Weyl’s Theorem for Algebraically Class $A(k)$ Operators

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Abstract. In this paper, we show that Weyl’s theorem holds for algebraically class $A(k)$, $k \in (0,1]$ operators and discuss some of its applications.

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1. Preliminaries

Let $B(H)$ denote the algebra of bounded linear operators on an infinite dimensional separable Hilbert space $H$. If $T \in B(H)$, we write $N(T)$ for the null space of $T$. Let $\alpha(T) = \text{dim} N(T)$ and $\beta(T) = \text{dim} N(T^*)$ where $T^*$ is the adjoint of $T$. Also, let $\sigma(T), \sigma_p(T), \sigma_a(T)$ and $\pi_{00}(T)$ denote respectively the spectrum, point spectrum, approximate point spectrum and the set of eigen values of $T$ of finite multiplicity. $T \in B(H)$ is called a Fredholm operator if $TH$ is closed and both $\alpha(T)$ and $\beta(T)$ are finite. Its index is given by
\( i(T) = \alpha(T) - \beta(T) \). The ascent of \( T \) is the least non-negative integer \( n \) such that \( N(T^n) = N(T^{n+1}) \) and its descent is the least non-negative integer \( n \) such that \( T^n(H) = T^{n+1}(H) \).

\( T \) is called Weyl if it is Fredholm of index zero and is called Browder if it is Fredholm of finite ascent and descent.

Weyl Spectrum of \( T \) is given by \( \sigma_w(T) = \{ \lambda \in C/T - \lambda \) is not weyl \} \) and its Browder spectrum is given by \( \sigma_b(T) = \{ \lambda \in C/T - \lambda \) is not Browder \}. \( p_{00}(T) = \sigma(T) - \sigma_b(T) \) is called the set of Riesz points of \( T \).

We say that Weyl’s theorem holds for \( T \in B(H) \) if \( \sigma(T) - \sigma_w(T) = \pi_{00}(T) \) and Browder’s theorem holds if \( \sigma(T) - \sigma_w(T) = p_{00}(T) \).

2. Weyl’s theorem for algebraically class \( A(k), k \in (0, 1] \) operators

An operator \( T \in B(H) \) is said to be hyponormal if \( T^*T \geq TT^* \) and \( p \)-hyponormal if \( (T^*T)^p \geq (TT^*)^p \) for a positive number \( p \). \( T \) is called log hyponormal if \( T \) is invertible and \( log(T^*T) \geq log(TT^*) \). Further, \( T \) is said to belong to class \( A(k) \) where \( k > 0 \) if \( (T^*T)^{2kT} \geq |T|^2 \). Class \( A(1) \) operator is called Class\( (A) \) operator and is defined by \( |T|^2 > |T|^2 \). \( T \) is said to be of algebraically class \( A(k) \), if there exists a non-constant complex polynomial \( p \) such that \( p(T) \) is of class \( A(k) \). The following implicatios hold

\[
\begin{align*}
\text{hyponormal} & \rightarrow \text{log - hyponormal} \rightarrow \text{class} A(k), k \in (0, 1] \\
\text{algebraically class } A(k), k \in (0, 1] & \rightarrow \text{class } A(k), k \in (0, 1] \\
\text{class } A(k), k \in (0, 1] & \rightarrow \text{class } A(k), k > 1
\end{align*}
\]

Hyponormal, log hyponormal and class \( A \) operators satisfy Weyl’s theorem [14, 11, 13]. In [12], it is shown that operators of class \( A(k), k > 1 \) with a limit condition satisfy Weyl’s theorem. In this work, we prove that algebraically class \( A(k), k \in (0, 1] \) operators satisfy Weyl’s theorem, without the limit condition.

**Lemma 2.1:**

If \( T \) is of class \( A(k) \) operator where \( k \in (0, 1] \) and \( M \) is an invariant subspace of \( T \), then \( T|_M \) is also a class \( A(k) \) operator where \( k \in (0, 1] \).
Proof: Let \( T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \) act on \( H = M \oplus M^\perp \) and \( P \) be the projection onto \( M \) so that \( A = TP \) on \( M \).

We have \( P(T^*|T|^{2k}T)^{1/k} \geq PT^*TP \). So, \( A^*A = PT^*TP \leq P(T^*|T|^{2k}T)^{1/k} \).

Using Hansen’s inequality we have
\[
A^*A \leq (PT^*|T|^{2k}TP)^{1/k} = (PT^*P|T|^{2k}PTP)^{1/k} \tag{1}
\]

Using Hansen’s inequality again, we get
\[
|A|^{2k} = (P|T|^2P)^k \geq P|T|^{2k}P
\]
and so \( A^*|A|^{2k}A \geq A^*(P|T|^{2k}P)A \) showing
\[
(A^*P|T|^{2k}PA)^{1/k} \leq (A^*|A|^{2k}A)^{1/k} \tag{2}
\]

By (1) and (2) \( A \) is of class \( A(k) \) operator where \( k \in (0, 1] \).

Lemma 2.2:
Let \( T \) be of class \( A(k), k > 0 \) and assume that \( \sigma(T) = \{\lambda\} \). Then \( T = \lambda I \).

Proof:
Case(i) Let \( \lambda = 0 \).

Every class \( A(k) \) operator is a normaloid and so \( T = 0 \).

Case(ii) Let \( \lambda \neq 0 \).

Since \( T \) is invertible of class \( A(k) \), \( T^{-1} \) is of class \( A(l) \) for \( l \geq k > 0 \) [15] and so \( T \) and \( T^{-1} \) are normaloids. But \( \sigma(T^{-1}) = \{\frac{1}{\lambda}\} \) giving \( \|T\||T^{-1}||=1 \). It follows from [9] that \( T \) is a convexiod. So \( w(T) = \{\lambda\} \) and hence \( T = \lambda \).

Lemma 2.3:
Every quasinilpotent algebraically class \( A(k), k > 0 \) operator is nilpotent.

Proof:
Suppose \( p(T) \) is class \( A(k), k > 0 \) for some non constant polynomial \( p \). We can write
\[
p(\lambda) - p(0) = a_0\lambda^m(\lambda - \lambda_1)......(\lambda - \lambda_n) \text{ where } m \neq 0 \text{ and } \lambda_i \neq 0 \text{ for every } 1 \leq i \leq n.
\]
Since \( \sigma(p(T)) = \sigma(T) \), the operator \( p(T) - p(0) \) is quasinilpotent and so \( \sigma(p(T) - p(0)) = \{0\} \). Then by lemma 2.2, \( p(T) - p(0) = 0 \) and so
\[
a_0T^m(T - \lambda_1)......(T - \lambda_n) = 0.
\]
But \( T - \lambda_i \) is invertible for every \( \lambda_i \neq 0 \) and so \( T^m = 0 \).
Lemma 2.4:

Let $T$ be an algebraically class $A(k)$, $k \in (0, 1]$ operator. Then $T$ is an isoloid.

Proof:

Suppose $p(T)$ is class $A(k)$. Let $\lambda \in \text{iso } \sigma(T)$. Then, using the spectral decomposition, we can write $T = T_1 \oplus T_2$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T_1) - \{\lambda\}$. Then we must have $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$ and so $p(T_1) - p(\lambda)$ is quasinilpotent.

By lemma 2.1, $p(T_1)$ is of class $A(k)$, $k \in (0, 1]$ and by lemma 2.2, we have $p(T_1) - p(\lambda) = 0$. Let $q(z) = p(z) - p(\lambda)$. Then $q(T_1) = 0$ and so $T_1$ is algebraically class $A(k)$, $k \in (0, 1]$. But $T_1 - \lambda$ is quasinilpotent and algebraically class $A(k)$. So, by lemma 2.3, $T_1 - \lambda$ is nilpotent. Then $\lambda \in \sigma_p(T_1)$ and so $\lambda \in \sigma_p(T)$. Thus $T$ is an isoloid.

Lemma 2.5:

If $T$ is of class $A(k)$, $k \in (0, 1]$ then $T - \lambda$ has finite ascent for $\lambda \in \sigma_p(T)$.

Proof:

$T$ satisfies the inequality $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in H$ and so $N(T^2) = N(T)$. Further, we have $N(T - \lambda) = N(T - \lambda)^*$. If $0 \neq \lambda \in \sigma_p(T)$ then $(T - \lambda)^2x = 0$ implies $(T - \lambda)^*(T - \lambda)x = 0$ and so $\|(T - \lambda)x\| = 0$ showing that $T - \lambda$ has ascent 1.

Lemma 2.6:

Let $T$ be an algebraically class $A(k)$ operator, $k \in (0, 1]$ and $\lambda \in \text{iso } \sigma(T)$. Then the ascent and descent of $T - \lambda$ are both equal to 1.

Proof:

Write $T = T_1 \oplus T_2$ on $H = H_1 \oplus H_2$ such that $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T_1) - \{\lambda\}$.

Let $p(T)$ be of class $A(k)$, $k \in (0, 1]$. Then $H_1$ is an invariant subspace for $p(T)$ and hence by lemma 2.1, $p(T_1)$ is of class $A(k)$ with $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$. Then $p(\lambda) \in p_{00}(p(T_1))[3]$. and so $\lambda \in p_{00}(T_1)[4]$. Since $\lambda$ does not belong to
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σ(T_2), we have λ ∈ p_{00}(T) and so T − λ is Browder.

We say that T ∈ B(H) has the single valued extension property (SVEP) if for every open set U ⊆ C the only analytic function f : U → H which satisfies the equation (T − λ)f(λ) = 0 is the constant function f ≡ 0. Trivially every operator T has SVEP at points of the resolvent set C − σ(T); also T has SVEP at λ ∈ iso σ(T).

Theorem 2.7:
If T ∈ B(H) is an algebraically class A(k), k ∈ (0, 1] operator, then T and T* satisfy Weyl’s theorem.

Proof:
Let p(T) be of class A(k), k ∈ (0, 1]. Then p(T) has SVEP and so T has SVEP [7]. Then T satisfies Browder’s theorem if and only if T* satisfies Browder’s theorem if and only if
\[ p_{00}(T) = σ(T) − σ_w(T) \subseteq π_{00}(T) \]
and \[ p_{00}(T^*) = σ(T^*) − σ_w(T^*) \subseteq π_{00}(T^*). \]

If λ ∈ π_{00}(T^*), then both T and T* have SVEP at λ and
\[ 0 < asc(T − λ)^* = dsc (T − λ) < ∞. \]

So, the ascent and descent of T − λ are finite and hence equal [2]. Then each of T − λ and (T − λ)^* is Fredholm of index zero and so
\[ π_{00}(T) \subseteq σ(T) − σ_w(T) \] and
\[ π_{00}(T^*) \subseteq σ(T^*) − σ_w(T^*). \]

So both T and T* satisfy Weyl’s theorem.

3. Application of Weyl’s theorem on class A(k), k ∈ (0, 1] operators

Theorem 3.1:
If T ∈ B(H) is an algebraically class A(k), k ∈ (0, 1] operator, then
\[ w(f(T)) = f(w(T)) \]
for every f ∈ H(σ(T)) where H(σ(T)) denotes the set of analytic functions on an open neighbourhood of σ(T).

Proof:
Since \( w(f(T)) \subseteq f(w(T)) \) is true with no restriction on \( T \), it suffices to show that \( f(w(T)) \subseteq w(f(T)) \).

Suppose \( \lambda \) does not belong to \( w(f(T)) \). Then \( f(T) - \lambda \) is Weyl and

\[
f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2) \ldots (T - \alpha_n)g(T)
\]

where \( c, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \) and \( g(T) \) is invertible. Since the operators on the righthand side of this equation commute, every \( T - \alpha_i \) is Fredholm. Now \( T \) has SVEP. It follows from \( [1] \) that \( i(T - \alpha_i) \leq 0 \) for each \( i = 1, 2, \ldots, n \). So, \( \lambda \) does not belongs to \( f(w(T)) \) and hence \( f(w(T)) = w(f(T)) \).

**Corollary 3.2**

If \( T \in B(H) \) is algebraically class \( A(k) \), \( k \in (0, 1] \), then for every \( f \in H(\sigma(T)) \), Weyl’s theorem holds for \( f(T) \).

**Proof:**

By \( [6] \), if \( T \) is an isoloid, then \( f(\sigma(T)) - \pi_{00}(T) = \sigma(f(T)) - \pi_{00}(f(T)) \) for every \( f \in H(\sigma(T)) \).

Then, by lemma 2.4 and theorem 3.1, we have

\[
\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T)) - \pi_{00}(f(T)) = f(w(T)) = w(f(T)).
\]

showing that Weyl’s theorem holds for \( f(T) \).

We now show that for algebraically class \( A(k) \), \( k \in (0, 1] \) operators, the spectral mapping theorem holds for the essential approximate point spectrum.

**Theorem 3.3:**

Let \( T \) or \( T^* \) be algebraically class \( A(k) \), \( k \in (0, 1] \). Then \( \sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) \) for every \( f \in H(\sigma(T)) \).

**Proof:**

For \( T \in B(H) \), by \( [10] \), we have \( \sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T)) \) for every \( f \in H(\sigma(T)) \) with no restrictions on \( T \). So, it suffices to show that \( f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T)) \). Suppose \( \lambda \) does not belong to \( \sigma_{ea}(f(T)) \) then

\[
f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2) \ldots (T - \alpha_n)g(T)
\]

where \( c, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \) and \( g(T) \) is invertible. Since \( i(T - \alpha_i) \leq 0 \) for each \( i = 1, 2, \ldots, n \), we have \( \lambda \) does not belong to \( f(\sigma_{ea}(T)) \) and so \( \sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) \).
Theorem 3.4:
Let $T$ be class $A(k)$ operator with $k \in (0, 1]$. If $\pi_{00}(T) = \phi$, then $T$ is extremely non compact.

Proof:
Since $T$ is normaloid and $\pi_{00}(T) = \phi$ by [5], $T$ is extremely non compact.

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