Some New Identities for Continued Fraction of Order Six

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Abstract

In this paper, we establish some new modular relations between a continued fraction of order six $U(q)$ and $U(q^n)$ for $n=6, 10, 14, 22$.

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1. INTRODUCTION

In Chapter 16 of his second notebook [2], Ramanujan develops the theory of theta-function and is defined by

$$(1.1) \quad f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1,$$
$$= (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}$$

where $(a; q)_0 = 1$ and $(a; q)_{\infty} = (1-a)(1-aq)(1-aq^2) \cdots$.

Following Ramanujan, we defined

$$(1.2) \quad \varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}},$$

$$(1.3) \quad \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

$$(1.4) \quad f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q)_{\infty}$$

and

$$(1.5) \quad \chi(q) := (-q; q^2)_{\infty}.$$

The ordinary hypergeometric series ${}_2F_1(a, b; c; x)$ is defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where

$$(a)_0 = 1, (a)_n = a(a+1)(a+2)\dots(a+n-1), \text{ for } n \geq 1, |x| < 1.$$

Let

$$z(r) := z(r; x) := {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$$

and

$$q_r := q_r(x) := \exp\left(-\pi \csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)}\right).$$

where $r = 2, 3, 4, 6$ and $0 < x < 1$.

Let n denote a fixed natural number, and assume that

$$(1.6) \quad n \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)},$$

where $r = 2, 3, 4$ or 6 . Then a modular equation of degree n in the theory of elliptic functions of signature r is a relation between α and β induced by (1.6). We often say that β is of degree n over α and $m(r) := \frac{z(r; \alpha)}{z(r; \beta)}$ is called the multiplier. We also use the notations $z_1 := z_1(r) = z(r; \alpha)$ and $z_n := z_n(r) = z(r; \beta)$ to indicate that β has degree n over α . When the context is clear, we omit the argument r in q_r , $z(r)$ and $m(r)$.

The class invariant G_n is defined by

$$(1.7) \quad G_n := 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(q) = (4\alpha(1-\alpha))^{-\frac{1}{24}}.$$

The celebrated Rogers-Ramanujan continued fraction is defined as

$$(1.8) \quad R(q) := \frac{q^{1/5} f(-q, -q^4)}{f(-q^2, -q^3)} = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots, \quad |q| < 1,$$

On page 365 of his Lost Notebook [8], Ramanujan recorded five identities showing the relationships between $R(q)$ and five continued fractions $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$, and $R(q^5)$. He also recorded these identities at the scattered

places of his Notebooks [7]. L. J. Rogers [9] established the modular equations relating $R(q)$ and $R(q^n)$ for $n=2,3,5$, and 11. The last of these equations cannot be found in Ramanujan's works.

The Ramanujan's cubic continued fraction $G(q)$ is defined as

$$(1.9) \quad G(q) := \frac{q^{1/3}f(-q, -q^5)}{f(-q^3, -q^3)} = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \cdots, \quad |q| < 1,$$

The continued fraction (1.9) was first introduced by Ramanujan in his second letter to G. H. Hardy [6]. He also recorded the continued fraction (1.9) on page 365 of his Lost Notebook [8] and claimed that there are many results for $G(q)$ similar the results obtained for the famous Rogers-Ramanujan continued fraction (1.8).

The Ramanujan Göllnitz-Gordon continued fraction [6, p. 44], [4], [8] is defined as follows:

$$(1.10) \quad H(q) := \frac{q^{1/2}f(-q^3, -q^5)}{f(-q, -q^7)} = \frac{q^{1/2}}{1} + \frac{q^2}{1+q^3} + \frac{q^4}{1+q^5} + \cdots, \quad |q| < 1,$$

Motivated by the above cited works on the continued fractions, in this paper, we established the modular relation between continued fraction $U(q)$ and $U(q^n)$ for $n = 6, 10, 14, 22$.

$$(1.11) \quad \begin{aligned} U(q) &:= q^{1/4} \frac{(q, q^6)(q^5, q^6)}{(q^2, q^6)(q^4, q^6)} \\ &= \frac{q^{-1/4}(1-q^2)}{(1-q^{3/2})+} \frac{(1-q^{1/2})(1-q^{7/2})}{q^{1/2}(1-q^{3/2})(1+q^3)+} \frac{(1-q^{5/2})(1-q^{13/2})}{q^{3/2}(1-q^{3/2})(1+q^6)+} \dots \end{aligned}$$

The continued fraction (1.11) was first established by K. R. Vasuki, N. Bhaskar and G. Sharath [10] $V(q)$ and $V(q^n)$ for $n=2, 3, 5, 7$ and 11.

2. PRELIMINARY RESULTS

In this section, we collect the necessary results required to prove our main results.

Lemma 2.1. [10] *If $x := U(q)$ and $y := U(q^2)$, then*

$$(2.1) \quad x^4 + 3x^4y^2 - y^2 + y^4 = 0.$$

Lemma 2.2. [10] *If $x := U(q)$ and $y := U(q^3)$, then*

$$(2.2) \quad x^3 - 3y^3x^2 + 3y^2x - y = 0.$$

Lemma 2.3. [10] *If $x := U(q)$ and $y := U(q^5)$, then*

$$(2.3) \quad y^6 + 9x^5y^5 - 5xy^5 + 5x^2y^4 - 5x^4y^2 + xy - 5x^5y - x^6 = 0.$$

Lemma 2.4. [10] *If $x := U(q)$ and $y := U(q^7)$, then*

$$(2.4) \quad \begin{aligned} &21x^5y^5 - 21x^7y^3 - 21x^3y^7 + xy - 7xy^5 + 14x^2y^6 \\ &- 7x^5y - 14x^6y^2 + y^8 - x^8 + 27x^7y^7 + 7x^3y^3 = 0. \end{aligned}$$

Lemma 2.5. [10] If $x := U(q)$ and $y := U(q^{11})$, then

$$\begin{aligned}
 (2.5) \quad & y^{12} + x^{12} - 495x^3y^7 + 297x^5y - 495x^7y^3 + 297x^3y^3 \\
 & - 308x^7y^7 - x^{11}y^{11} - 22x^3y^{11} + 66y^{10}x^2 - 243xy \\
 & - 924x^5y^5 + 297xy^5 + 11x^{11}y^7 + 92x^6y^6 - 66x^9y \\
 & - 165x^5y^9 + 495x^8y^4 + 495x^4y^8 + 11x^7y^{11} - 66xy^9 \\
 & + 11x^9y^9 - 165x^9y^5 - 22x^{11}y^3 + 66x^{10}y^2 = 0.
 \end{aligned}$$

3. RELATION BETWEEN $U(q)$ AND $U(q^n)$

Theorem 3.1. If $u := U(q)$ and $v := U(q^6)$, then

$$\begin{aligned}
 (3.1) \quad & (-9v^2 - 27v^6 - 1 - 27v^4)u^{12} + (30v^2 + 27v^8 + 63v^6 \\
 & + 42v^4 - 81v^{10} - 81v^{12})u^8 + (-27v^6 - 12v^2 - 51v^4 \\
 & + 27v^{10} + 63v^8)u^4 + v^2 + 5v^4 + 3v^6 - 9v^8 = 0.
 \end{aligned}$$

Proof. Using the equations (2.1), then we have to express y in terms of x .

$$(3.2) \quad y = \sqrt{\frac{1}{2} - \frac{3}{2}x^4 - \frac{1}{2}m}.$$

where $m = \sqrt{9x^8 - 10x^4 + 1}$

Replace q to q^2 in the equation (2.2) and using the above equation (3.2), we arrive at the equation (3.1). $\square$

Theorem 3.2. If $u := U(q)$ and $v := U(q^{10})$, then

$$\begin{aligned}
 (3.3) \quad & (18v^2 + 1 + 1458v^{10} + 135v^4 + 729v^{12} + 540v^6 + 1215v^8)u^{24} \\
 & + (-30v^4 + 39366v^{20} + 1845v^6 + 19683v^{22} + 10935v^{18} - 29160v^{16} \\
 & - 2916v^{10} - 29160v^{14} - 10368v^{12} - 195v^2)u^{20} + (300v^2 + 4860v^{18} \\
 & + 169695v^{16} - 180v^{10} - 116640v^{20} - 43740v^{22} + 14385v^8 - 945v^4 \\
 & - 3030v^6 - 66015v^{12} + 41310v^{14})u^{16} + (-24030v^{18} + 890v^6 + 3840v^{10} \\
 & + 85860v^{20} + 1060v^4 - 17160v^8 + 31590v^{22} - 154440v^{16} - 130v^2 \\
 & - 11520v^{14} + 84040v^{12})u^{12} + (-20v^6 - 1530v^{10} - 7335v^{12} - 8100v^{22} \\
 & - 160v^4 + 9090v^{18} + 20v^2 - 8505v^{20} + 60v^{14} + 14385v^{16} + 2095v^8)u^8 \\
 & + (-30v^{20} - 40v^8 + 108v^{14} - 5v^6 + 585v^{22} + 120v^{10} - 128v^{12} - 615v^{18} \\
 & - v^2 + 6v^4)u^4 + v^{24} - 20v^{18} - 6v^{14} + 15v^{20} - 6v^{22} + 15v^{16} + v^{12} = 0.
 \end{aligned}$$

Proof. Using the equations (3.2) and replace q to q^2 in the equation (2.3), we arrive at the equation (3.3). $\square$

Theorem 3.3. *If $u := U(q)$ and $v := U(q^{14})$, then*

(3.4)

$$\begin{aligned}
& (-13608v^{10} - 1512v^6 - 24v^2 - 1 - 20412v^{12} - 6561v^{16} - 252v^4 - 5670v^8 \\
& - 17496v^{14})u^{32} + (101304v^8 + 980v^2 - 857304v^{18} - 4251528v^{28} \\
& - 2480058v^{26} + 2204496v^{20} + 3306744v^{24} + 5373459v^{22} - 7616v^4 \\
& - 540189v^{14} + 106974v^{10} - 104328v^{12} - 1594323v^{30} - 1248048v^{16} \\
& - 10563v^6)u^{28} + (-3024v^2 - 40987296v^{18} - 2171736v^{10} - 37343754v^{20} \\
& + 22228668v^{24} - 12124728v^{26} + 118244v^6 - 3011526v^{12} + 7154v^4 \\
& + 33985980v^{22} + 16226028v^{14} + 826686v^{28} + 17186904v^{16} + 4960116v^{30} \\
& + 102284v^8)u^{24} + (5866000v^{12} - 542136v^8 - 5786802v^{30} + 13608v^4 \\
& + 60505704v^{20} - 24807888v^{16} + 3495975v^{10} + 3213v^2 + 75997467v^{18} \\
& - 173824v^6 - 76007484v^{22} + 29714769v^{26} - 54255096v^{24} + 13226976v^{28} \\
& - 27250482v^{14})u^{20} + (-3433836v^{12} - 1718948v^{10} + 33483618v^{24} \\
& + 3123036v^{30} - 30904524v^{20} - 1428v^2 - 10961244v^{28} + 11408684v^{16} \\
& + 46411596v^{22} - 42649068v^{18} + 80052v^6 + 14216356v^{14} + 413378v^8 \\
& - 15036v^4 - 19452636v^{26})u^{16} + (-2814721v^{14} + 9083494v^{18} + 746984v^{12} \\
& - 4879224v^{24} + 312788v^{10} - 13587v^6 + 4693248v^{26} + 5866000v^{20} \\
& + 1102248v^{28} + 2016v^4 + 294v^2 - 780759v^{30} - 74424v^8 - 10487925v^{22} \\
& - 2756432v^{16})u^{12} + (616v^6 - 15540v^{10} - 28v^2 + 723912v^{22} + 14v^4 \\
& + 3388v^8 + 168672v^{14} + 64386v^{28} + 212184v^{16} + 81648v^{30} + 102284v^{24} \\
& - 354732v^{26} - 334614v^{20} - 51226v^{12} - 600964v^{18})u^8 + (-8v^4 + 392v^{14} \\
& - 273v^{10} + v^2 - 1288v^{20} + 3521v^{26} - 7616v^{28} - 3962v^{22} - 1712v^{16} \\
& + 11256v^{24} - 2940v^{30} + 336v^{12} + 2223v^{18} + 56v^8 + 14v^6)u^4 - 28v^{20} \\
& - v^{16} + 56v^{22} + 8v^{18} - 28v^{28} + 56v^{26} - 70v^{24} - v^{32} + 8v^{30} = 0.
\end{aligned}$$

Proof. Using the equations (3.2) and replace q to q^2 in the equation (2.4), we arrive at the equation (3.4). $\square$

Theorem 3.4. *If $u := U(q)$ and $v := U(q^{22})$, then*

$$\begin{aligned}
& (986040816v^{40} + 866052v^{46} + 5787668202778v^{20} + 172185640v^{42} \\
& + 169882124472v^{34} - 25567718v^{44} - 9293998296v^{38} - 3107697009288v^6 \\
& + 36295472877496v^{18} + 2121602538968v^{16} + 7169816404080v^{26} \\
& - 8124260320v^{36} - 1484627578016v^{30} - 4645309792068v^{24} - 67057229360v^{32}
\end{aligned}$$

$$\begin{aligned}
& + 7526562463596v^8 + 1153816174106v^{28} - 33822367464468v^{14} \\
& - 21407327495664v^{22} + 16249149212472v^{10} - 90004495680v^2 \\
& - 10144434386148v^{12} - 1689359173194v^4)u^{40} + (-3273211848740v^{24} \\
& + 738329332176v^{14} - 40302183174v^{34} - 10034825676276v^{20} \\
& - 13900658361876v^{12} + 878301864495v^6 - 101694712680v^4 - 3740v^{46} \\
& + 2954425676v^{36} - 11067373615958v^{22} + 236555666724v^{30} + 320144v^{44} \\
& + 1034347215v^{38} - 20009092v^{42} + 1049419250820v^8 + 13667984126888v^{18} \\
& + 959647496676v^{26} + 1784257035924v^{28} - 199694118756v^{32} \\
& + 24665123615220v^{16} + 52161040v^{40} - 5381586440208v^{10} + 49220622v^2)u^{12} \\
& + (-715417510156v^{24} - 33360756984v^{34} - 3865351731692v^{16} - 177147v^{46} \\
& + 4000764020v^{36} + 3875886471700v^{22} - 1939322249040v^8 + 190669714684v^6 \\
& + 4596069155997v^{14} - 32563650724v^{32} + 375181768896v^4 + 3811754547520v^{12} \\
& + 2179277770780v^{20} - 1440521010324v^{26} + 4958612v^{44} - 242137108v^{40} \\
& + 298590324054v^{30} - 32418804v^{42} - 5774844512356v^{18} - 1743267899032v^{10} \\
& + 170319058652v^{28} + 1801417464v^{38} + 40572575868v^2)u^{44} + (-672672v^{46} \\
& + 1259642544930v^4 - 213474556752v^{34} - 9163994670848v^{24} - 257049452v^{42} \\
& - 154715374417744v^{18} + 27681311067504v^{14} + 114690778770656v^{20} \\
& + 610216920792v^8 + 330768452168v^{32} + 137744426798440v^{22} - 60498408v^{40} \\
& + 92105320977390v^{12} - 30187877125560v^{26} - 194969404867512v^{16} \\
& + 3021711821168v^{30} - 17484707970v^{36} + 10551429891684v^6 + 25208370v^{44} \\
& + 12279205620v^{38} - 1703798470176v^{28} - 15520421664v^2 \\
& + 5496853742820v^{10})u^{24} + (-1153986564v^{40} + 1215992658672v^4 + 459756v^{46} \\
& - 2697448160v^{38} - 3520165727716v^{26} - 18738939663123v^8 + 182245316v^{42} \\
& + 77704695629262v^{12} + 822524400988v^{30} + 69665179936v^{36} - 15497108v^{44} \\
& + 10478952028842v^{28} - 1225146773983v^{32} + 30959507291192v^{18} \\
& - 50016236089561v^{24} - 141321547999313v^{16} + 47705515968792v^{10} \\
& - 66064283852688v^{14} + 2264361946264v^{22} + 120301867133228v^{20} \\
& - 9479924411760v^6 - 88508190672v^2 - 41371403616v^{34})u^{32} + (3487088v^{44} \\
& + 159366000917640v^{16} - 2239286841198v^6 + 35971419v^{42} - 723099316v^{40} \\
& - 361512569928v^{30} - 23820883717257v^{10} - 103497180951340v^{22} - 114356v^{46} \\
& - 88146271283136v^{20} - 631047903112v^{32} + 3536210601v^2 + 554962826v^{38}
\end{aligned}$$

$$\begin{aligned}
& -13174667920764v^8 - 1216990926544v^{24} - 62092692238104v^{12} \\
& + 1166586125364v^4 + 4090465135592v^{28} - 2599813287924v^{14} \\
& + 16879599634590v^{26} + 116117584643866v^{18} + 29466128204v^{36} \\
& - 42618798387v^{34})u^{20} + (3095081649v^{16} - 108948114v^{20} - 14271253308v^{12} \\
& - 7766502216v^{10} - 399142932v^{18} - 6018313554v^4 - 531441v^{24} + 14875836v^{22} \\
& - 1 + 766634616v^{14} + 18641257569v^8 + 13214331852v^6 - 7872923556v^2)u^{48} \\
& + (+43706520v^{44} - 14346731102964v^{12} + 77402091902322v^{14} - 1395306v^{46} \\
& - 44249452367285v^{10} + 33337235568692v^{22} - 55314629077356v^{20} \\
& + 1909836815336v^{30} + 28409245036860v^{24} + 15071777424v^{38} \\
& - 843039580v^{40} - 325887925v^{42} + 112874578740v^2 + 692025090900v^{32} \\
& + 48686814300476v^{16} - 68629953513708v^{18} - 6639961932596v^{28} \\
& + 2047048773264v^4 - 239874223632v^{34} - 3699702295920v^8 \\
& - 22233126036v^{36} - 9475829967054v^{26} + 9963537817676v^6)u^{36} \\
& + (154696957v^{40} - 48884627296404v^{18} - 3528733232948v^{28} + 174064v^{46} \\
& + 43951142923578v^{20} - 12308846722v^{36} - 5960592v^{44} + 437056008367v^{32} \\
& + 105771886704v^{34} + 42663462395532v^{22} + 3572185950396v^8 \\
& - 4066106645832v^{26} - 85811470969983v^{16} - 3412629704v^{38} \\
& + 59293520v^{42} - 659156837224v^{30} - 487546443252v^4 + 3762312617719v^{24} \\
& + 38057905046304v^{12} - 2478573368172v^6 - 7934890634208v^{14} \\
& + 21122195260320v^{10} - 527425668v^2)u^{16} + (5849753404v^{34} - 2598156v^2 \\
& + 1562491386720v^{12} - 2182595382v^4 - 275815463088v^{26} - 18470269672v^{30} \\
& 10560v^{46} - 118426v^{44} - 38293932600v^6 + 833720325372v^{24} + 1272984v^{42} \\
& + 297597214728v^{10} - 392074238358v^{28} - 2180590723872v^{18} - 617364v^{40} \\
& - 2590541276400v^{16} + 1655953978800v^{22} - 80969064v^{38} + 554586385176v^{14} \\
& - 870137732v^{36} - 92837286864v^8 - +44924792088v^{32} + 638073143754v^{20})u^8 \\
& + (193630617296952v^{16} - 112694973711156v^{12} + 1832097764v^{40} + 873477v^{46} \\
& + 639181712840v^{32} - 4253381815617v^6 + 330211819068v^{34} - 517405899879v^{14} \\
& - 15661888572462v^{10} - 15255667669v^{38} - 30543436v^{44} - 83914926220730v^{22} \\
& + 45403410492v^2 + 25510822958500v^{26} - 3777649924752v^4 + 184473146v^{42} \\
& - 134940248337048v^{20} - 45154052504v^{36} + 24868553257836v^8 \\
& - 6309227266496v^{28} + 33996057356464v^{24} + 83226321847256v^{18}
\end{aligned}$$

$$\begin{aligned}
(3.5) \quad & -3582730793678v^{30})u^{28} - v^{24} - 66v^{28} + 12v^{26} + 1884v^{30} + 109329v^{32} + 12v^{46} \\
& - 66v^{44} + 844388v^{36} + 109329v^{40} + 1884v^{42} + 824472v^{38} + 824472v^{34} - v^{48} \\
& + (396v^{46} + 3752892v^6 - 109516212v^8 + 80572480v^{36} - 6465174408v^{10} \\
& - 392747839v^{34} - 66585987564v^{24} + 102271662702v^{18} - 48413750220v^{12} \\
& + 70095543804v^{16} - 708588v^4 - 83988v^{42} - 34071600888v^{14} + 59049v^2 \\
& + 631860748v^{30} - 1095864v^{38} - 3414039948v^{32} - 81496125348v^{22} + 6336v^{44} \\
& - 1558480v^{40} + 19467071940v^{26} + 19796882652v^{20} + 28507213020v^{28})u^4 = 0.
\end{aligned}$$

Proof. Using the equations (2.1) and (2.5), we arrive at the equation (3.5). $\square$

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