On the Direct Product of Intuitionistic Fuzzy Subgroups

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Abstract

In this paper, we first discuss some properties of intuitionistic fuzzy subgroup of a group. Then we obtained some equivalent characterizations of the direct product of intuitionistic fuzzy subgroups by means of \((\alpha, \beta)\)-Cut sets.

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1. Introduction

After the introduction of the concept of fuzzy set by Zadeh [8] several researches were conducted on the generalization of the notion of fuzzy set. The idea of Intuitionistic fuzzy set was given by Atanassov [2]. In this paper we discuss some results relating to the intuitionistic fuzzy subgroup of a group by means of their \((\alpha, \beta)\)–cut sets. Then we obtained some equivalent characterizations of the direct product of intuitionistic fuzzy subgroups.
2. Preliminaries

Definition (2.1) [4] An IFS $A = \{ < x, \mu_A(x), \nu_A(x) > : x \in G \}$ of a group $G$ is said to be **intuitionistic fuzzy subgroup** of $G$ (In short IFSG) of $G$ if

(i) $\mu_A(xy) \geq \min \{ \mu_A(x), \mu_A(y) \}$

(ii) $\mu_A(x^{-1}) = \mu_A(x)$

(iii) $\nu_A(xy) \leq \max \{ \nu_A(x), \nu_A(y) \}$

(iv) $\nu_A(x^{-1}) = \nu_A(x)$ for all $x, y \in G$

Proposition (2.2) [4] An IFS $A = \{ < x, \mu_A(x), \nu_A(x) > : x \in G \}$ of a group $G$ is intuitionistic fuzzy subgroup of $G$ if and only if $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}$ and $\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}$ holds $\forall x, y \in G$

Definition (2.3) [5] Let $A$ be intuitionistic fuzzy set of a universe set $X$. Then $(\alpha, \beta)$-cut of $A$ is a crisp subset $C_{\alpha, \beta}(A)$ of the IFS $A$ is given by

$C_{\alpha, \beta}(A) = \{ x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \}$, where $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$.

Theorem (2.4) [5]: If $A$ is IFS of a group $G$. Then $A$ is IFSG of $G$ if and only if $C_{\alpha, \beta}(A)$ is a subgroup of group $G$ for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$

Definition (2.5) [5]: Let $G$ be a group and $A$ be IFSG of group $G$. Let $x \in G$ be a fixed element. Then for every element $g \in G$, we define

(i) $(xA)(g) = (\mu_{xA}(g), \nu_{xA}(g))$, where $\mu_{xA}(g) = \mu_A(x^{-1}g)$ and $\nu_{xA}(g) = \nu_A(x^{-1}g)$.

Then $xA$ is called intuitionistic fuzzy left coset of $G$ determined by $A$ and $x$

(ii) $Ax(g) = (\mu_{Ax}(g), \nu_{Ax}(g))$, where $\mu_{Ax}(g) = \mu_A(gx^{-1})$ and $\nu_{Ax}(g) = \nu_A(gx^{-1})$.

Then $Ax$ is called the intuitionistic fuzzy right coset of $G$ determined by $A$ and $x$.

Definition (2.6) [5]: An IFSG $A$ of a group $G$ is IFNSG of $G$ if and only if

$xA = Ax$ for all $x \in G$

Theorem (2.7) [5]: Let $A$ be intuitionistic fuzzy subgroup of a group $G$ and $x$ be any fixed element of $G$. Then for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, we have

(i) $x \cdot C_{\alpha, \beta}(A) = C_{\alpha, \beta}(xA)$

(ii) $C_{\alpha, \beta}(A) \cdot x = C_{\alpha, \beta}(Ax)$

Proposition (2.8): If $A$ is IFS of a group $G$. Then $A$ is IFNSG of $G$ if and only if $C_{\alpha, \beta}(A)$ is a normal subgroup of group $G$, for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$

Proof. Now $A$ is IFNSG of $G$ $\iff$ $xA = Ax$ for all $x \in G$

$\iff C_{\alpha, \beta}(xA) = C_{\alpha, \beta}(Ax)$, for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$

$\iff x \cdot C_{\alpha, \beta}(A) = C_{\alpha, \beta}(A) \cdot x$, for all $x \in G$

$\iff C_{\alpha, \beta}(A)$ is normal subgroup of $G$

Definition (2.9): Let $A$ and $B$ be two IFSG of group $G$. Then $A$ and $B$ are said to be **intuitionistic fuzzy conjugate subgroups** of $G$ if for some $g \in G$

$\mu_A(x) = \mu_B(g^{-1}xg)$ and $\nu_A(x) = \nu_B(g^{-1}xg)$, for every $x \in G$.  

Definition (2.10) [1] Let A be an intuitionistic fuzzy subset in a set S, the strongest intuitionistic fuzzy relation on S, that is an intuitionistic fuzzy relation on A is V given by
\[ \mu_V(x, y) = \min \{ \mu_A(x), \mu_A(y) \} \]
and
\[ \nu_V(x, y) = \max \{ \nu_A(x), \nu_A(y) \} , \]
for all \( x, y \in S \).

3. Direct product of Intuitionistic Fuzzy Sets and their properties

Definition (3.1) Let \( A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \) and \( B = \{ < y, \mu_B(y), \nu_B(y) > : y \in Y \} \) be any two IFS’s of X and Y respectively. Then the Cartesian product of A and B is denoted by \( A \times B \) and is defined as;
\[ A \times B = \{ <( x, y) , \mu_{A \times B}(x, y), \nu_{A \times B}(x, y) > : x \in X \text{ and } y \in Y \} \]
Where \( \mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\} \) and \( \nu_{A \times B}(x, y) = \max\{\nu_A(x), \nu_B(y)\} \).

Proposition (3.2) If A and B be two IFS of X and Y respectively. Then
\( C_{\alpha, \beta}( A \times B ) = C_{\alpha, \beta}( A ) \times C_{\alpha, \beta}( B ) , \) for all \( \alpha, \beta \in [0,1] \) with \( 0 \leq \alpha + \beta \leq 1 \).

Proof. Let \( (x, y) \in C_{\alpha, \beta}( A \times B ) \) be any element
\[ \Leftrightarrow \mu_{A \times B}(x, y) \geq \alpha \quad \text{and} \quad \nu_{A \times B}(x, y) \leq \beta \]
\[ \Leftrightarrow \min\{\mu_A(x), \mu_B(y)\} \geq \alpha \quad \text{and} \quad \max\{\nu_A(x), \nu_B(y)\} \leq \beta \]
\[ \Leftrightarrow \mu_A(x) \geq \alpha , \mu_B(y) \geq \alpha \quad \text{and} \quad \nu_A(x) \leq \beta , \nu_B(y) \leq \beta \]
\[ \Leftrightarrow \mu_A(x) \geq \alpha , \nu_A(x) \leq \beta \quad \text{and} \quad \mu_B(y) \geq \alpha , \nu_B(y) \leq \beta \]
\[ \Leftrightarrow x \in C_{\alpha, \beta}( A ) \quad \text{and} \quad y \in C_{\alpha, \beta}( B ) \]
\[ \Leftrightarrow (x, y) \in C_{\alpha, \beta}( A ) \times C_{\alpha, \beta}( B ) \]
Hence \[ C_{\alpha,\beta}(A \times B) = C_{\alpha,\beta}(A) \times C_{\alpha,\beta}(B) \]

**Theorem (3.3)** Let A and B be IFSG of group \( G_1 \) and \( G_2 \) respectively. Then

\( A \times B \) is also IFSG of group \( G_1 \times G_2 \).

**Proof.** Since A and B be IFSG of group \( G_1 \) and \( G_2 \) respectively.

Therefore \( C_{\alpha,\beta}(A) \) and \( C_{\alpha,\beta}(B) \) are subgroup of group \( G_1 \) and \( G_2 \) respectively

for all \( \alpha, \beta \in [0,1] \) with \( 0 \leq \alpha + \beta \leq 1 \). [ by theorem (2.4)]

\[ \Rightarrow C_{\alpha,\beta}(A) \times C_{\alpha,\beta}(B) \text{ is subgroup of group } G_1 \times G_2 \]

\[ \Rightarrow C_{\alpha,\beta}(A \times B) \text{ is subgroup of group } G_1 \times G_2 \]

\[ \Rightarrow A \times B \text{ is IFSG of group } G_1 \times G_2 \text{ [ by theorem (2.4)]} \]

**Theorem (3.4)** Let A and B be IFNSG of group \( G_1 \) and \( G_2 \) respectively. Then

\( A \times B \) is also IFNSG of group \( G_1 \times G_2 \).

**Proof.** Since A and B be IFNSG of group \( G_1 \) and \( G_2 \) respectively. Then by

Proposition (2.8) \( C_{\alpha,\beta}(A), C_{\alpha,\beta}(B) \) are normal subgroup of \( G_1, G_2 \) respectively

\[ \Rightarrow C_{\alpha,\beta}(A) \times C_{\alpha,\beta}(B) \text{ is normal subgroup of group } G_1 \times G_2 \]

\[ \Rightarrow C_{\alpha,\beta}(A \times B) \text{ is normal subgroup of group } G_1 \times G_2 \]

\[ \Rightarrow A \times B \text{ is IFNSG of group } G_1 \times G_2 \text{ [ by Proposition (2.8)]} \]

**Remark(3.5)** If A and B are IFS of group \( G_1 \) and \( G_2 \) respectively. If \( A \times B \) is

also IFSG of group \( G_1 \times G_2 \), then it is not necessarily that both A and B should be

IFSG of \( G_1 \times G_2 \).
Example (3.6): Let $G_1 = \{ e_1, a \}$, where $a^2 = e_1$ and let $G_2 = \{ e_2, x, y, xy \}$, where $x^2 = y^2 = e_2$ and $xy = yx$. Then

$G_1 \times G_2 = \{ (e_1, e_2), (e_1, x), (e_1, y), (e_1, xy), (a, e_2), (a, x), (a, y), (a, xy) \}$

Let $A = \{ < e_1, 0.7, 0.2 >, < a, 0.6, 0.3 > \}$ and $B = \{ < e_2, 0.9, 0.1 >, < x, 1, 0 >, < y, 0.8, 0.2 >, < xy, 0.7, 0.2 > \}$ be IFS of $G_1$ and $G_2$ respectively. Then

$A \times B = \{ < (e_1, e_2), 0.7, 0.2 >, < (e_1, x), 0.7, 0.2 >, < (e_1, y), 0.7, 0.2 >, < (e_1, xy), 0.7, 0.2 >, < (a, e_2), 0.6, 0.3 >, < (a, x), 0.6, 0.3 >, < (a, y), 0.6, 0.3 >, < (a, xy), 0.6, 0.3 > \}$

Here $A \times B$ is also IFSG of group $G_1 \times G_2$, where as $A$ is IFSG of $G_1$ but $B$ is not IFSG of $G_2$ as $C_{0.8, 0.2}(B) = \{ x, y \}$ is not subgroup of $G_2$.

Proposition (3.7): Let $A$ and $B$ be IFS’s of the groups $G_1$ and $G_2$, respectively. Suppose that $e_1$ and $e_2$ are the identity element of $G_1$ and $G_2$, respectively. If $A \times B$ is an IFSG of $G_1 \times G_2$, then at least one of the following two statements must holds.

(i) $\mu_B(e_2) \geq \mu_A(x)$ and $\nu_B(e_2) \leq \nu_A(x)$, for all $x$ in $G_1$,

(ii) $\mu_A(e_1) \geq \mu_B(y)$ and $\nu_A(e_1) \leq \nu_B(y)$, for all $y$ in $G_2$.

Proof: Let $A \times B$ be an IFSG subgroup of $G_1 \times G_2$.

If possible, let the statements (i) and (ii) does not holds.

Then we can find $x$ in $G_1$ and $y$ in $G_2$ such that $\mu_A(x) > \mu_B(e_2)$, $\nu_A(x) < \nu_B(e_2)$ and $\mu_B(y) > \mu_A(e_1)$, $\nu_B(y) < \nu_A(e_1)$. Thus we have

$\mu_{A \times B}(x, y) = \min \{ \mu_A(x), \mu_B(y) \} > \min \{ \mu_A(e_1), \mu_B(e_2) \} = \mu_{A \times B}(e_1, e_2)$

and,

$\nu_{A \times B}(x, y) = \max \{ \nu_A(x), \nu_B(y) \} < \max \{ \nu_A(e_1), \nu_B(e_2) \} = \nu_{A \times B}(e_1, e_2)$.

Which implies that $A \times B$ is not an IFSG of $G_1 \times G_2$. A contradiction.

Hence either $\mu_B(e_2) \geq \mu_A(x)$ and $\nu_B(e_2) \leq \nu_A(x)$, holds for all $x$ in $G_1$ or $\mu_A(e_1) \geq \mu_B(y)$ and $\nu_A(e_1) \leq \nu_B(y)$, holds for all $y$ in $G_2$.

Proposition (3.8): Let $A$ and $B$ be IFS of the groups $G_1$ and $G_2$ respectively such that $\mu_A(x) \leq \mu_B(e_2)$ and $\nu_A(x) \geq \nu_B(e_2)$ holds for all $x \in G_1$, $e_2$ being the identity element of $G_2$. If $A \times B$ is an IFSG of $G_1 \times G_2$, then $A$ is IFSG of group $G_1$. 

Proof. Let \( x, y \in G_1 \) be any element. Then \((x, e_2)\) and \((y, e_2)\) are in \(G_1 \times G_2\). Since \( \mu_A(x) \leq \mu_B(e_2) \) and \( \nu_A(x) \geq \nu_B(e_2) \) holds for all \( x \) in \( G_1 \), we get,

\[
\mu_A(xy^{-1}) = \min\{\mu_A(xy^{-1}), \mu_B(e_2 e_2)\} = \mu_{AXB}(xy^{-1}, e_2 e_2)
\]

\[
\nu_A(xy^{-1}) = \max\{\nu_A(xy^{-1}), \nu_B(e_2 e_2)\} = \nu_{AXB}(xy^{-1}, e_2 e_2)
\]

Thus \( \mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y)\} \), also \( \nu_A(xy^{-1}) = \max\{\nu_A(x), \nu_A(y)\} \).

Therefore, \( \nu_A(xy^{-1}) \leq \max\{\nu_A(x), \nu_A(y)\} \). Hence \( A \) is an IFSG of \( G_1 \).

**Corollary (3.9)** Let \( A \) and \( B \) be IFS of the groups \( G_1 \) and \( G_2 \) respectively such that \( \mu_B(y) \leq \mu_B(e_1) \) and \( \nu_B(y) \geq \nu_B(e_1) \) holds for all \( y \in G_2 \), \( e_1 \) being the identity element of \( G_1 \). If \( A \times B \) is an IFSG of \( G_1 \times G_2 \), then \( B \) is IFSG of group \( G_2 \).

From Theorem (3.4) and Proposition (3.7) and (3.8), we have the following:

**Corollary (3.10)** Let \( A \) and \( B \) be IFS of the groups \( G_1 \) and \( G_2 \) respectively. If \( A \times B \) is an IFSG of \( G_1 \times G_2 \), then either \( A \) is IFSG of \( G_1 \) or \( B \) is IFSG of group \( G_2 \).

**Theorem (3.11)** Let \( A, C \) be IFSG’s of the groups \( G_1 \) and \( G_2 \), \( B, D \) be IFSG of the group \( G_2 \) respectively such that \( A \) and \( C \) are intuitionistic fuzzy conjugate subgroups of \( G_1 \) and \( B, D \) are intuitionistic fuzzy conjugate subgroups of \( G_2 \). Then the IFSG \( A \times B \) of \( G_1 \times G_2 \) is conjugate to the IFSG \( C \times D \) of \( G_1 \times G_2 \).

**Proof.** Since \( A \) and \( C \) are intuitionistic fuzzy conjugate subgroups of \( G_1 \), \( \exists s \in G_1 \) such that \( \mu_A(x) = \mu_C(g_1^{-1} x g_1) \) and \( \nu_A(x) = \nu_C(g_1^{-1} x g_1) \), \( \forall x \in G_1 \).

Also \( B, D \) are intuitionistic fuzzy conjugate subgroups of \( G_2 \), \( \exists t \in G_2 \) such that \( \mu_B(y) = \mu_D(g_2^{-1} y g_2) \) and \( \nu_B(y) = \nu_D(g_2^{-1} y g_2) \), \( \forall y \in G_2 \).

Now \( \mu_{AXB}(x, y) = \min\{\mu_A(x), \mu_B(y)\} = \min\{\mu_C(g_1^{-1} x g_1), \mu_D(g_2^{-1} y g_2)\} \)

\[
= \mu_{CxD}((g_1^{-1} x g_1), (g_2^{-1} y g_2)) = \mu_{CxD}(g_1^{-1}, g_2^{-1})(x, y)(g_1, g_2)
\]

Similarly, \( \nu_{AXB}(x, y) = \max\{\nu_A(x), \nu_B(y)\} = \max\{\nu_C(g_1^{-1} x g_1), \nu_D(g_2^{-1} y g_2)\} \)

\[
= \nu_{CxD}((g_1^{-1} x g_1), (g_2^{-1} y g_2)) = \nu_{CxD}(g_1^{-1}, g_2^{-1})(x, y)(g_1, g_2)
\]
Hence the IFSG $A \times B$ of $G_1 \times G_2$ is conjugate to the IFSG $C \times D$ of $G_1 \times G_2$.

**Theorem (3.12)** Let $A$ be an IFS of a group of $G$ and $V$ be the strongest intuitionistic fuzzy relation on $G$. Then $A$ is an IFSG of $G$ if and only if $V$ is an IFSG of $G \times G$.

**Proof.** Firstly, let $A$ be IFSG of group $G$.
Let $x = (x_1, x_2), y = (y_1, y_2)$ be any two element of $G \times G$, we have
\[
\mu_V(xy^{-1}) = \mu_V[(x_1, x_2)(y_1, y_2)^{-1}] = \mu_V[(x_1, x_2)(y_1^{-1}, y_2^{-1})] = \mu_V[(x_1y_1^{-1}, x_2y_2^{-1})] \\
\geq \min\{\mu_A(x_1y_1^{-1}), \mu_A(x_2y_2^{-1})\} \\
= \min\{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_A(x_2), \mu_A(y_2)\}\} \\
= \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_A(y_1), \mu_A(y_2)\}\} \\
= \min\{\min\{\mu_V(x_1, x_2), \mu_V(y_1, y_2)\}\} \\
= \min\{\mu_V(x), \mu_V(y)\}
\]

Similarly, we have
\[
\nu_V(xy^{-1}) = \nu_V[(x_1, x_2)(y_1, y_2)^{-1}] = \nu_V[(x_1, x_2)(y_1^{-1}, y_2^{-1})] = \nu_V[(x_1y_1^{-1}, x_2y_2^{-1})] \\
\leq \max\{\nu_A(x_1y_1^{-1}), \nu_A(x_2y_2^{-1})\} \\
= \max\{\max\{\nu_A(x_1), \nu_A(y_1)\}, \max\{\nu_A(x_2), \nu_A(y_2)\}\} \\
= \max\{\max\{\nu_A(x_1), \nu_A(x_2)\}, \max\{\nu_A(y_1), \nu_A(y_2)\}\} \\
= \max\{\nu_V(x_1, x_2), \nu_V(y_1, y_2)\} \\
= \max\{\nu_V(x), \nu_V(y)\}
\]

Thus $V$ is IFSG of $G \times G$

**References**


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