On Completely Semiprime $Q$-Fuzzy Ideals in Ordered Semigroups

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Abstract

In this paper, we first introduce the new concept of completely semiprime $Q$-fuzzy ideals of an ordered semigroup $S$, which is an extension of completely semiprime ideals of ordered semigroup $S$, and investigate some its related properties. Especially, we characterize an ordered semigroup that is a semilattice of simple ordered semigroups in terms of completely semiprime $Q$-fuzzy ideals of an ordered semigroups.

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1 Introduction

Let $S$ be a nonempty set. A fuzzy subset of $S$ is, by definition, an arbitrary mapping $f : S \rightarrow [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. The important concept of fuzzy set put forth by Zadeh in 1965 [9] has opened up a new domain that has kept insights and applications in a wide range of scientific fields. A theory of fuzzy sets on ordered semigroups has been recently developed [1-5]. Following the terminology given by Zadeh, if $S$ is an ordered semigroup, fuzzy sets in ordered semigroup $S$ have been first considered by Kehayopulu and Tsingelis in [1], then they defined ”fuzzy” analogous for several notations, which have proven useful in the theory of ordered semigroups. In this paper we attempt to introduce and give a detailed investigation of completely semiprime $Q$-fuzzy ideals of an ordered semigroup $S$. Moreover, we investigate some its related properties. Especially, we characterize an ordered semigroup that is a semilattice of simple ordered semigroups in terms of completely semiprime $Q$-fuzzy ideals of ordered semigroups.
2 Preliminary Notes

Throughout this paper, we denote by $\mathbb{Z}^+$ the set of all positive integers. In sequel we denote, by $S$, an ordered semigroup, that is, a semigroup $S$ with an order relation $\leq$ such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. Let $Q$ be nonempty set. A function $f$ from $S \times Q$ to real closed interval $[0, 1]$ is called a $Q$-fuzzy subset of $S$. Let $A$ be an nonempty subset of $S$. We denote by $f_A$ the characteristic mapping of $A$, that is, the mapping of $S \times Q$ into $[0, 1]$ define by

$$f_A(x, q) := \begin{cases} 
 1 & \text{if } x \in A, \\
 0 & \text{if } x \notin A.
\end{cases}$$

Then $f_A$ is a $Q$-fuzzy subset of $S$.

A nonempty subset $A$ of an ordered semigroup $S$ is called a left (resp. right) ideal of $S$ if

1. $SA \subseteq A$ (resp. $AS \subseteq A$), and
2. If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

If $A$ is both a left and a right ideal of $S$, then it is called an (two side) ideal of $S$ [7]. We denote by $I(a)$ the two sided ideal of $S$ generated by $a(a \in S)$. Then $I(a) = (a \cup Sa \cup aS \cup SaS]$. An ideal $I$ of $S$ is called prime (also called weakly prime) if for any two ideals $A, B$ of $S$ such that $AB \subseteq I$, $A \subseteq I$ or $B \subseteq I$; $I$ is called completely semiprime (also called semiprime) if for any element $a$ of $S$ such that $a^2 \in I$, then $a \in I$.

**Definition 2.1** An ordered semigroup $S$ is simple if for every ideal $I$ of $S$, we have $I = S$.

Let $S$ be an ordered semigroup. A $Q$-fuzzy subset $f$ of $S$ is called a $Q$-fuzzy left ideal of $S$ if

1. $x \leq y \Rightarrow f(x, q) \geq f(y, q)$.
2. $f(xy, q) \geq f(y, q)$,
for all $x, y \in S, q \in Q$.

Let $S$ be an ordered semigroup. A $Q$-fuzzy subset $f$ of $S$ is called a $Q$-fuzzy right ideal of $S$ if

1. $x \leq y \Rightarrow f(x, q) \geq f(y, q)$.
2. $f(xy, q) \geq f(x, q)$,
for all $x, y \in S, q \in Q$.

A $Q$-fuzzy subset $f$ of $S$ is called a fuzzy ideal of $S$ if it is both a $Q$-fuzzy left and $Q$-fuzzy right ideal of $S$.

**Definition 2.2** Let $f$ be any function from a set $S$ to a set $T$ and $\mu$ any $Q$-fuzzy subset of $T$. Then $f^{-1}(\mu)$, the pre-image of $\mu$ under $f$, is a $Q$-fuzzy subset of $S$, defined by $(f^{-1}(\mu))(x, q) = \mu(f(x), q)$ for all $x \in S, q \in Q$. 

Definition 2.3 Let $S, T$ be two ordered semigroups, a mapping $f : S \to T$ is called isotone if $x, y \in S, x \leq y$ implies $f(x) \leq f(y)$ in $T$. $f$ is called a homomorphism if it is isotone and satisfies that $f(xy) = f(x)f(y)$ in $T$, for all $x, y \in S$.

3 Main Results

Definition 3.1 A $Q$-fuzzy $f$ of an ordered semigroup $S$ is called completely semiprime if $f(a, q) \geq f(a^2, q)$ for all $a \in S, q \in Q$.

The following theorem shows that the concept of $Q$-fuzzy completely semiprimality in an ordered semigroup is an extension of completely semiprimality.

Theorem 3.2 Let $A$ be a nonempty subset of an ordered semigroup $S$. Then the following statements are equivalent:

(1) $A$ is completely semiprime.
(2) The characteristic function $f_A$ of $A$ is completely semiprime.

Proof. $\Rightarrow$. Let $a \in S, q \in Q$. If $a^2 \in A$, then, since $A$ is completely semiprime, we have $a \in A$. Thus $f_A(a, q) = f_A(a^2, q)$. If $a^2 \notin A$, then we have $f_A(a, q) \geq 0 = f_A(a^2, q)$. Therefore we have $f_A(a, q) \geq f_A(a^2, q)$ for all $a \in S, q \in Q$, and $f_A$ is a completely semiprime.

$\Leftarrow$. Let $a^2 \in A, a \in S, q \in Q$. Then, since $f_A$ is completely semiprime, we have $f_A(a, q) \geq f_A(a^2, q) \geq 1$. Since $f_A$ is a $Q$-fuzzy subset of $S$ and $f_A(a, q) \leq 1$ for any $a \in S, q \in Q$, so we have $f_A(a, q) = 1$, which implies that $a \in A$. It thus follows that $A$ is completely semiprime.

Theorem 3.3 Let $f$ be any $Q$-fuzzy ideals of an ordered semigroup $S$. Then the following statements are equivalent:

(1) $f$ is completely semiprime.
(2) $\forall a \in S, \forall q \in Q$ \(f(a, q) = f(a^2, q)\).
(3) $\forall a \in S) (\forall q \in Q) (\forall n \in \mathbb{Z}^+) f(a, q) = f(a^n, q)$.

Proof. It is clear that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2).

(1)$\Rightarrow$(2). Let $a$ be any element of $S$ and $q$ any element of $Q$. Then, since $f$ is a completely semiprime $Q$-fuzzy ideal of $S$, we have

$$f(a, q) \geq f(a^2, q) \geq f(a, q),$$

and so we have $f(a, q) = f(a^2, q)$.

(2)$\Rightarrow$(3). We prove this result by induction. Clearly, the result holds for $n = 2$. Let $k \geq 2$ be any positive integer. Let $f(a^n, q) = f(a, q)$ holds for
∀a ∈ S, ∀q ∈ Q and ∀n ∈ ℤ⁺, 1 ≤ n ≤ k. We claim that \( f(a^{k+1}, q) = f(a, q) \). Indeed:

Case 1. If \( k \) is odd, let \( k = 2m + 1 \). Then \( f(a^{k+1}, q) = f((a^{m+1})^2, q) = f(a^{m+1}, q) \). Since \( m+1 < k \), by the induction hypothesis, \( f(a^{m+1}, q) = f(a, q) \).

Case 2. If \( k \) is even, let \( k = 2m \). Then again by the induction hypothesis, we have
\[
0 \leq f(a, q) \leq f(a^{k+1}, q) = f((a^{m+1})^2, q) = f(a^{m+1}, q) = f(a, q),
\]
which implies that \( f(a^{k+1}, q) = f(a, q) \). This proves the result.

The following theorem gives a characterization of completely semiprime \( Q \)-fuzzy ideals of an ordered semigroup by ordered \( Q \)-fuzzy points.

**Theorem 3.4** Let \( f \) be a \( Q \)-fuzzy ideal of an ordered semigroup \( S \). Then \( f \) is completely semiprime if and only if for any ordered \( Q \)-fuzzy points \( a_λ \in S(∀\lambda \in (0, 1]) \), \( a_λ^2 \in f \) implies \( a_λ \in f \).

**Proof.** Let \( f \) be a \( Q \)-fuzzy ideal of an ordered semigroup \( S \) and \( a ∈ S, q ∈ Q \). Then \( f(a, q) ≥ f(a^2, q) \). If \( a_λ^2 \in f, \lambda ∈ (0, 1] \), then \( f(a^2, q) ≥ \lambda \), and so \( f(a, q) ≥ \lambda \) which implies \( a_λ \in f \).

Conversely, let \( a \) be any element of \( S \). Put \( \lambda = f(a^2, q) \). If \( \lambda ∈ (0, 1] \), since \( a_λ^2 \in f \), then, by hypothesis, we have \( a_λ \in f \). Which implies \( f(a, q) ≥ \lambda = f(a^2, q) \). This completes the proof.

**Proposition 3.5** If \( f \) is a completely semiprime \( Q \)-fuzzy ideal of an ordered semigroup \( S \), then \( f(ab, q) = f(ba, q) \) for all \( a, b ∈ S \) and for all \( q ∈ Q \).

**Proof.** Suppose that \( f \) is a completely semiprime \( Q \)-fuzzy ideal of an ordered semigroup \( S \) and \( ∀a, b ∈ S, ∀q ∈ Q \). Then, by Theorem 3.3, we have
\[
f(ab, q) = f((ab)^2, q) = f(abab, q) ≥ f(ba, q).
\]
Similarly, \( f(ba, q) ≥ f(ab, q) \). It thus follows that \( f(ab, q) = f(ba, q) \).

**Theorem 3.6** Let \( f : S → T \) be a homomorphism of ordered semigroups and \( μ \) a completely semiprime \( Q \)-fuzzy ideal of \( T \). Then \( f^{-1}(μ) \) is a completely semiprime \( Q \)-fuzzy ideal of \( S \).
Proof. First we show that $f^{-1}(\mu)$ is a $Q$-fuzzy ideal of ordered semigroup $S$. Indeed: Let $x, y \in S, q \in Q$ and $x \leq y$. Then, since $f$ is a homomorphism of ordered semigroups from $S$ to $T$, we have $f(x) \leq f(y)$. Since $\mu$ is a $Q$-fuzzy ideal of ordered semigroup $T$, and so $\mu(f(x), q) \geq \mu(f(y), q)$, i.e., $f^{-1}(\mu)(x, q) \geq f^{-1}(\mu)(x, q)$. Furthermore, for any $x, y \in S, q \in Q$, we have

$$f^{-1}(\mu)(xy, q) = \mu[f(xy), q] \geq \mu(f(x), q) \vee \mu(f(y), q) = f^{-1}(\mu)(x, q) \vee f^{-1}(\mu)(y, q).$$

Moreover, $f^{-1}(\mu)$ is completely semiprime. Indeed: For any $a \in S, q \in Q$, we have

$$f^{-1}(\mu)(a^2, q) = \mu[f(a^2), q] = [\mu[f(a)]^2, q] = \mu[f(a), q] = f^{-1}(\mu)(a, q).$$

Therefore, by Theorem 3.3, $f^{-1}(\mu)$ is a completely semiprime $Q$-fuzzy ideal of $S$.

Lemma 3.7 Let $S$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then $A$ is a left ideal (resp. right) ideal of $S$ if and only if the characteristic mapping $f_A$ of $A$ is a $Q$-fuzzy left (resp. right) ideal of $S$.

An ordered semigroup $(S; \cdot, \leq)$ is called intra-regular if, for each element $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$.

Proposition 3.8 An ordered semigroup $S$ is intra-regular if and only if $(\forall a \in S, \forall q \in Q)f(a, q) = f(a^2, q)$, for every $Q$-fuzzy ideal $f$ of $S$.

Proof. $\Rightarrow$. Let $f$ be a $Q$-fuzzy ideal of $S$ and $a \in S, q \in Q$. Then, by hypothesis, there exist $x, y \in S$ such that $a \leq xa^2y$, and

$$f(a, q) \geq f(xa^2y, q) \geq f(a^2y, q) \geq f(a^2, q) \geq f(a, q),$$

which implies that $f(a, q) = f(a^2, q)$.

$\Leftarrow$. By Lemma 3.7, $f_{I(a^2)}$ is a $Q$-fuzzy ideal of $S$. By hypothesis, we have $f_{I(a^2)}(a, q) = f_{I(a^2)}(a^2, q) = 1$, so $a \in I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S)$. Thus $a \leq t$ for some $t \in a^2 \cup Sa^2 \cup a^2S \cup Sa^2S$. If $t = a^2$, then $a \leq a^2 \leq a^4 \in Sa^2S$, that is $a \in (Sa^2S)$. If $t = xa^2$ for some $x \in S$, then $a \leq xa^2 \leq x(xa^2)a = x^2a^2a \in Sa^2S$. If $t = a^2y$ for some $y \in S$, then $a \leq a^2y \leq a(a^2y)y = aa^2y^2 \in Sa^2S$. If $t \in Sa^2S$, then $a \in (Sa^2S)$. Thus $S$ is intra-regular.

Theorem 3.9 Let $S$ be an ordered semigroup. Then the following statements are equivalent:
(1) $S$ is intra-regular.
(2) $S$ is a semilattice of simple semigroups.
(3) Every ideal of $S$ is completely semiprime.
(4) Every $Q$-fuzzy ideal of $S$ is completely semiprime.

Proof. The equivalence of (1), (2) and (3) is due to Remark 2 in [8], and of (1) and (4) is due to Theorem 3.3 and Proposition 3.8.

References


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