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The Benz Theorem and Aleksandrov Problem for Preserving Distance Mapping

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Abstract

In this paper, some isometric extending mappings which rooted Aleksandrov problem (one of is Benz's Theorem)are obtained. i.e. about mapping that have some property can be extend to isometry in whole space.

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1 Introduction

In 1970, Aleksandrov,A.D.[1] first advance DOPP problem: how to let a DOPP mapping between metric spaces to be an isometry. "One Distance Preserving Property (denoted by DOPP)" (*Suppose that* (X, d), (Y, d) *are two metric linear spaces. The mapping* $f : X \to Y$ *is said to have DOPP if* $d(x, y) = 1$ *for any* $x,\ y \in X$, *then* $d(f(x), f(y)) = 1$.)

In 1983, T.Rassias, B.Mielnik, [2] gave a series results about DOPP and SDOPP mapping (both f and f^{-1} are all preserves one distance) can be isometry. Benz's work is important significance to DOPP problem. 1985, Benz gave a result as*Let* X, Y *be real normed linear spaces such that* $dimX > 2$ *and* Y *is strictly convex.*

In this paper we establish an generalized Benz isometric extension problem.

2 Main results on Benz Theorem

In this section we do not assume the above condition of "strictly convex and $dimX \geq 2$*," and show the following refined theorem.*

Theorem 2.1. *Let X and Y are two normed spaces, and $f : X \to Y$ is a surjection,and satisfies that*
(1) $||x-y|| \leq 1$ then $||f(x)-f(y)|| \leq ||x-y||$,
(2) $||x-y|| \geq \alpha$, then $||f(x)-f(y)|| \geq \alpha$.
Then, f is an isometry.

Proof.
i). $||f(x)-f(y)|| \leq ||x-y||$ for all $x, y \in X$.
Let

$$||x-y|| \leq \frac{m}{n}$$

It is obviously for $m = 1$. Next, we show that is true for $m \geq 2$, we find a sequence $x = z_0, z_1, \ldots, z_m = y$, such that $||z_i - z_{i+1}|| \leq \frac{1}{n}$ $(i = 0, 1, 2, \ldots, m-1)$. Clearly, we have

$$||f(x)-f(y)|| \leq \sum ||f(z_i)-f(z_{i+1})|| \leq \sum ||z_i - z_{i+1}|| \leq \frac{m}{n}.$$

It follows that $||f(x)-f(y)|| \leq ||x-y||$.

ii).f preserves α. In fact, suppose that $||x-y|| = \alpha$. We have

$$||f(x)-f(y)|| \leq ||x-y||$$

By i), and by the condition (2),

$$||f(x)-f(y)|| \geq ||x-y||,$$

so

$$||f(x)-f(y)|| = ||x-y|| = \alpha,$$

this show that f preserves α.

iii).The following will give that $||f(x)-f(y)|| = ||x-y||$, as $||x-y|| \leq \alpha$.
In fact, by i)

$$||f(x)-f(y)|| \leq ||x-y||,$$

Assume that

$$||f(x)-f(y)|| < ||x-y||,$$

let $z = x + \frac{\alpha}{||x-y||}(y-x)$, then $||z-x|| = \alpha$, $||z-y|| = \alpha - ||x-y||$ This means that

$$\alpha = ||f(z)-f(x)|| \leq ||f(z)-f(y)||+||f(x)-f(y)|| < \alpha-||x-y||+||x-y|| = \alpha,$$

hence a contradiction. This implies that $||f(x) - f(y)|| = ||x - y||$.

iv). In this part, it affirm that f *preserve that the distance* $\frac{n}{2}\alpha$.
Let $||x - z|| = \frac{\alpha}{2}n$, *then*

$$||f(z) - f(x)|| \leq \frac{\alpha}{2}n.$$

Let

$$u = f(x) + \frac{\alpha}{2} \cdot \frac{f(z) - f(x)}{||f(z) - f(x)||},$$

then there exists a $v \in X$ *such that* $f(v) = u$ *by* f *is a surjection.*
Then $\|u - f(x)\| = \frac{\alpha}{2} < \alpha$, *and the condition (2) implies that* $||v - x|| < \alpha$, *again by part ii)* $||v - x|| = ||u - f(x)|| = \frac{\alpha}{2}$. *It must be have that*

$$||u - f(z)|| \geq \frac{\alpha}{2}(n - 1).$$

Otherwise, if $||u - f(z)|| < \frac{\alpha}{2}(n - 1)$, *then we can find a sequence* $v_i \in X, i = 1, 2, \ldots, n - 1$ *such that* $v_0 = v, v_{n-1} = z$ *and* $||f(v_i) - f(v_{i+1})|| < \frac{\alpha}{2} < \alpha$ *by* f *is a surjective, thus* $||v_i - v_{i+1}|| < \alpha$ *and*

$$||v_i - v_{i+1}|| = ||f(v_i) - f(v_{i+1})|| < \frac{\alpha}{2}.$$

This implies that $||v - z|| < \frac{n-1}{2}\alpha$, *moreover that* $||x - z|| < \frac{n}{2}\alpha$, *which is a contradiction with* $||x - z|| = \frac{n}{2}\alpha$. *So*

$$\frac{\alpha}{2}(n - 1) \leq ||u - f(z)|| = ||f(z) - f(x)|| - \frac{\alpha}{2}.$$

This implies that

$$||f(z) - f(x)|| = \frac{\alpha}{2}n.$$

v). f *is an isometry from* X *to* Y.
In fact, for any $x, y \in X$, *there exists* n *such that* $||x - y|| < \frac{\alpha}{2}n$.
Assume that

$$||f(x) - f(y)|| < ||x - y||,$$

let $z = x + \frac{\frac{\alpha}{2}n}{||x-y||}(y - x)$. *So* $||z - x|| = \frac{\alpha}{2}n$, $||z - y|| = \frac{\alpha}{2}n - ||x - y||$. *Thus by v),*

$$||f(z) - f(x)|| = \frac{\alpha}{2}n.$$

$$\frac{\alpha}{2}n = ||f(x) - f(z)|| \leq ||f(x) - f(y)|| + ||f(y) - f(z)|| < \frac{\alpha}{2}n$$

This a contradiction implies that

$$f(x) - f(y)|| = ||x - y||.$$

Theorem 2.2. *Let X, Y be two normed spaces, and the mapping $f : X \to Y$ satisfying the (DOPP), $||f(x)-f(y)|| \leq ||x-y||$ for $x, y \in X$ with $||x-y|| \leq 1$ and distance n preserving property($n > 0$ is any integer,i.e.$||f(x) - f(y)|| = n$ with $||x - y|| = n$). Then f is an isometry from X to Y.*

Proof. *First, we prove that if $||x - y|| \leq 1$, then $||f(x) - f(y)|| = ||x - y||$. Assume that*

$$||f(x) - f(y)|| < ||x - y||.$$

Set

$$z = x + \frac{1}{||y - x||}(y - x).$$

Clearly,

$$||z - x|| = 1 \text{ and } ||z - y|| = 1 - ||x - y|| \leq 1.$$

Due to the DOPP, it follows that

$$1 = ||f(z)-f(x)|| \leq ||f(z)-f(y)||+||f(x)-f(y)|| < 1-||x-y||+||x-y|| = 1$$

Hence $||f(x) - f(y)|| = ||x - y||$.
Second, for any $x, y \in X$ with $||x - y|| > 1$, there exists an integer n with $n < ||x - y|| \leq n + 1$

$$z = x + n\frac{1}{||y - x||}(y - x).$$

Then $||z - x|| = n, ||z - y|| = ||x - y|| - n < 1$ since f satisfies DOPP.
It follows that $||f(z) - f(x)|| = n$ and $||f(z) - f(y)|| = ||z - y||$ by the above proof.
This implies that

$$||f(z) - f(x)|| + ||f(z) - f(y)|| = n + ||x - y|| - n = ||x - y||,$$

then

$$||f(x) - f(y)|| \leq ||x - y||.$$

Next we'll prove that

$$||f(x) - f(y)|| = ||x - y||.$$

Suppose that

$$||f(x) - f(y)|| < ||x - y||,$$

Let

$$z_1 = x + (n+1)\frac{1}{||y-x||}(y-x).$$

Thus $||z_1 - x|| = n+1,\ ||z_1 - y|| = n+1-||x-y|| \leq 1$ *and*

$$n+1 = ||f(z_1)-f(x)|| \leq ||f(z_1)-f(y)||+||f(y)-f(x)|| < n+1-||x-y||+||x-y|| = n+1.$$

This is a contradiction and we complete the proof.

$$||f(x) - f(y)|| = ||x - y||.$$

Remark 2.3. *The above Theorems true for Lipschitz condition* $||f(x) - f(y)|| \leq ||x - y||$ *for any* $x, y \in X$.

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