A Note on Minimal Quasi-Ideals in Ternary Semigroups

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Abstract

A nonempty subset $Q$ of a ternary semigroup $S$ is called a quasi-ideal of $S$ if $[SSQ] \cap [SQS] \cap [QSS] \subseteq Q$ and $[SSQ] \cap [SSQSS] \cap [QSS] \subseteq Q$. A quasi-ideal of $S$ is said to be minimal if it does not properly contain any quasi-ideal of $S$. In this note, we characterize when a quasi-ideal in a ternary semigroup $S$ is a minimal quasi-ideal in $S$.

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1 Introduction

Let $S$ be a nonempty set. Then $S$ is called a ternary semigroup if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1 x_2 x_3]$, such that

$[[x_1 x_2 x_3] x_4 x_5] = [x_1 [x_2 x_3 x_4] x_5] = [x_1 x_2 [x_3 x_4 x_5]]$

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for all \(x_1, x_2, x_3, x_4, x_5 \in S\). Hereafter, let \([\ ]\) denotes the ternary operation on \(S\) if \(S\) is a ternary semigroup.

Ternary algebraic systems, called triplexes, have been introduced by Lehmer in 1932 (see [4]). This turns out to be commutative ternary groups. Ternary semigroups were first introduced by Banach who showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup (see [7]). Siomon studied ideals and radicals on ternary semigroups (see [6]). The authors in [1] investigated some properties of quasi-ideals and bi-ideals in ternary semigroups. In [2], the notion of ideal extensions of ordered ternary semigroups have been introduced.

Let \( (S, \cdot) \) be a semigroup. For \(x_1, x_2, x_3 \in S\), define a ternary operation on \(S\) by \([x_1, x_2, x_3] = x_1 \cdot x_2 \cdot x_3\). Then \(S\) is a ternary semigroup.

For nonempty subsets \(A_1, A_2, A_3\) of a ternary semigroup \(S\), let
\[
[A_1A_2A_3] = \{[x_1x_2x_3] \mid x_1 \in A_1, x_2 \in A_2, x_3 \in A_3\}.
\]
For \(x \in S\), let \(xA_1A_2 = \{\{x\}A_1A_2\}\). For any other cases can be defined analogously.

Let \(S\) be a ternary semigroup. A nonempty subset \(T\) of \(S\) is said to be a ternary subsemigroup of \(S\) if \([TTT] \subseteq T\).

Let \(S\) be a ternary semigroup and \(I\) a nonempty subset of \(S\). Then \(I\) is called a left (middle, right) ideal of \(S\) if
\[
[SSI] \subseteq I \quad ([SIS] \subseteq I, [ISS] \subseteq I)\]
If \(I\) is a left, right and middle ideal of \(S\), then \(I\) is called an ideal of \(S\). A left (middle, right) ideal of \(S\) is said to be minimal if it does not properly contain any left (middle, right) ideal of \(S\). Note that, for \(a \in S\), \([SSa], [SaS] \cup [SSaSS]\) and \([aSS]\) are left, middle and right ideals of \(S\), respectively.

A nonempty subset \(Q\) of a ternary semigroup \(S\) is called a quasi-ideal of \(S\) if the following conditions hold:

\[(i)\] \([SSQ] \cap [SQS] \cap [QSS] \subseteq Q,\]
\[(ii)\] \([SSQ] \cap [SSQSS] \cap [QSS] \subseteq Q.\]

It was proved in [1] that if \(L, M\) and \(R\) are left, middle and right ideals of \(S\) then the intersection \(L \cap M \cap R\) is a quasi-ideal of \(S\). If \(Q\) is a quasi-ideal of \(S\), then \(Q\) is a ternary subsemigroup of \(S\). Indeed,
\[
[QQQ] \subseteq [SSQ] \cap [SQS] \cap [QSS] \subseteq Q.
\]

A quasi-ideal of \(S\) is said to be minimal if it does not properly contain any quasi-ideal of \(S\).

A ternary semigroup \(S\) is called a quasi-simple ternary semigroup if \(S\) is the unique quasi-ideal of \(S\).
In this note, we characterize when a quasi-ideal in a ternary semigroup $S$ is a minimal quasi-ideal in $S$. The similar results have been done on $\Gamma$-semigroups (see [3]).

## 2 Main Results

The main result of this paper is contain in Theorem 2.1.

**Theorem 2.1** Let $S$ be a ternary semigroup. A subset $Q$ of $S$ is a minimal quasi-ideal of $S$ if and only if $Q$ is the intersection of a minimal left ideal $L$, a minimal middle ideal $M$ and a minimal right ideal $R$ of $S$.

**Proof.** Assume that a subset $Q$ of $S$ is a minimal quasi-ideal of $S$. Let $a \in Q$. We have $[SSa], [SaS] \cup [SSaSS]$ and $[aSS]$ are left, middle and right ideals of $S$, respectively. Then $[SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS]$ is a quasi-ideal of $S$. Since

$$[SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS] = ([SQS] \cap [SSQSS] \cap [QSS]) \subseteq Q$$

and $Q$ is a minimal quasi-ideal of $S$, so we obtain

$$[SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS] = Q.$$  

Next, we shall show that $[SSa]$ is a minimal left ideal of $S$. Let $L$ be a left ideal of $S$ such that $L \subseteq [SSa]$. Since

$$L \cap [SaS] \cap [aSS] \subseteq [SSa] \cap [SaS] \cap [aSS] \subseteq Q,$$

so $L \cap [SaS] \cap [SSa] = Q$ which implies that $Q \subseteq L$. Since

$$[SSa] \subseteq [SSQ] \subseteq [SSL] \subseteq L,$$

so we have $L = [SSa]$. Similarly, $[aSS]$ is a minimal right ideal of $S$. To show that $[SaS] \cup [SSaSS]$ is a minimal middle ideal of $S$, let $M$ be a middle ideal of $S$ such that $M \subseteq [SaS] \cup [SSaSS]$. Then

$$[SSa] \cap M \cap [aSS] \subseteq [SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS] \subseteq Q.$$  

This implies that $[SSa] \cap M \cap [aSS] = Q$. Then $Q \subseteq M$. Since

$$[SaS] \cup [SSaSS] \subseteq [SQS] \cup [SSQSS] \subseteq [SMS] \cup [SSMSS] \subseteq M,$$
we obtain \( M = [SaS] \cup [SSaSS] \).

Conversely, assume that \( Q \) is the intersection of a minimal left ideal \( L \), a minimal middle ideal \( M \) and a minimal right ideal \( R \) of \( S \). Then \( Q \) is a quasi-ideal of \( S \). Let \( Q' \) be a quasi-ideal of \( S \) such that \( Q' \subseteq Q \). We have \([SSQ'] \subseteq [SSQ] \subseteq [SSL] \subseteq L\),

\[
[SSQ'] \cup [SSQ'SS] \subseteq [SQS] \cup [SSQSS] \subseteq [SMS] \cup [SSMSS] \subseteq M
\]

and \([Q'SS] \subseteq [QSS] \subseteq [RSS] \subseteq R \). Since \([SSQ'], [SQS] \cup [SSQ'SS] \) and \([Q'SS] \) are left, middle and right ideals of \( S \), respectively, we get \( L = [SSQ'] \), \( M = [SQ'S] \cup [SSQ'SS] \) and \( R = [Q'SS] \). Therefore,

\[
Q = L \cap M \cap R = [SSQ'] \cap [SQ'S] \cap [Q'SS] \subseteq Q'.
\]

This proves that \( Q \) is a minimal quasi-ideal of \( S \).

Using Theorem 2.1, we have the following.

**Corollary 2.2** A ternary semigroup \( S \) contains a minimal quasi-ideal if and only if \( S \) contains a minimal left ideal, a minimal middle ideal and a minimal right ideal.

**Theorem 2.3** Let \( S \) be a ternary semigroup and \( Q \) a quasi-ideal of \( S \). The following are equivalent.

(i) \( Q \) is a minimal quasi-ideal of \( S \).

(ii) \( Q = [SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS] \) for all \( a \in Q \).

(iii) \( Q = ([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS]) \cup \{a\} \) for all \( a \in Q \).

**Proof.** (i) \(\Rightarrow\) (ii). Assume that \( Q \) is a minimal quasi-ideal of \( S \). Let \( a \in Q \). Since \([SSa], [SaS] \cup [SSaSS] \) and \([aSS] \) are left, middle and right ideals of \( S \), respectively, we obtain \([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS] \) is a quasi-ideal of \( S \). By assumption and

\[
[SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS] \subseteq [SSQ] \cap ([SQS] \cup [SSQSS]) \cap [QSS] \subseteq Q,
\]

we have that \([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS] = Q \).

That (ii) \(\Rightarrow\) (iii) is clear.

(iii) \(\Rightarrow\) (i) Assume that \( Q = ([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS]) \cup \{a\} \) for all \( a \in Q \). Let \( Q' \) be a quasi-ideal of \( S \) such that \( Q' \subseteq Q \). Let \( a \in Q' \), then \( a \in Q \). By assumption,

\[
Q = ([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS]) \cup \{a\}
\subseteq ([SSQ] \cap ([SQS] \cup [SSQSS]) \cap [Q'SS]) \cup \{a\}
\subseteq Q' \cup \{a\}
= Q'.
\]
Thus $Q$ is a minimal quasi-ideal of $S$.

Let $S$ be a ternary semigroup and $a \in S$. In [1], the following are defined.

1. The principal left ideal generated by $a$ is $(a)_L = \{a\} \cup [SSa]$.
2. The principal right ideal generated by $a$ is $(a)_R = \{a\} \cup [aSS]$.
3. The principal middle ideal generated by $a$ is $(a)_M = \{a\} \cup [SaS] \cup [SSaSS]$.
4. The principal ideal generated by $a$ is $(a) = \{a\} \cup [SSa] \cup [SaS] \cup [SSaSS] \cup [aSS]$.

The relations $L$, $R$, $M$, $J$, $H$ and $D$ are defined on $S$ by, for $a, b \in S$,

\[
\begin{align*}
    aLb & \iff (a)_L = (b)_L \\
    aRb & \iff (a)_R = (b)_R \\
    aMb & \iff (a)_M = (b)_M \\
    aJb & \iff (a) = (b)
\end{align*}
\]

$H = L \cap M \cap R$ and $D = LMR = RML$. The relations $L$, $R$, $M$, $J$ and $H$ are equivalence relations on $S$.

Define the relation $Q$ on a ternary semigroup $S$ by, for all $a, b \in S$, $aQb$ if

\[
\begin{align*}
    ([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS]) \cup \{a\} = \\
    ([SSb] \cap ([SbS] \cup [SSbSS]) \cap [bSS]) \cup \{b\}.
\end{align*}
\]

Note that $Q$ is an equivalence relation on $S$.

**Theorem 2.4** $H = Q$.

**Proof.** Let $a, b \in S$ be such that $aHb$. Then $aLb$, $aMb$ and $aRb$. Since $\{a\} \cup [SSa] = \{b\} \cup [SSb]$, $\{a\} \cup [SaS] \cup [SSaSS] = \{b\} \cup [SbS] \cup [SSbSS]$ and $\{a\} \cup [aSS] = \{b\} \cup [bSS]$, we have $aQb$.

Let $a, b \in S$ be such that $aQb$. If $a = b$, then $aHb$. Assume that $a \neq b$. Then

\[
\begin{align*}
    b \in [SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS], \quad a \in [SSb] \cap ([SbS] \cup [SSbSS]) \cap [bSS].
\end{align*}
\]

Since $b \in [SSa]$ and $a \in [SSb]$, $aLb$. Similarly, $aMb$ and $aRb$. Then $aHb$.

**Theorem 2.5** Let $S$ be a ternary semigroup and $Q$ a quasi-ideal of $S$. Then $Q$ is a minimal quasi-ideal of $S$ if and only if $Q$ is a $H$-class.
Proof. Assume that $Q$ is a minimal quasi-ideal of $S$. Let $a, b \in Q$. By Theorem 2.3,
\[
Q = ([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS]) \cup \{a\}
\]
\[= ([SSb] \cap ([SbS] \cup [SSbSS]) \cap [bSS]) \cup \{b\}.
\]
Then $aHb$ which implies $Q$ is a $H$-class.

Conversely, assume that $Q$ is a $H$-class. Then for all $a, b \in Q$, we have
\[
([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS]) \cup \{a\}
\]
\[= ([SSb] \cap ([SbS] \cup [SSbSS]) \cap [bSS]) \cup \{b\}.
\]
Let $x, y \in Q$. Then $y \in ([SSx] \cap ([SxS] \cup [SSxSS]) \cap [xSS]) \cup \{x\}$. So, $Q \subseteq ([SSx] \cap ([SxS] \cup [SSxSS]) \cap [xSS]) \cup \{x\}$. Since
\[
([SSx] \cap ([SxS] \cup [SSxSS]) \cap [xSS]) \cup \{x\}
\]
\[\subseteq ([SSQ] \cap ([SQS] \cup [SSQSS]) \cap [QSS]) \cup Q \subseteq Q
\]
it follows that $([SSa] \cap ([SaS] \cup [SSaSS]) \cap [aSS]) \cup \{a\} = Q$ for all $a \in Q$. By Theorem 2.3, $Q$ is minimal.

Using the fact that every quasi-simple ideals is minimal and Theorem 2.5, we obtain the following.

**Corollary 2.6** Let $S$ be a ternary semigroup and $Q$ a quasi-ideal of $S$. Then $Q$ is quasi-simple if and only if $Q$ is a $H$-class.

**References**


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