge of \( K_4 \).

Since \( G \) is connected to the following possibilities have been made.

i) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to one of \( \{w,x\} \).

ii) There exists a vertex \( u_i \) in \( K = K_{n-4} \) which is adjacent to one of \( \{y,z\} \).

In all the cases it can be verified that no graph exists.

Sub case 2 (XI): \( \langle S \rangle = K_4 \).

Let the vertices of \( K_4 \) be \( w,x,y,z \). Since \( G \) is connected to the following possibilities have been made.

i) One of the vertices \( u_i \) in \( K = K_{n-4} \) adjacent to all the vertices of \( S \).

ii) One of the vertices \( u_i \) in \( K = K_{n-4} \) adjacent to three vertices of \( S \) and one vertex is adjacent to \( u_j \).

iii) One of the vertices \( u_i \) in \( K = K_{n-4} \) adjacent to two vertices of \( S \) and remaining two vertices are adjacent to \( u_j \).

iv) One of the vertices \( u_i \) in \( K = K_{n-4} \) adjacent to two vertices of \( S \), one vertex is adjacent to \( u_j \) and another vertex is adjacent to \( u_k \).
v) All the vertices are adjacent to the distinct vertices of \( K = K_{n-4} \).
i) One of the vertices \( u_i \) in \( K = K_{n-4} \) adjacent to all the vertices of \( S \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{w,x,y,z,u_1,u_j\} \ i\neq j \)
forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2,n = 8 \), and \( \chi = 4, \gamma_{cp} = 6 \), \( K = K_{n-4} = K_4 \). Let
the vertices of \( K_4 \) be \( u_1,u_2,u_3,u_4 \). If \( d(w) = 1,d(x) = 1 \), \( d(y) = 1,d(z) = 1 \) then \( G \sim G_4 \)
(4,0,0,0). In all the other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{w,x,y,z,u_1\} \) forms a
\( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2,n = 7, \gamma_{cp} = 5, \chi = 3 \), \( K = K_{n-4} = K_3 \). Let
the vertices of \( K_3 \) be \( u_1,u_2,u_3 \). If \( d(w) = 2,d(x) = 1 \), \( d(y) = 1,d(z) = 1 \) then \( G \sim G_{27} \). If \( d(w) = 2,d(x) = 2 \), \( d(y) = 1,d(z) = 1 \)
then \( G \sim G_{28} \). If \( d(w) = 2,d(x) = 2 \), \( d(y) = 2,d(z) = 1 \) then \( G \sim G_{29} \). If \( d(w) = 2,d(x)
= 2 \), \( d(y) = 2,d(z) = 2 \) then \( G \sim G_{30} \). In all the other cases no new graph exists.

ii) One of the vertices \( u_i \) in \( K = K_{n-4} \) adjacent to three vertices of \( S \) say \( w,x,y \) and one
vertex \( z \) is adjacent to \( u_j \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{w,x,y,z,u_1,u_j\} \ i\neq j \)
forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2,n = 7, \gamma_{cp} = 5, \chi = 3, \chi = 3 \), \( K = K_{n-4} = K_3 \). Let
the vertices of \( K_3 \) be \( u_1,u_2,u_3 \). If \( d(w) = 1,d(x) = 1,d(y) = 1 \), \( d(z) = 1 \) then \( G \sim G_3 \)
(3,1,0,0). In all the other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{w,x,y,z,u_1\} \) forms a
\( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2,n = 7, \gamma_{cp} = 5, \chi = 3, \chi = 3 \), \( K = K_{n-4} = K_4 \). Let
the vertices of \( K_4 \) be \( u_1,u_2,u_3,u_4 \). If \( d(w) = 1,d(x) = 1,d(y) = 1,d(z) = 1 \) then \( G \sim K_4 \)
(3,1,0,0). In all the other cases no new graph exists.

iii) One of the vertices \( u_i \) in \( K = K_{n-4} \) adjacent to two vertices of \( S \) say \( w,x \) and
remaining two vertices \( \{y,z\} \) is adjacent to \( u_j \).

(a) Suppose \( K = K_{n-4} \) has even number of vertices, then \( \{w,x,y,z,u_1,u_j\} \ i\neq j \)
forms a \( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2,n = 7, \gamma_{cp} = 5, \chi = 3, \chi = 3 \), \( K = K_{n-4} = K_4 \). Let
the vertices of \( K_4 \) be \( u_1,u_2,u_3,u_4 \). If \( d(w) = 1,d(x) = 1,d(y) = 1,d(z) = 1 \) then \( G \sim K_4 \)
(2,2,0,0). In all the other cases no new graph exists.

(b) Suppose \( K = K_{n-4} \) has odd number of vertices, then \( \{w,x,y,z,u_1\} \) forms a
\( \gamma_{cp} \) set of \( G \). Since \( \gamma_{cp} = n-2,n = 7, \gamma_{cp} = 5, \chi = 3, \chi = 3 \), \( K = K_{n-4} = K_4 \). Let
the vertices of \( K_4 \) be \( u_1,u_2,u_3,u_4 \). If \( d(w) = 1,d(x) = 1,d(y) = 1,d(z) = 1 \) then \( G \sim K_4 \)
(2,2,0,0). In all the other cases no new graph exists.
G. Mahadevan and B. Ayisha

If $d(w) = 2, d(x) = 2, d(y) = 1, d(z) = 1$ then $G \cong G_{28}$. If $d(w) = 2, d(x) = 2,$ $d(y) = 1, d(z) = 1$ then $G \cong G_{31}$. If $d(w) = 2, d(x) = 1, d(y) = 1, d(z) = 1$ then $G \cong G_{32}$. In all the other cases no new graph exists.

iv) One of the vertices $u_i$ in $K = K_{n-4}$ adjacent to two vertices of $S$ say $w, x$ and one vertex say $y$ is adjacent to $u_i$ and another vertex say $z$ is adjacent to $u_k$.$i \neq j \neq k$.

(a) Suppose $K = K_{n-4}$ has even number of vertices, then $\{w,x,y,z,u_i, u_j\}$ forms a $\gamma_{cp}$ set of $G$. Since $\gamma_{cp} = n-2, n = 8$ and $\chi = 4, \gamma_{cp} = 6$. Let the vertices of $K_4$ be $u_1, u_2, u_3, u_4$. If $d(w) = 1, d(x) = 1, d(y) = 1, d(z) = 1$ then $G \cong K_4(2,1,1,0)$. In all the other cases no new graph exists.

(b) Suppose $K = K_{n-4}$ has odd number of vertices, then $\{w,x,y,z,u_i\}$ forms a $\gamma_{cp}$ set of $G$. Since $\gamma_{cp} = n-2, n = 7,$ $\chi = 3, \gamma_{cp} = 5$. Let the vertices of $K_3$ be $u_1, u_2, u_3$. If $d(w) = 1, d(x) = 1, d(y) = 1, d(z) = 1$ then $G \cong K_3(2P_2, P_2, P_2)$. If $d(w) = 1, d(x) = 2, d(y) = 2, d(z) = 2$ then $G \cong G_{19}$. If $d(w) = 1, d(x) = 1, d(y) = 2, d(z) = 1$ then $G \cong G_{34}$. If $d(w) = 2, d(x) = 1, d(y) = 2, d(z) = 1$ then $G \cong G_{34}$. In all the other cases no new graph exists.

v) All the vertices $\{w,x,y,z\}$ are adjacent to the distinct vertices of $K = K_{n-4}$.

(a) Suppose $K = K_{n-4}$ has even number of vertices, then $\{w,x,y,z,u_i, u_j\}$ forms a $\gamma_{cp}$ set of $G$. Since $\gamma_{cp} = n-2, n = 8$ and $\chi = 4, \gamma_{cp} = 6$. Let the vertices of $K_4$ be $u_1, u_2, u_3, u_4$. If $d(x) = 2, d(y) = 1, d(z) = 1$ then $G \cong K_4(2,1,1,0)$. If $d(w) = 2, d(x) = 2, d(y) = 2, d(z) = 1$ then $G \cong G_{19}$. If $d(w) = 2, d(x) = 1, d(y) = 2, d(z) = 1$ then $G \cong G_{19}$. If $d(w) = 2, d(x) = 1, d(y) = 2, d(z) = 1$ then $G \cong G_{19}$. If $d(w) = 2, d(x) = 1, d(y) = 2, d(z) = 1$ then $G \cong G_{19}$. In all the other cases no new graph exists.

Case 3: $\gamma_{cp} = n-4$ and $\chi = n-2$

Then $<S> = K_2$ or $K_2$. $G$ contains a clique $K = K_{n-2}$ (or) does not contains a clique $K = K_{n-2}$. $G$ contains a clique $K = K_{n-2}$. If $G$ contains a clique $K = K_{n-2}$.

Subcase 3(I): $<S> = K_2$

Let the vertices of $K_2$ be $\{x,y\}$. Since $G$ is connected, there exists a vertex say $u_i$ in $K = K_{n-2}$ which is adjacent to $x$ (or equivalently $y$).

(a) Assume $K = K_{n-2}$ has even number of vertices, then $\{y,u_i\}$ for $i \neq j$ in $K = K_{n-2}$ forms a $\gamma_{cp}$ set of $G$. Since $\gamma_{cp} = n-4, n = 6, \chi = 4, \gamma_{cp} = 2$. Let the vertices of $K_4$ be $u_1, u_2, u_3, u_4$. If $d(x) = 2, d(y) = 1$, then $G \cong K_4(P_3)$. If $d(x) = 2, d(y) = 2$, then $G \cong G_{35}$. If $d(x) = d(y) = 2$, then $G \cong G_{36}$. If $d(x) = 2, d(y)$
been made, then $G = \tilde{G}_{36}$. If $d(x) = 3$, $d(y) = 2$, then $G = \tilde{G}_{37}$. If $d(x) = 3$, $d(y) = 2$, then $G = \tilde{G}_{38}$. If $d(x) = 2$, $d(y) = 3$, then $G = \tilde{G}_{37}$. If $d(x) = 2$, $d(y) = 3$, then $G = \tilde{G}_{38}$. If $d(x) = 2$, $d(y) = 4$, then $G = \tilde{G}_{39}$. If $d(x) = 3$, $d(y) = 2$, then $G = \tilde{G}_{39}$. If $d(x) = 3$, $d(y) = 1$, then $G = \tilde{G}_{40}$. If $d(x) = 3$, $d(y) = 1$, then $G = \tilde{G}_{40}$. If $d(x) = 3$, $d(y) = 3$, then $G = \tilde{G}_{41}$. If $d(x) = 3$, $d(y) = 3$, then $G = \tilde{G}_{42}$. If $d(x) = 3$, $d(y) = 3$, then $G = \tilde{G}_{42}$. If $d(x) = 3$, $d(y) = 3$, then $G = \tilde{G}_{43}$. If $d(x) = 4$, $d(y) = 3$, then $G = \tilde{G}_{44}$. If $d(x) = 4$, $d(y) = 1$, then $G = \tilde{G}_{45}$. If $d(x) = 3$, $d(y) = 2$, then $G = \tilde{G}_{46}$. If $d(x) = 4$, $d(y) = 4$, then $G = \tilde{G}_{47}$. In all the other cases no new graph exists.

Subcase 3(II): $\langle S \rangle = K_2$

Let the vertices of $K_2$ be $\{x, y\}$. Since $G$ is connected the following possibilities have been made,

i) The vertices $x, y$ are adjacent to $u_i$ in $K = K_{n-2}$.

ii) $x$ is adjacent to $u_i$ and $y$ is adjacent to $u_j \neq j$.

i) The vertices $x, y$ are adjacent to $u_i$ in $K = K_{n-2}$.

(a) Suppose $K = K_{n-2}$ has even number of vertices, then $\{x, y, u_i\}$ forms a $\gamma_{cp}$ set of $G$. Since $\gamma_{cp} = n-4$, $n = 7, \gamma_{cp} = 3, \chi = 5, K = K_{n-2} = K_5$. Let the vertices of $K_5$ be $u_1, u_2, u_3, u_4, u_5$. If $d(x) = 2$, $d(y) = 1$, then $G = \tilde{K}_5 (P_3)$. If $d(x) = 3$, $d(y) = 1$, then $G = \tilde{G}_{48}$. If $d(x) = 4$, $d(y) = 1$, then $G = \tilde{G}_{49}$. If $d(x) = 5$, $d(y) = 1$, then $G = \tilde{G}_{50}$. If $d(x) = 2$, $d(y) = 2$, then $G = \tilde{G}_{51}$. If $d(x) = 2$, $d(y) = 3$, then $G = \tilde{G}_{52}$. If $d(x) = 3$, $d(y) = 2$, then $G = \tilde{G}_{53}$. If $d(x) = 4$, $d(y) = 2$, then $G = \tilde{G}_{54}$. If $d(x) = 2$, $d(y) = 5$, then $G = \tilde{G}_{54}$. If $d(x) = 4$, $d(y) = 2$, then $G = \tilde{G}_{55}$. If $d(x) = 4$, $d(y) = 4$, then $G = \tilde{G}_{56}$. If $d(x) = 3$, $d(y) = 4$, then $G = \tilde{G}_{56}$. In all the other cases no new graph exists.

(b) Suppose $K = K_{n-2}$ has odd number of vertices, then $\{x, y, u_i\}$ forms a $\gamma_{cp}$ set of $G$. Since $\gamma_{cp} = n-4$, $n = 7, \gamma_{cp} = 3, \chi = 5, K = K_{n-2} = K_5$. Let the vertices of $K_5$ be $u_1, u_2, u_3, u_4, u_5$. If $d(x) = 1$, $d(y) = 1$, then $G = \tilde{K}_5 (2, 0, 0, 0, 0)$. If $d(x) = 2$, $d(y) = 1$, then $G = \tilde{G}_{57}$. If $d(x) = 3$, $d(y) = 1$, then $G = \tilde{G}_{58}$. If $d(x) = 4$, $d(y) = 1$, then $G = \tilde{G}_{59}$. In all the other cases no new graph exists.

ii) $x$ is adjacent to $u_i$ and $y$ is adjacent to $u_j \neq j$.

(a) Suppose $K = K_{n-2}$ has even number of vertices, then $\{x, y, u_i, u_j\}$ forms a $\gamma_{cp}$ set of $G$. Since $\gamma_{cp} = n-4$, $n = 8, \gamma_{cp} = 4, \chi = 6, K = K_{n-2} = K_6$. Let the vertices of $K_6$ be $u_1, u_2, u_3, u_4, u_5, u_6$. If $d(x) = 1$, $d(y) = 1$, then $G = \tilde{K}_6 (2, 0, 0, 0, 0, 0)$. In all the other cases no new graph exists.
(b) Suppose $K = K_{n-2}$ has odd number of vertices, then $\{x,y,u_i\}$ forms a $\gamma_c$ set of $G$. Since $\gamma_c = n-4$, $n = 7$,$\gamma_c = 3, \chi = 5, K = K_{n-2} = K_5$. Let the vertices of $K_5$ be $u_1,u_2,u_3,u_4,u_5$. If $d(x) = 1$, $d(y) = 1$, then $G \cong K_5 (1,1,0,0,0)$. If $d(x) = 2$, $d(y) = 1$, then $G \cong G_{60}$. If $d(x) = 3$, $d(y) = 1$, then $G \cong G_{61}$. If $d(x) = 4,d(y) = 1$, then $G \cong G_{62}$. If $d(x) = 2$, $d(y) = 2$, then $G \cong G_{63}$. If $d(x) = 2$, $d(y) = 3$, then $G \cong G_{64}$. If $d(x) = 3$, $d(y) = 2$, then $G \cong G_{64}$. In all the other cases no new graph exists.

If $G$ does not contain a clique $K$ on $n-2$ vertices, then it can be verified that no graph exists.

**Case 4:** $\gamma_c = n-6$ and $\chi = n$.

Since $\chi = n$, $G = K_n$. (i) If $K_n$ has even number of vertices then $\gamma_c = 2$, $n = 8$. So , $G \cong K_8$.

If $K_n$ has odd number of vertices then $\gamma_c = 1$, $n = 7$. So $G \cong K_7$.

**REFERENCES**


**Received: September, 2012**