An Approximation Problems on $L_2(R)$
by Wavelet Neural Network

Ma Yumei$^1$, Nan Dong$^2$ and Liu Lijun$^1$

$^1$ Dalian Nationalities University
Department of Mathematics
Dalian liaoning, 116600, China
mayumei@dlmu.edu.cn (Ma Yumei)

$^2$ Beijing University of Technology
Department of Mathematics
Beijing, 100124, China

Abstract

Suppose that $\sigma$ is a sigmoidal function, Li Xin showed that a linear combination of dilates and translates of generates a near tight wavelet frame for $L_2(R)$. This paper extend Li Xin’s result and firstly generalize the definite field $I_\pi$ to $R^n$, secondly show the approximate problem on a compact set of $L_2(R)$.

Mathematics Subject Classification: 92B20, 46T30, 46Fxx

Keywords: Neural Network, frame of wavelet

1 Introduction

Many problems concerning the applications of neural networks can be directly converted into the problems of approximating multivariate functions by compositions (or superpositions) of the activation functions of neural networks. Mathematically, a neural network with one hidden layer is expressed as a superposition of an activation function $\sigma$ as follows

$$N(x) = \sum_{k=1}^{n} c_k \sigma(<w_k, x> + \theta_k), \quad x \in R^n, \quad s \geq 1,$$

where for $1 \leq k \leq n$, $\theta_k \in R$ are the thresholds, $w_k \in R^n$ connection weights, and $c_k \in R$ coefficients. Various density or complexity results concerning the approximation of multivariate functions, defined on an arbitrary
functions in $L^1$ univariate wavelet frames by neural networks will be provided to approximate $V$ Let $K$ exists a compact set $f$ uniformly for all $x, h \in g \in C(R^n)$, with $f \in C(R^n)$, such that $\|f - g\|_{L^p} < \varepsilon$. Here $C_c(R^n) = \{f \in C(R^n)\}$. And thus any $L^2(R)$ functions can be reconstructed by the wavelet frame series, and a procedure of approximating functions was proposed from wavelet recovery formulas. Pati and Krishnaprasa first constructed wavelet frames for $L^2(R)$ by using the squashing function $\sigma(x) = (1 + e^{-px})^{-1}$, $p > 0$. By setting

$$\varphi(x) = \sigma(x + d) - \sigma(x - d), \quad d > 0,$$

and

$$\psi(x) = \varphi(x + q) - \varphi(x - q), \quad q > 0,$$

it is shown that $\psi$ generates a wavelet frame for $L_2(R)$. And thus any $L^2(R)$ functions can be reconstructed by the wavelet frame series.

The following lemmas will be needed at next section.

**Lemma 1.1.** [2] Suppose that $f \in L^p(R^n)$, then for any $\varepsilon > 0$ we have $g \in C_c(R^n)$ such that $\|f - g\|_{L^p} < \varepsilon$. Here $C_c(R^n) = \{f \in C(R^n)\}$. There exists a compact set $K \in R^n$, with $f = 0$ on $R^n \setminus K$.

**Lemma 1.2.** (Fejer) [2] Let $\phi : R^1 \mapsto R^1$ is an integral, such that $\int_{R^n} \phi(x)dx = 1$ $\varepsilon > 0$, $\phi_\varepsilon(x) = (\frac{1}{2\pi}) \phi(\frac{x}{\varepsilon})$. Define $\phi_\varepsilon : R^n \mapsto R$, if

1. $f \in L^p(R^n), p \in [1, \infty), \text{ Then } \varepsilon \to 0, \|\phi_\varepsilon * f - f\|_p \to 0$;
2. $f \in L^\infty(R) \cap L^1(R), K \subset R$, $\limsup_{y \to 0} |f(x + y) - f(x)| = 0$,

then $f * \phi_\varepsilon$ is convergency uniformly on $K$.

**Lemma 1.3.** [3] The $V$ is a compact set of $L^p(K)$ if and only if

1. $V$ is a closed set of $L^p(K)$
2. there is a $A > 0$, with $\|f\|_{L^p(K)} \leq A$ for all $f \in V$
3. for any $h > 0$, define $f_h$ is as following

$$f_h(x) = \frac{1}{m(B(x, h))} \int_{B(x, h) \cap K} f(t)dt,$$

Here $B(x, h) \subseteq R^n$ is a ball that the center of a circle at $x$ and radius $h$ $mB(x, h)$ is the volume of $B(x, h)$. Then $\|f_h - f\|_{L^p(K)}$ is convergency to zero uniformly for all $f \in V$ with $h \to 0$.

**Lemma 1.4.** [4] Let $K$ is a compact set of $R^n$, $V$ is a compact of $L^p(K)$. Let $V_h = \{f_h : f \in V\}$, then $V_h$ is a compact of $C(K)$ for any $h$. Ma Yumei, Nan Dong and Liu Lijun

604
2 The Approximation on Dimension $n$ Real Space

In this section, the result of Xin Li\(^{[1]}\) are extended to a compact not only $K = I_{pl} \subset R^s$.

**Theorem 2.1.** If $\sigma$ satisfies the condition that above, then for all $f \in L^2(R^n)$, $\varepsilon > 0$, there exists $K \subset R^n$, that is finite measurable set and positive integers $N, M$, constant $\lambda_i$, $d_{ij} \in R^1$, and vector $y_{ij} \in R^n$, $c_{ij}(f)$ that depend on $f$, $i = 1, 2, ..., M$, $j = 1, 2, ..., N$ such that:

$$
\left( \int_{K_0} |f(x) - \sum_{j=1}^{N} c_j(F)N(x-x_j)|^2 \, dx \right)^{\frac{1}{2}} < \varepsilon
$$

**Proof.** Set $f \in L^2(R^n)$, and $\varepsilon > 0$ by Lemma 1.1, there exists a compact set $K \subset R^n$, $F(x) \in C_c(R^n)$ with $F \mid_{R^n\setminus K} = 0$, such that:

$$
\|f - F\|_{L^2} < \frac{\varepsilon}{4}
$$

Let $h(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\|x\|^2}{2}}$, $h_\delta(x) = \delta^{-n} h(\frac{x}{\delta})$. Then for any $x \in R^n$ we have that

$$
F * h_\delta(x) = \int_{R^n} F(t)h_\delta(x-t)\, dt = \int_{K} F(t)h_\delta(x-t)\, dt
$$

and the Riemann sum of $F * h_\delta(x)$, such that

$$
|F * h_\delta(x) - \sum_{j=1}^{N} c_j(F)e^{\frac{1}{4}\|x-x_j\|^2}| < \frac{\varepsilon}{4}
$$

again by Lemma 1.2, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$
(\int_{R^n} |F * h_\delta(x) - F(x)|^2\, dx)^{\frac{1}{2}} < \frac{\varepsilon}{4}
$$

Because that $e^{-\frac{\|x-x_j\|^2}{\delta^2}} \in L^2(R^n) \subset L^2(I_m)$, by Li Xin\(^{[2]}\) we can find $\exists \lambda_p(t)$ with

$$
N(x) = \frac{2}{A+B} \sum_{|m|<p} a_m(f) \sum_{j,k \in Z} < \lambda_p, \psi_{n,j,k} > \psi_{n,j,k}(< m, x >)
$$

such that $\|e^{-\frac{\|x-x_j\|^2}{\delta^2}} - N(x-x_j)\|_{L^2(K)} < d_{n,h_\delta} \frac{\varepsilon}{4}$ Here, $d_{n,h_\delta} = 1 + \frac{2}{\sqrt{2\pi}n^\frac{1}{4}} \|h_\delta\|_{L^2(I_m)}$

we got the constant $L = \sum_{j=1}^{N} |c_j(F)|$ and vector $x_j \in R^n, j = 1, 2, ..., N$, such that

$$
\|f(x) - \sum_{j=1}^{N} c_j(F)N(x-x_j)\|_{L^2(K)}
For any $N < \infty$ and $\varepsilon > 0$, we can find a wavelet frame $\{\psi_{n,j,k}\}_{n,j,k}$ independent of $f$, furthermore structure a $N(x)$ that approximate $f$ on $K$.

**Theorem 2.2.** $\sigma$ satisfies the condition above, let $K = I_\pi \subset R^s$, $V$ is a compact of $L^p(K)$ ($p \geq 2$), then for $\varepsilon > 0$, there exist $A, B, M, p$, constant $a_m, \lambda_p$ and the wavelet frame of $\{\psi_{n,j,k}\}_{n,j,k}$ that independent on $f (f \in V)$, but the coefficient $c_i(f)$ are depend on $f$, such that for all $f \in V$, with

$$\|f(x) - \sum_{i=1}^{M} c_i(f) \cdot N(x - x_i)\|_{L^p(I_\pi)} < \varepsilon.$$ 

Here, $N(x) = \frac{2}{A + B} \sum_{m} a_m(f) \sum_{j,k \in Z} \lambda_p \psi_{n,j,k} > \psi_{n,j,k}(< m, x >)$

$$A \left\| e^{-\|x\|_{R^s}} \right\|^2_{L^2(R^s)} \leq \sum_{j,k \in Z} \left| \langle \psi_{n,j,k}, e^{-\|x\|} \rangle \right|^2 \leq B \left\| e^{-\|x\|_{R^s}} \right\|^2_{L^2(R^s)}.$$ 

**Proof.** For any $\varepsilon > 0$, by (3) of Lemma 1.3, there exists $h_0 > 0$, with $0 < h < h_0$, for all $f \in V$, we have $||f_h(x) - f(x)||_{L^2(K)} < \frac{\varepsilon}{4}$. Let $h(x) = \frac{1}{\sqrt{\pi}} e^{-\|x\|^2 / 2}$, $h_\delta(x) = \delta^{-n} h(\frac{x}{\delta})$, then for any $x \in R^n$ we have that

$$f_h \ast h_\delta(x) = \int_{R^n} f_h(t) h_\delta(x - t) dt = \int_K f_h(t) h_\delta(x - t) dt$$

(1)

Then for each $0 < h < h_0$, Lemma 1.2 and Lemma 1.4 implies that

$$||f_h \ast h_\delta - f||_{L^2(K)} < \frac{\varepsilon}{4}$$

far all $f_h \in V_h$: $||f - f_h \ast h_\delta||_{L^2(K)} < \frac{\varepsilon}{2}$

for each $h < h_0$, $\delta < \delta_0$, write the Riemann sum of $f_h \ast h_\delta(x) \sum_{j=1}^{M} f(t_j) h_\delta(x - t_j)m(\Delta t_j)$. Here: $\bigcup_{j=1}^{M} \Delta t_j$ is a partition of $K$, $t_j \in \Delta t_j$. Then

$$\begin{align*}
f_K f_h(t) h_\delta(x - t) dt & - \sum_{j=1}^{M} f_h(t_j) h_\delta(x - t) m(\Delta t_j) \\
= & \sum_{j=1}^{M} \int_{\Delta t_j} [f_h(t) - f_h(t_j)] h_\delta(x - t) dt + \sum_{j=1}^{M} f_h(t_j) \int_{\Delta t_j} [h_\delta(x - t) - h_\delta(x - t_j)] dt \\
= & I_1 + I_2,
\end{align*}$$
An approximation problems on $L_2(R)$

for $\forall \varepsilon > 0$, since that $V_h$ is a compact, then there exists $\delta_0 > 0$, with

$$\max\{\text{diam}(\Delta t_j)\} < \delta_0$$

for all $f_h \in V_h$

$$||I_1||_{L^2(K)}, ||I_2||_{L^2(K)} < \frac{\varepsilon}{8}. \quad (2)$$

Thus, for all $f \in V$, $||f(x) - \sum_{j=1}^{M} f_h(t_j)m(\Delta t_j)h_\delta(x-t_j)||_{L^2(K)} < \left(\frac{2}{\delta} + d_{n,h}\right)\varepsilon$

Because that $h_\delta(x-x_j) = e^{-\frac{\|x-x_j\|^2}{\delta^2}} \in L^2(I_\sigma)$, there exists the wavelet frame $\{\psi_{n,j,k}\}_{n,j,k}$, such that

$$\|N(x-x_j) - e^{-\frac{\|x-x_j\|^2}{\delta^2}}\|_{L^2(x)} < d_{n,e} \frac{\varepsilon}{4} \quad (3)$$

Here, $N(x) = \frac{2}{A+B} \sum_{|m| < p} a_m(e^{-\frac{\|x\|^2}{\delta}}) \sum_{j,k \in Z} < \lambda_p, \psi_{n,j,k} > \psi_{n,j,k}(< m, x >)$

Let $N = \max\{N_j\}_{j=1,2,\ldots,M}$, combing (1)-(3) we have that

$$\|f(x) - \sum_{j=1}^{M} c_j(f)N(x-x_j)\|_{L^p(I_\sigma)} < \varepsilon$$

Here, $N(x)$ is independent on $f(x)$ but only depend on $e^{-\|x\|^2}$.

References


Received: September, 2011