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On Four Dimensional Semi-C-reducible Landsberg Space

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Abstract. In the present paper we have work out the h-connection vectors of four-dimensional Landsberg and Berwald spaces, four dimensional semi C-reducible Landsberg spaces and necessary and sufficient condition under which the four-dimensional semi-C-reducible Landsberg space to be a Berwald space.

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1. Introduction

A theory of intrinsic orthonormal frame field on n-dimensional Finsler space has been studied by Matsumoto and Miron ([2], [3]) and is called 'Miron frame' by Matsumoto. A four dimensional Finsler space with Miron frame has been studied in [5], [6] and [7].

Let F^4 be a four dimensional Finsler space with fundamental function L. The metric tensor g_{ij} and (h) hv-torsion tensor C_{ijk} of F^4 are defined by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad C_{ijk} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}.$$

Let $\{e^i_\lambda\}$; λ = 1, 2, 3, 4; be the Miron frame of F4, where $e^i_{1)} = l^i = \frac{y^i}{L}$ is the normalized supporting element, $e^i_{2)} = m^i = \frac{C^i}{C}$ is the normalized torsion vector, $e^i_{3)} = n^i$ and $e^i_{4)} = p^i$ are constructed by $g_{ij} e^i_{\lambda)} e^i_{\mu)} = \delta_{\lambda\mu}$. Hence C is the length of torsion vector $C_i = C_{ijk} g^{jk}$. The greek letters λ, μ, ν varies from 1 to 4 throughout the chapter.

The (h) hv-torsion tensor C_{ijk} of F4 is written as [7],

(1.1) $$LC_{ijk} = C_{\lambda\mu\nu} e_{\lambda)i} e_{\mu)j} e_{\nu)k}$$
$$= C_{222} m_i m_j m_k + C_{233} \Pi_{(ijk)} (m_i n_j n_k) + C_{244} \Pi_{(ijk)} (m_i p_j p_k)$$
$$+ C_{322} \Pi_{(ijk)} (m_i m_j n_k) + C_{333} (n_i n_j n_k) + C_{344} \Pi_{(ijk)} (n_i p_j p_k)$$
$$+ C_{422} \Pi_{(ijk)} (m_i m_j p_k) + C_{433} \Pi_{(ijk)} (n_i n_j p_k) + C_{444} (p_i p_j p_k)$$
$$+ C_{234} \Pi_{(ijk)} \{(m_i (n_j p_k + n_k p_j)\}$$

where $\Pi_{(ijk)}$ denote the cyclic permutation of indices I, j, k and summation, for instance

$$\Pi_{(ijk)} (A_i B_j C_k) = A_i B_j C_k + A_j B_k C_i + A_k B_i C_j .$$

If we put, $C_{222} = H$, $C_{233} = I$, $C_{244} = K$, $C_{333} = J$, $C_{344} = J'$, $C_{444} = H'$, $C_{433} = I'$, $C_{234} = K'$,

Then we have [7],

$$H + I + K = LC, \qquad C_{322} = -(J + J'), \qquad C_{422} = -(H' + I')$$

And hence (1.1) may be written as

(1.2) $$LC_{ijk} = H m_i m_j m_k + J n_i n_j n_k + H p_i p_j p_k$$
$$+ I \Pi_{(ijk)} (m_i n_j n_k) + K \Pi_{(ijk)} (m_i p_j p_k) + J \Pi_{(ijk)} (n_i p_j p_k)$$
$$-(J + J') \Pi_{(ijk)} (n_i m_j m_k) + I \Pi_{(ijk)} (n_i n_j p_k)$$
$$-(H + I') \Pi_{(ijk)} (m_i m_j p_k) + K' \Pi_{(ijk)} \{(m_i (n_j p_k + n_k p_j)\} .$$

The eight scalars H, I, J, K, H′, I′, J′, K′ are called the main scalars of a four dimensional Finsler space.

Definition 1.1. [4] A Finsler space F^n is called a Landsberg space if the Berwald connection BΓ is h-metrical.

Definition 1.1. [4] In terms of Cartan's connection CΓ, a Landsberg space is characterized by

(1) the (v) hv-torsion tensor P^i_{jk} vanishes identically, namely

$$P^i_{jk} (= C^i_{jk|0}) = 0$$

or (2) the hv-curvature tensor P^h_{ijk} vanishes identically, where the suffix '0' indicates the contraction by supporting element y^i.

Definition 1.2. [2] In a Finsler space F^n, if the connection coefficient G^i_{jk} of Berwald connection BΓ are function of position alone, then space is called a Berwald space.

Theorem 1.2. [2] In terms of the Cartan connection CΓ, a Berwald space is characterized by $C_{ijk|h} = 0$.

2. The h-connection vectors and the tensor $C_{ijk|h}$

Let us consider the Miron frame $\{e^i_\lambda\}$ of F^4. If a tensor T^i_j of (1, 1) type is given, then [2],

$$T^i_j = T_{\lambda\mu} e^i_{\lambda)} e_{\mu)j}.$$

Now, we denote the h-covariant derivative with respect to Cartan connection $(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ by |n. Let $H_{\lambda)\mu\nu}$ be scalar components of h-covariant derivative $e^i_{\lambda)|j}$ of vector $e^i_{\lambda)}$ belonging to the Miron frame $\{e^i_{\lambda)}\}$ of F^4, i.e.,

(2.1) $$e^i_{\lambda)|j} = H_{\lambda)\mu\nu}\, e^i_{\mu)}\, e_{\nu)j}.$$

The scalar $H_{\lambda)\mu\nu}$ satisfy the following condition ([7], [2])

$$H_{1)\mu\nu} = 0 \quad \text{and} \quad H_{\lambda)\mu\nu} = -H_{\mu)\lambda\nu}.$$

If we assume,

(2.2) $$H_{2)3\mu} = h_\mu, \qquad H_{2)4\mu} = j_\mu, \quad \text{and} \quad H_{3)4\mu} = K_\mu,$$

then from (3.1), we obtain

(2.3) $$l^j_{|j} = 0, \quad m^i_{|j} = n^i h_j + p^i j_j, \quad n^i_{|j} = -m^i h_j + p^i K_j,$$
$$p^i_{|j} = -m^i j_j - n^i K_j,$$

where

(2.4) $$h_j = h_\lambda e_{\lambda)i}, \qquad j_i = j_\lambda e_{\lambda)i}, \qquad K_i = K_\lambda e_{\lambda)i}.$$

Definition 2.1. The vectors h_i, j_i and K_i defined in (2.4) are called the h-connection vectors of a four dimensional Finsler space.

Now, if we denote the scalar components of $T^i_{j|k}$ by $T_{\lambda\mu,\nu}$, that is,

(2.5) $$T^i_{j|k} = T_{\lambda\mu,\nu}\, e^i_{\lambda)}\, e_{\mu)j}\, e_{\nu)k},$$

then we obtain

(2.6) $$T_{\lambda\mu,\nu} = (\delta_K T_{\lambda\mu}) e^K_{\nu)} + T_{\delta\mu} H_{\delta)\lambda\nu} + T_{\lambda\delta} H_{\delta)\mu\nu},$$

where, $\delta_k = \partial_k - \Gamma^{*j}_{0k}\, \dot{\partial}_r$.

Further, if S is a scalar field of F4 then $S_{|k} = S_{,\lambda}\, e_{\lambda)k}$, where $S_{,\lambda} = (\delta_k S)\, e_{\lambda)}{}^{k}$ and is called h-scalar derivative of S.

Differentiating equation (1.2) h-covariantly and using equation (2.3), we have

(2.7) $$LC_{ijk|h} = {}^{1}A_h m_i m_j m_k + {}^{2}A_h n_i n_j n_k + {}^{3}A_h p_i p_j p_k + {}^{4}A_h \Pi_{(ijk)}(m_i m_j n_k)$$
$$+{}^{5}A_h \Pi_{(ijk)}(m_i m_j p_k) + {}^{6}A_h \Pi_{(ijk)}(m_i n_j n_k) + {}^{7}A_h \Pi_{(ijk)}(m_i p_j p_k)$$
$$+{}^{8}A_h \Pi_{(ijk)}(n_i n_j p_k) + {}^{9}A_h \Pi_{(ijk)}(n_i p_j p_k)$$
$$+{}^{10}A_h \Pi_{(ijk)}\{(m_i(n_j p_k + n_k p_j)\},$$

where

$${}^{1}A_h = H_{|h} + 3(J + J')h_h + 3(H' + I')j_h,$$
$${}^{2}A_h = J_{|h} + 3Ih_h - 3I'K_h,$$
$${}^{3}A_h = H'_{|h} + 3Kj_h + 3J'K_h,$$
$${}^{4}A_h = -(J + J')_{|h} - 2K'j_h + (H - 2I)h_h + (H' + I')K_h,$$
(2.7)′ $${}^{5}A_h = -(H' + I')_{|h} - 2K'h_h + (H - 2K)j_h - (J + J')K_h,$$
$${}^{6}A_h = I_{|h} - (3J + 2J')h_h - I'j_h - 2K'K_h$$
$${}^{7}A_h = K_{|h} - J'h_h - (3H' + 2I')J_h + 2K'K_h,$$
$${}^{8}A_h = I'_{|h} + 2K'h_h + Ij_h + (J - 2J')K_h,$$
$${}^{9}A_h = J'_{|h} + Kh_h + 2K'j_h - (H' - 2J')K_h,$$
$${}^{10}A_h = K'_{|h} - (H' + 2I')h_h - (J + 2J')j_h + (I - K)K_h.$$

We denote the scalar components of the vector ${}^{r}A_h$ by ${}^{r}A_\mu$; r = 1, 2, …, 10, i.e.,

(2.8) $${}^{r}A_h = {}^{r}A_\mu\, e_{\mu)h}, \qquad r = 1, 2, \ldots, 10,$$

for instant, the scalar component of ${}^{1}A_h$ are given by

(2.8)′ $${}^{1}A_\mu = H_{,\mu} + 3(J + J')h_\mu + 3(H' + I')j_\mu.$$

In the view of equation (2.6), (2.2) and (2.8), the explicit form of $C_{\alpha\beta\gamma,\delta}$ can be written as,

(2.9) $$C_{222,\mu} = {}^{1}A_\mu, \quad C_{322,\mu} = {}^{4}A_\mu, \quad C_{422,\mu} = {}^{5}A_\mu$$
$$C_{233,\mu} = {}^{6}A_\mu, \quad C_{333,\mu} = {}^{2}A_\mu, \quad C_{433,\mu} = {}^{8}A_\mu,$$
$$C_{244,\mu} = {}^{7}A_\mu, \quad C_{344,\mu} = {}^{9}A_\mu, \quad C_{444,\mu} = {}^{3}A_\mu,$$
$$C_{234,\mu} = {}^{10}A_\mu.$$

Contraction of equation (2.7) by y^i gives

(2.10) $$P_{ijk} = C_{ijk|0} = {}^1A_1 m_i m_j m_k + {}^2A_1 n_i n_j n_k + {}^3A_1 p_i p_j p_k + {}^4A_1 \Pi_{(ijk)}(m_i m_j n_k) + {}^5A_1 \Pi_{(ijk)}(m_i m_j p_k) + {}^6A_1 \Pi_{(ijk)}(m_i n_j n_k) + {}^7A_1 \Pi_{(ijk)}(m_i p_j p_k) + {}^8A_1 \Pi_{(ijk)}(n_i n_j p_k) + {}^9A_1 \Pi_{(ijk)}(n_i p_j p_k) + {}^{10}A_1 \Pi_{(ijk)}\{(m_i (n_j p_k + n_k p_j)\}.$$

Now, we consider the four dimensional Landsberg space. Since the (v) hv-torsion tensor P_{ijk} of a Landsberg space vanishes identically, therefore we have ${}^rA_1 = 0$ for each r = 1, 2, …., 10. Moreover $C_{ijk|h} = C_{ijh|k}$ is satisfied in a Landsberg space. So equation (2.5) implies $C_{\lambda\mu\nu,\delta} = C_{\lambda\mu\delta,\nu}$. In the view of equation (2.9) this can be explicitely written as

(2.11) $$\begin{aligned} &{}^1A_3 = {}^4A_2, \quad {}^1A_4 = {}^5A_2, \quad {}^6A_2 = {}^4A_3, \quad {}^6A_3 = {}^2A_2, \\ &{}^3A_2 = {}^7A_4, \quad {}^3A_3 = {}^9A_4, \quad {}^4A_3 = {}^6A_2, \quad {}^5A_4 = {}^7A_2, \\ &{}^2A_4 = {}^8A_3, \quad {}^8A_4 = {}^9A_3, \quad {}^4A_4 = {}^5A_3 = {}^{10}A_2, \\ &{}^6A_4 = {}^8A_2 = {}^{10}A_3, \quad {}^7A_3 = {}^9A_2 = {}^{10}A_4. \end{aligned}$$

In view of equations (2.10) and (2.11), equation (2.7) gives

(2.12) $$\begin{aligned} LC_{ijk|h} = {}&{}^1A_2 m_i m_j m_k m_h + {}^2A_3 n_i n_j n_k n_h + {}^3A_4 p_i p_j p_k p_h \\ &+ B_1 \Pi_{(ijkh)}(m_i m_j m_k n_h) + B_2 \Pi_{(ijkh)}(m_i m_j m_k p_h) \\ &+ B_3 \Pi_{(ijkh)}(m_i n_j n_k n_h) + B_4 \Pi_{(ijkh)}(n_i n_j n_k p_h) \\ &+ B_5 \Pi_{(ijkh)}(m_i p_j p_k p_h) + B_6 \Pi_{(ijkh)}(n_i p_j p_k p_h) \\ &+ B_7 \Pi_{(ijkh)}(m_i m_j n_k n_h) + B_8 \Pi_{(ijkh)}(m_i m_j p_k p_h) \\ &+ B_9 \Pi_{(ijkh)}(n_i n_j p_k p_h) + B_{10} \Pi_{(ijkh)}(m_i m_j n_k p_h) \\ &+ B_{11} \Pi_{(ijkh)}(m_i n_j n_k p_h) + B_{12} \Pi_{(ijkh)}(m_i n_j p_k p_h), \end{aligned}$$

where, $B_1 = {}^1A_3 = {}^4A_2$, $B_2 = {}^1A_4 = {}^5A_2$, $B_3 = {}^6A_3 = {}^2A_2$,
$B_4 = {}^2A_4 = {}^8A_3$, $B_5 = {}^3A_2 = {}^7A_4$, $B_6 = {}^3A_3 = {}^9A_4$,
$B_7 = {}^4A_3 = {}^6A_2$, $B_8 = {}^5A_4 = {}^7A_2$, $B_9 = {}^8A_4 = {}^9A_3$,
$B_{10} = {}^4A_4 = {}^5A_3 = {}^{10}A_2$, $B_{11} = {}^6A_4 = {}^8A_2 = {}^{10}A_3$,
$B_{12} = {}^7A_3 = {}^9A_2 = {}^{10}A_4$.

The notation $\Pi_{(ijkh)}$ in the expression (2.12) stands for the possible permutation of indices i, j, k, h and summation, for instance,

$$\Pi_{(ijkh)}(m_i m_j m_k n_h) = m_i m_j m_k n_h + m_j m_k m_h n_i + m_k m_h m_i n_j + m_h m_i m_j n_k$$

$$\Pi_{(ijkh)}(m_i m_j n_k n_h) = m_i m_j n_k n_h + m_j m_k n_h n_i + m_k m_h n_i n_j + m_h m_i n_j n_k + m_i m_k n_j n_h + m_j m_h n_i n_k$$

$$\Pi_{(ijkh)}(m_i m_j n_k p_h) = m_i m_j n_k p_h + m_j m_k n_h p_i + m_k m_h n_i p_j + m_h m_i n_j p_k + m_i m_j n_h p_k + m_j m_k n_i p_h + m_k m_h n_j p_i + m_h m_i n_k p_j$$

$$+m_i m_k n_j p_h + m_i m_k n_h p_j + m_j m_h n_i p_k + m_j m_i n_k p_h .$$

Proposition 2.1. The h-covariant derivative of (h) hv-torsion vector C_{ijk} of a four dimensional Landsberg space can be written in the form (2.12).

Now, we consider a four dimensional Berwald space. From theorem (1.2) such a space is characterized by $C_{\alpha\beta\gamma,\delta} = 0$. In view of equation (2.7)′, we have

$$C_{322,\delta} + C_{333,\delta} + C_{344,\delta} = (H + I + K)\, h_\delta = 0$$

which implies $h_\delta = 0$. Similarly

$$C_{422,\delta} + C_{433,\delta} + C_{444,\delta} = (H + I + K)\, j_\delta = 0,$$

which implies $j_\delta = 0$. Hence from $C_{222,\delta} = 0$, we get $H_{,\delta} = 0$, and also

$$J_{,\delta} = 3I'K\square, \qquad I_{,\delta} = 2K'k_\delta, \qquad K_{,\delta} = -2K'k_\delta,$$

$$H'_{,\delta} = -3J'k_\delta, \quad I'_{,\delta} = -(J - 2J')k_\delta, \quad J'_{,\delta} = (H' - 2I')k_\delta,$$

$$K'_{,\delta} = (I - K)\, k_\delta,$$

which gives $(H + I + K)_{,\delta} = (LC)_{,\delta} = 0$. Summarizing above results, we have:

Theorem 2.1. In a four dimensional Berwald space, the h-connection vectors h_i and j_i vanish identically. Also main scalar H and the unified main scalar LC are h-covariantly constant. Furthermore, if h-connection vector, K_i vanishes then all the main scalars are h-covariantly constant.

To solve the h-connection vectors explicitely in terms of main scalar, in the next section we consider a four dimensional semi-C-reducible Landsberg space F^4.

3. Semi-C-reducible Landsberg space F^4

Definition 3.1. [4] A Finsler space F^n ($n \geq 3$) with the non-zero length C of the torsion vector C^i is called semi-C-reducible if the (h) hv-torsion tensor C_{ijk} is of the form

$$C_{ijk} = \frac{p}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) + \frac{q}{C^2} C_i C_j C_k ,$$

where p and q (= 1 – p) do not vanish.

It has been seen ([6]) that a four dimensional Finsler space F^4, is semi-C-reducible if and only if

(3.1) $$H' = I' = K' = J = J' = 0.$$

Thus, for a four dimensional semi-C-reducible Landsberg space, equation (2.8), (2.11) and (3.1) gives,

$$H_{,3} = (H - 2I)\, h_2, \qquad H_{,4} = (H - 2K)\, j_2,$$

$$I_{,2} = (H - 2I)\, h_3, \qquad I_{,3} = 3Ih_2,$$

(3.2) $$I_{,4} = Ij_2 = (I - K)\, K_3, \qquad K_{,2} = (H - 2K)\, j_4,$$

$$K_{,3} = Kh_2 = (I - K)\, K_4, \qquad K_{,4} = 3Kj_2, \qquad 3Ih_4 = Ij_3$$

$$(H - 2I)h_4 = (I - K)\, K_2 = (H - 2K)\, j_3, \qquad Kh_3 = Ij_4,$$

$$Kh_4 = 3Kj_3.$$

Now we assume that non of main scalars H, I, K vanish. To solve the connection vectors we divide the four dimensional semi-C-reducible Finsler space in following five classes

(I) $H \neq 2I \neq 2K$, (II) $H \neq 2K = 2I$

(III) $H = 2I \neq 2K$, (IV) $H = 2K = 2I$,

(V) $H = 2K \neq 2I$.

For a semi-C-reducible Landsberg space, equations (2.10) and (2.8)′ give,

$$H_{,1} = I_{,1} = K_{,1}, \quad h_1 = j_1 = 0 \quad \text{and} \quad K_1 = 0 \quad \text{provided} \quad K = I.$$

Under the restriction of class (I) using the fact H + I + K = LC it follows from relation (3.2) that

$$h_2 = \frac{H_{,3}}{H-2I} = \frac{I_{,3}}{3I} = \frac{K_{,3}}{K} = \frac{(LC)_{,3}}{LC}$$

$$h_3 = \frac{I_{,2}}{H-2I}, \qquad h_4 = 0,$$

$$j_2 = \frac{H_{,4}}{H-2I} = \frac{I_{,4}}{I} = \frac{K_{,4}}{3K} = \frac{(LC)_{,4}}{LC}$$

$$j_3 = 0, \qquad j_4 = \frac{K_{,2}}{H-2K}$$

$$K_2 = 0, \qquad K_3 = \frac{Ij_2}{I-K}, \qquad K_4 = \frac{Kh_2}{I-K}$$

and the identity

$$\frac{KI_{,2}}{H-2I} = \frac{IK_{,2}}{H-2K}$$

Holds. Therefore, we have

Theorem 3.1. The h-connection vectors and h-covariant derivative of (h) hv-torsion tensor C_{ijk} of a four dimensional semi-C-reducible Landsberg space of class (I) are given

$$h_i = h_2\, m_i + \frac{I_{,2}}{H-2I}\, n_i, \qquad j_1 = j_2\, m_i + \frac{K_{,2}}{H-2K}\, p_i,$$

$$K_i = \frac{I}{I-K}\,[Ij_2\, n_i + K\, h_2\, p_i]$$

and

$$LC_{ijk|h} = H_{,2}\, m_i m_j m_k m_h + \frac{3II_{,2}}{H-2I}\, n_i n_j n_k n_h + \frac{3KK_{,2}}{H-2K}\, p_i p_j p_k p_h$$
$$+ H_{,3}\, \Pi_{(ijkh)}(m_i m_j m_k n_h) + H_{,4}\, \Pi_{(ijkh)}(m_i m_j m_k p_h)$$
$$+ I_{,3}\, \Pi_{(ijkh)}(m_i n_j n_k n_h) + K_{,4}\, \Pi_{(ijkh)}(m_i p_j p_k p_h)$$
$$+ I_{,2}\, \Pi_{(ijkh)}(m_i m_j n_k n_h) + K_{,2}\, \Pi_{(ijkh)}(m_i m_j p_k p_h)$$

$$+ A\Pi_{(ijkh)}(n_i n_j p_k p_h) + I_{,4}\, \Pi_{(ijkh)}(m_i n_j n_k p_h)$$
$$+ K_{,3}\, \Pi_{(ijkh)}(m_i n_j p_k p_h)$$

where

$$h_2 = \frac{H_{,3}}{H-2I} = \frac{I_{,3}}{3I} = \frac{K_{,3}}{K} = \frac{(LC)_{,3}}{LC}$$

$$j_2 = \frac{H_{,4}}{H-2I} = \frac{I_{,4}}{I} = \frac{K_{,4}}{3K} = \frac{(LC)_{,4}}{LC}$$

and $$A = \frac{KI_{,2}}{H-2I} = \frac{IK_{,2}}{H-2K}.$$

By the similar process for classes (II), (III) and (IV) of semi-C-reducible Landsberg space F^4, we obtain

Theorem 3.2. The h-connection vectors and h-covariant derivative of (h) hv-torsion tensor C_{ijk} of a four dimensional semi-C-reducible Landsberg space of class (II) are given by

$$h_i = \frac{I_{,2}}{H-2I}\, n_i, \qquad j_1 = \frac{I_{,2}}{H-2I}\, p_i, \qquad K_i \text{ is arbitrary and}$$

$$LC_{ijk|h} = H_{,2}\, m_i m_j m_k m_h + \frac{3II_{,2}}{H-2I}(n_i n_j n_k n_h + p_i p_j p_k p_h) +$$
$$I_{,2}\Pi_{(ijkh)}[(m_i m_j (n_k n_h + p_k p_h)] + \frac{II_{,2}}{H-2I}\Pi_{(ijkh)}(n_i n_j p_k p_h).$$

Theorem 3.3. The h-connection vectors and h-covariant derivative of (h) hv-torsion tensor C_{ijk} of four dimensional semi-C-reducible Landsberg space of class (III) are given by

$$h_i = \frac{IK_{,2}}{2K(I-K)}\, n_i, \qquad j_1 = \frac{K_{,2}}{2(I-K)}\, p_i, \qquad K_i = 0, \text{ and}$$

$$LC_{ijk|h} = \frac{3II_{,2}}{2(K-I)}\, n_i n_j n_k n_h + \frac{3K^2 I_{,2}}{2I(K-I)}\, p_i p_j p_k p_h$$
$$+ I_{,2}\Pi_{(ijkh)} m_i m_j n_k n_h + \frac{KI_{,2}}{2(K-I)}\Pi_{(ijkh)}(n_i n_j p_k p_h).$$

Further we consider class (V). In this case equation (3.2) gives

$$I_{,2} = I_{,3} = I_{,4} = 0, \qquad h_2 = h_4 = j_2 = j_3 = 0, \quad \text{and} \quad h_3 = j_4.$$

Thus, we have

Theorem 3.4. If the (h) hv-torsion tensor of a four dimensional Finsler space F^4 of class (V) is written in the form

$$LC_{ijk} = 2I\, m_i m_j m_k + I\,\Pi_{(ijk)}[(m_i (n_j n_k + p_j p_k)]$$

then the main scalar I is h-covariant constant. The h-connection vectors and h-covariant derivative of (h) hv-torsion tensor C_{ijk} for such space are given by

$h_i = Dn_i, \quad j_i = Dp_j, \quad K_i$ is arbitrary vector and

$$LC_{ijk|h} = DI\,[3n_i\, n_j\, n_k\, n_h + 3p_i\, p_j\, p_k\, p_h + \Pi_{(ijkh)}(n_i\, n_j\, p_k\, p_h)],$$

where D is arbitrary scalar.

Finally we shall find a condition for a semi-C-reducible Landsberg space to be Berwald space. From theorem (1.2), equation (2.7) and (2.8), we have for a four dimensional semi-C-reducible Berwald space,

(3.3) $\quad H_{|h} = I_{|h} = K_{|h} = 0.$

Further, if the condition (3.3) satisfies in a four dimensional semi-C-reducible Landsberg space, then using (3.2) and (2.8) in equation (2.7), we have $C_{ijk|h} = 0$ provided H = 2I = 2K does not hold. Thus, we have

Theorem 3.5. A necessary and sufficient condition for a four dimensional semi-C-reducible Landsberg space to be Berwald space is that all the non-zero main scalars H, I, K are h-covariant constant provided H = 2I = 2K does not hold.

Since $H_{,1} = 0$ is satisfied in semi-C-reducible Landsberg space, from Ricci identity

$$H_{|i}|_j - H|_{j|i} = - H_{|t}\, C^t_{ij} - H|_t\, P^t_{ij},$$

we have $\quad H_{|i} = - H|_{i|0}.$

Similarly, we obtain $I_{|i} = - I|_{i|0}$ and $K_{|i} = - K|_{i|0}$. Thus in view of theorem (3.5), we have

Theorem 3.6. In a four dimensional semi-C-reducible Landsberg space F^4 the following condition are equivalent to each other

(1) F^4 is a Berwald space.

(2) All the main scalar H, I, K are h-covariant constant.

(3) Main scalar H, I, K satisfy the relations $H|_{i|0} = I|_{i|0} = K|_{i|0} = 0$ provided H = 2I = 2K does not hold.

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