On Fuzzy Subgroups of Finite Abelian Groups

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Abstract

In this paper, we determine the number of fuzzy subgroups of finite abelian group $G = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_n}$, where $p_1, p_2, \dots, p_n$ are all distinct primes. We have used the equivalence relation on the fuzzy subgroups of any group G defined by Sulaiman and Abdul Ghafur [3] in order to determine all fuzzy subgroups of the above mentioned group. We also determined the number of fuzzy subgroups of abelian group $G = Z_{p^n} \times Z_q$ under the same equivalence relation.

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1 Introduction

In 1965, Zadeh first introduced fuzzy set, while Mordeson et.al called him a pioneer of work on fuzzy subsets. After that paper, several aspects of fuzzy subsets were studied. In 1971, Rosenfeld [11] introduced fuzzy sets in the realm of group theory and formulated the concepts of fuzzy subgroups of a group. An increasing number of properties from classical group theory have been generalized.

One of the most important problem of fuzzy theory is to classify the fuzzy subgroups of a finite group [4]. Several problems have treated the particular case finite abelian group. Laszlo studied the construction of fuzzy subgroups of groups of the orders one to six. Zhang and Zhou [14] have determined the number of fuzzy subgroups of cyclic groups of the order p^n where p is a prime number. Murali and Makamba [8], considering a similar problem, found the number of fuzzy subgroups of groups of the order $p^n q^m$ where p and q are distinct primes. In [13], Tarnaucanu and Bentea established the recurrence

relation verified by the number of fuzzy subgroups of finite cyclic groups. Their result is improving Murali's work in [10, 9].

Raden Sulaiman and Abd Ghafur Ahmed [3], worked on the particular case of finite abelian cyclic groups $G = Z_p \times Z_q \times Z_r \times Z_s$, namely with p, q, r, s are distinct prime numbers. However, their approach is different from Tarnauceanu in [13]. Using their approach, we have generalized the case of fuzzy subgroups of finite abelian groups $G = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_n}$ where $p_1, p_2, \dots, p_n$ are all distinct primes.

2 Preliminaries

In this section we summarize the preliminary definition and result that are required later in this paper. In this section, a group G is assumed to be a finite group.

Definition 2.1 *Let X be a nonempty set. A fuzzy subset of X is a function μ from X into $[0, 1]$.*

Definition 2.2 *A fuzzy subset of G is called a fuzzy subgroup of G if*

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\}, \quad \forall x, y \in G$$

$$\mu(x^{-1}) \geq \mu(x), \quad \forall x \in G.$$

Theorem 2.3 [4]. *A fuzzy subset μ of G is a fuzzy subgroup of G if and only if there is a chain of subgroups of G ,*

$$P_1(\mu) \leq P_2(\mu) \leq \dots \leq P_n(\mu) = G$$

such that μ can be written as

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1(\mu) \\ \theta_2, & x \in P_2(\mu) \setminus P_1(\mu) \\ \vdots & \\ \theta_n, & x \in P_n(\mu) \setminus P_{n-1}(\mu) \end{cases}$$

Example 2.4 *Consider the group $G = Z_{48}$. Define functions $\mu, \gamma, \alpha, \beta$ as follows :*

$$\mu(x) = \begin{cases} 1, & x \in \{0, 4, 8, 12, 14, 16, 20, 22\} \\ \frac{2}{5}, & x \in \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 15, 17, 19, 21\} \end{cases}$$

$$\gamma(x) = \begin{cases} 1, & x \in \{0, 6, 12, 18, 24, 30, 36, 42\} \\ \frac{1}{2}, & x \in \{2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 34, 38, 40, 44, 46\} \\ \frac{1}{4}, & x \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \dots, 47\} \end{cases}$$

$$\alpha(x) = \begin{cases} \frac{1}{2}, & x \in \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44\} \\ \frac{2}{3}, & x \in \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46\} \\ 0, & x \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, \dots, 47\} \end{cases}$$

$$\beta(x) = \begin{cases} 1, & x \in \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44\} \\ \frac{1}{2}, & x \in \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46\} \\ \frac{1}{4}, & x \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, \dots, 47\} \end{cases}$$

Note that $P_1(\mu) = \{0, 4, 8, 12, 14, 16, 20, 22\}$ is not a subgroup of Z_{48} . According to Theorem 2.3, μ is not a fuzzy subgroup of Z_{48} , where as α , β and γ are fuzzy subgroups of Z_{48} .

3 Fuzzy Subgroups of $G = Z_{p_1} \times Z_{p_2} \times \dots \times Z_{p_n}$

Since without any equivalence relation on fuzzy subgroups of group G , the number of fuzzy subgroups is infinite, even for the trivial group $\{e\}$. So [3, 4] defined the equivalence relation on the set of all fuzzy subgroups of G . This Definition differs from that of Murali and Makamba [4,5], but it is equivalent to the Definitions of Dixit [11], Zhang [8] and Tarnauceanu [7]. We prefer to use this Definition given below:

Definition 3.1 [4]. Let μ, γ be fuzzy subgroups of G of the form

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1 \\ \theta_2, & x \in P_2 \setminus P_1(\mu) \\ \vdots & \\ \theta_m, & x \in P_m \setminus P_{m-1}(\mu) \end{cases}, \quad \gamma(x) = \begin{cases} \lambda_1, & x \in M_1 \\ \lambda_2, & x \in M_2 \setminus M_1(\gamma) \\ \vdots & \\ \lambda_n, & x \in M_n \setminus M_{n-1}(\gamma) \end{cases}$$

with

$\theta_i, \lambda_j \in [0, 1], \theta_k > \theta_l, \lambda_k > \lambda_l$ for $k < l$ and $\bigcup_{1 \leq i \leq m} P_i = G, \bigcup_{1 \leq j \leq n} M_j = G$.

Then we define that μ and γ are equivalent if $m = n$ and $A_i = B_i, \forall i \in \{1, 2, 3, \dots, n\}$.

It is easy to check that this relation is indeed an equivalence relation. Two fuzzy subgroups of G are said to be different if they are not equivalent.

Example 3.2 Let α, β, γ be fuzzy subgroups as in example 2.4.

Since $P_1(\gamma) \neq P_1(\beta), \forall i \in 1, 2, 3$. Thus α is equivalent to β whereas γ is not equivalent to β .

Lemma 3.3 [3] The number of fuzzy subgroups is equal to the number of chain on the lattice subgroups of G .

Theorem 3.4 [3] *Let H be a subgroup of G , and let the set of all subgroups of G which contain H (but are not equal to H) be $H_1, H_2, H_3, \dots, H_n = G$. Then*

$$O(F_{P_1=H}) = \sum_{i=1}^n O(F_{P_1=H_i})$$

Theorem 3.5 [3] *Let e be identity element of a group G . Then*

$$O(F_G) = 2.O(F_{P_1=\{e\}}).$$

Theorem 3.6 [3] *Let $G = Z_p \times Z_q \times Z_r \times Z_s \times Z_t$, where p, q, r, s and t are distinct primes. Then the number of fuzzy subgroups of G are 1080.*

Proof: Since the order of the group $G = Z_p \times Z_q \times Z_r \times Z_s \times Z_t$ is $pqrst$. Hence every nontrivial subgroups of G must be of order

$$p, q, r, s, t, pq, pr, ps, pt, r, qs, qt, rs, rt, pqr, pq, pqt, prs,$$

$$prt, pst, qst, rst, qrt, pqrs, pqrt, pqst, prst,qrst.$$

Hence every subgroup of G can be obtained from $G = Z_p \times Z_q \times Z_r \times Z_s \times Z_t$ by changing k pieces of Z_p, Z_q, Z_r, Z_s and Z_t to be 0, where $0 \leq k \leq 5$. For $k = 0$, the subgroup is G and for $k = n$ that subgroup is $\{e\} = \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\}$. Thus we have 30 nontrivial subgroups of G , namely

$$\begin{aligned} H_1 &= Z_p \times Z_q \times Z_r \times Z_s \times \{0\}, H_2 = Z_p \times Z_q \times Z_r \times \{0\} \times Z_t \\ H_3 &= Z_p \times Z_q \times \{0\} \times Z_s \times Z_t, H_4 = Z_p \times \{0\} \times Z_r \times Z_s \times Z_t \\ H_5 &= \{0\} \times Z_q \times Z_r \times Z_s \times Z_t \\ H_6 &= Z_p \times Z_q \times Z_r \times \{0\} \times \{0\}, H_7 = Z_p \times Z_q \times \{0\} \times Z_s \times \{0\} \\ H_8 &= Z_p \times \{0\} \times Z_r \times Z_s \times \{0\}, H_9 = \{0\} \times Z_q \times Z_r \times Z_s \times \{0\} \\ H_{10} &= Z_p \times \{0\} \times Z_r \times \{0\} \times Z_t, H_{11} = Z_p \times \{0\} \times \{0\} \times Z_s \times Z_t \\ H_{12} &= \{0\} \times Z_q \times Z_r \times Z_s \times \{0\}, H_{13} = \{0\} \times Z_q \times Z_r \times \{0\} \times Z_t \\ H_{14} &= \{0\} \times Z_q \times \{0\} \times Z_s \times Z_t, H_{15} = \{0\} \times \{0\} \times Z_r \times Z_s \times Z_t \\ H_{16} &= Z_p \times Z_q \times \{0\} \times \{0\} \times \{0\}, H_{17} = Z_p \times \{0\} \times \{0\} \times Z_s \times \{0\} \\ H_{18} &= Z_p \times \{0\} \times Z_r \times \{0\} \times \{0\}, H_{19} = Z_p \times \{0\} \times \{0\} \times \{0\} \times Z_t \\ H_{20} &= \{0\} \times Z_q \times \{0\} \times Z_s \times \{0\}, H_{21} = \{0\} \times Z_q \times Z_r \times \{0\} \times \{0\} \\ H_{22} &= \{0\} \times Z_q \times \{0\} \times \{0\} \times Z_t, H_{23} = \{0\} \times \{0\} \times Z_r \times Z_s \times Z_t \\ H_{24} &= \{0\} \times \{0\} \times \{0\} \times Z_s \times Z_t, H_{25} = \{0\} \times \{0\} \times Z_r \times Z_s \times \{0\} \\ H_{26} &= Z_p \times \{0\} \times \{0\} \times \{0\} \times \{0\}, H_{27} = \{0\} \times Z_q \times \{0\} \times \{0\} \times \{0\} \\ H_{28} &= \{0\} \times \{0\} \times Z_r \times \{0\} \times \{0\}, H_{29} = \{0\} \times \{0\} \times \{0\} \times Z_s \times \{0\} \\ H_{30} &= \{0\} \times \{0\} \times \{0\} \times \{0\} \times Z_t \end{aligned}$$

According to [3], we have $O(F_{P_1=G}) = 1$ and

$$O(F_{P_1=H_1}) = O(F_{P_1=H_2}) = O(F_{P_1=H_3}) = O(F_{P_1=H_4}) = O(F_{P_1=H_5}) = 1$$

$$O(F_{P_1=H_6}) = O(F_{P_1=H_7}) = \cdots = O(F_{P_1=H_{15}}) = 3$$

$$O(F_{P_1=H_{16}}) = O(F_{P_1=H_{17}}) = \cdots = O(F_{P_1=H_{25}}) = 13$$

and

$$O(F_{P_1=H_{26}}) = O(F_{P_1=H_{27}}) = \cdots = O(F_{P_1=H_{30}}) = 75$$

Then by [3], we obtain

$$O(F_{P_1=\{e\}}) = 5 \times 75 + 5 \times 1 + 10 \times 3 + 10 \times 13 = 540$$

and therefore by 3.5,

$$O(F_G) = 2 \times 540 = 1080$$

Theorem 3.7 Let $G = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_n}$ where $p_1, p_2, \dots, p_n$ are n distinct primes. Then the number of subgroups of G present in any layer of subgroup lattice of G are nC_r , $1 \leq r \leq n$, where n is number of distinct primes in a group G and r denotes the r^{th} layer on the subgroup lattice of the group.

Proof: Since order of G is $p_1, p_2, \dots, p_n$, where all these p_i 's are n distinct primes. Then the non-trivial subgroups of G must be of order

$$p_1, p_2, \dots, p_n,$$

$$p_1p_2, p_1p_3, \dots, p_{n-1}p_n,$$

$$p_1p_2p_3, p_1p_3p_4, \dots, p_{n-2}p_{n-1}p_n,$$

$$\vdots$$

$$p_1p_2 \cdots p_{n-1}, \dots, p_2, p_3, \dots, p_n,$$

Therefore, the non-trivial subgroups of G that we have in $(n-1)^{th}$ layer above identity in subgroups lattice diagram are

$$H_1 = Z_{p_1} \times \{0\} \times \{0\} \times \{0\} \times \{0\}_n$$

$$H_2 = \{0\} \times Z_{p_2} \times \{0\} \times \{0\} \times \{0\}_n$$

$$\vdots$$

$$H_i = \{0\} \times \{0\} \times Z_{p_i} \times \{0\} \times \{0\}_n$$

$$\vdots$$

$$H_n = \{0\} \times \{0\} \times \{0\} \times \{0\} \times Z_{p_n}$$

Thus by inclusion exclusion principal, we have $\binom{n}{n-1} = n$. Similarly for $(n-2)^{nd}$ layer, we have

$$\begin{aligned}
H_{12} &= Z_{p_1} \times Z_{p_2} \times \{0\} \times \cdots \times \{0\}_n, \\
H_{13} &= Z_{p_1} \times \{0\} \times Z_{p_3} \times \{0\} \times \cdots \times 0_n, \\
&\vdots \\
H_{1n} &= Z_{p_1} \times \{0\} \times \cdots \times \{0\} \times Z_{p_n}, \\
H_{23} &= \{0\} \times Z_{p_2} \times Z_{p_3} \times \{0\} \times \cdots \times 0_n, \\
H_{24} &= \{0\} \times Z_{p_2} \times \{0\} \times Z_{p_4} \times \{0\} \times \cdots \times \{0\}_n, \\
&\vdots \\
H_{(n-1)n} &= \{0\} \times \{0\} \times \cdots \times \{0\} \times Z_{p_{n-1}} \times Z_{p_n}
\end{aligned}$$

So, it is easy to see that number of subgroups of G on $(n-2)^{nd}$ layer is

$$\binom{n}{n-2} = \frac{n!}{2!(n-2)!}$$

The subgroups of G in r^{th} layer on the lattice subgroup by the inclusion exclusion principle are given by,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Similarly, in 1^{st} layer below group G , we have $\binom{n}{n-1} = n$.

Corollary 3.8 *Let H be a subgroup of $G = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_n}$ where $p_1, p_2, \dots, p_n$ are n distinct primes, in k^{th} layer ($1 \leq k \leq n$), below G in the lattice diagram. Then the number of subgroups containing a particular subgroup H , in r^{th} layer are ${}^k C_r$, where $1 \leq r \leq k$. Furthermore, the total number of subgroups of G containing H are $\sum_{i=1}^k {}^k C_i$.*

Proof: Let H_n be the identity subgroup of G present in n^{th} layer i.e., $H = \{e\} = \{0\} \times \{0\} \times \cdots \times \{0\}$ upto n times, then by above theorem, the number of subgroups containing H_n are ${}^n C_r$, $\forall r = 1, 2, \dots, n-1$. & applying the theorem 3.7, follows the result. Now, for H_{n-1} be a subgroup of G present in $(n-1)^{th}$ layer, then H_{n-1} must be of the form

$$H_{n-1} = \left\{ \prod_{i=1}^n Z_{p_i} \mid Z_{p_i} = 0 \text{ for } (n-1) \text{ positions} \right\}$$

Now H_{n-1} is contained in all those subgroups in r^{th} layer ($1 \leq r \leq n-2$) which has same positions for non-zero Z_{p_i} as in H_{n-1} . Then the remaining zeroes of H_n can be arranged in ${}^{n-1}C_r$ ways for each layer above and each of these new arranged subgroups of r^{th} layer contain H_{n-1} . Thus summing all these ${}^{n-1}C_r$ subgroups in each r^{th} layer, we obtain total number of subgroups containing H_{n-1} . Similarly for the subgroup H_k present in k^{th} layer of the form

$$H_k = \left\{ \prod_{i=1}^n Z_{p_i} \mid Z_{p_i} = 0 \text{ for } k \text{ positions} \right\}$$

is contained in those subgroups of r^{th} layer which has the same positions for non-zero Z_{p_i} as in H_k . Therefore, the remaining zeroes can be arranged in ${}^k C_r$ ways for each layer above and $\sum_{r=1}^k {}^k C_r$ will give total number of subgroups containing H_k .

Remark 3.9 The subgroup H_2 present in 2^{nd} layer below G of the lattice diagram is of the form

$$H_2 = \left\{ \prod_{i=1}^n Z_{p_i} \mid Z_{p_i} = 0 \text{ for } 2 \text{ positions} \right\}$$

and therefore can be arranged in ${}^2 C_r$ ways ($1 \leq r < 2$), the result follows. And the subgroup

$$H_1 = \left\{ \prod_{i=1}^n Z_{p_i} \mid Z_{p_i} = 0 \text{ for } 1 \text{ position} \right\}$$

is only contained in G .

Corollary 3.10 The number of fuzzy subgroups of $G = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_n}$ where $p_1, p_2, \dots, p_n$ are n distinct primes, is given by

$$O(F_G) = 2 \cdot O(F_{p_1=H_n}),$$

where

$$O(F_{p_1=H_n}) = \sum_{r=0}^{n-1} {}^n C_r \left\{ \sum_{i=0}^{r-1} {}^r C_i \left(\sum_{j=0}^{i-1} {}^i C_j (\dots (1 + {}^2 C_1)) \right) \right\},$$

where H_n is any subgroup present in n th layer and $n > 1$.

Proof: We use mathematical induction on n . $O(H_1) = 1, O(H_2) = 1 + {}^2 C_1 = 1 + 2 = 3$, so the result hold for all $n \leq k$,

$$O(F_{p_1=H_k}) = \sum_{r=0}^{k-1} {}^k C_r \left\{ \sum_{i=0}^{r-1} {}^r C_i \left(\sum_{j=0}^{i-1} {}^i C_j (\dots (1 + {}^2 C_1)) \right) \right\},$$

where H_k is subgroup present in k^{th} layer. We prove for $n = k + 1$. The order of H_{k+1} will be sum of orders of all subgroups containing it by 3.4 & 3.10, i.e.,

$$O(H_{k+1}) = \sum_{r=0}^k {}^{k+1}C_r \times O(H_r).$$

Where H_r is subgroup present in r^{th} layer ($0 \leq r \leq k$) Therefore,

$$O(H_{k+1}) = \sum_{r=0}^k {}^{k+1}C_r \times \left\{ \sum_{i=0}^{r-1} {}^rC_i \left(\sum_{j=0}^{i-1} {}^iC_j (\dots (1 + {}^2C_1)) \right) \right\}$$

Hence the result follows.

Corollary 3.11 *Let $G = Z_{p_1} \times Z_{p_2} \times \dots \times Z_{p_n}$ be an abelian group, then the number of fuzzy subgroups of G are*

$$O(F_G) = 2 \left[\sum_{i_1=0}^{n-1} \binom{n}{i_1} \left\{ \sum_{i_2=0}^{i_1-1} \binom{i_1}{i_2} \left(\sum_{i_3=0}^{i_2-1} \binom{i_2}{i_3} (\dots \left(\sum_{i_n=0}^{i_{n-1}-1} \binom{i_{n-1}}{i_n} \right) \right) \right\} \right],$$

4 Fuzzy Subgroups of an Abelian Group of Order $p^n q$

An Abelian Group of order $p^n q$, where p and q are distinct primes and n is any natural number, is a cyclic group of the form $G = Z_{p^n} \times Z_q$. Using the definition of equivalence as defined above, we determine the number of fuzzy subgroups of G .

Theorem 4.1 *The number of distinct fuzzy subgroups of $Z_{p^n} \times Z_q$ are given by*

$$O(F_G) = 2 \times O(H_n)$$

where,

$$O(H_n) = 2^{n-1}(n+2).$$

Proof: Since we have two types of subgroup of G of order p^l and $p^l q$ ($0 \leq l \leq k$). It is clear from lattice diagram that the subgroup of order H_l are shown in first column of lattice diagram of G . Therefore, the order of H_l is given as 2^l and ($0 \leq l \leq k$). Now for the second type of subgroup the order is given by $O(H_l) = 2^{l-1}(l+2)$. We prove it by induction on n . For $n=0$ & 1 , we have by 3.10, $1 = O(H_0) = 2^{0-1}(0+2)$ below G and $3 = O(H_1) = 2^{1-1}(1+2)$. Assume the statement is true for $n = k$ i.e., $O(H_k) = 2^{k-1}(k+2)$. now we shall show that it is true for $n = k+1$ i.e.,

$$O(H_{k+1}) = 2^k(\overline{k+1} + 2)$$

Since we know by theorem 3.4 that order of H_{k+1} will be the sum of orders of all those subgroups which contain H_{k+1} . Therefore, adding orders of H'_k s in k^{th} layer and sum of all those subgroups in $k - 1^{th}$ to zeroth layer below G , we have the order of H_{k+1} ,

$$\begin{aligned} \frac{2^k(k+2)}{2} + \frac{2^k(k+2)}{2} + 2^k &= \frac{2^k(2k+4)}{2} + 2^k \\ &= 2^k(k+2) + 2^k = 2^k(k+2+1) \\ &= 2^k(\overline{k+1} + 2). \end{aligned}$$

Hence the result follows.

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