Hartley Transform for \( L^p \) Boehmians and Spaces of Ultradistributions

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Abstract. The Hartley transform is first extended to a space of Boehmians where its properties are established. Having all its desired properties, the extended Hartley transform of a Boehmian is well defined in the space of continuous functions satisfying continuity conditions with respect to \( \delta \) and \( \Delta \)-convergence. To the context of ultradistributions the complex Hartley transform is obtained as an adjoint operator.

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1. Introduction

The Hartley transform of a function is a spectral transform closely related to the Fourier transform [2,3]. The advantages of the Hartley transform over the Fourier transform are in the analysis of real signals as it avoids the use of complex arithmetic which result in faster digital algorithms. As the Fourier transform is continued into the complex plane, the Hartley transform can also be continued into the complex plane and further the Fourier and Hartley transforms have complementary symmetry properties with respect to their real and imaginary axis.

The Hartley transform \((Hf)(u)\) of \( f(x) \) is defined by [2, 3, 4]

\[
(Hf)(u) = \int_{-\infty}^{\infty} f(x) \cas(2\pi ux) \, dx.
\]
The kernel of the Hartley transform is often written as \[2,3\]
\[
cas(x) = \cos x + \sin x = \frac{1 - i}{2} \exp(ix) + \frac{1 + i}{2} \exp(-ix).
\]

So that
\[
\exp(i x) = \frac{1 + i}{2} \cas(x) + \frac{1 - i}{2} \cas(-x).
\]

This may be used to show that
\[
\int_{-\infty}^{\infty} \cas(2\pi ux) \cas(2\pi y) \, du = \delta(x - y),
\]
where \(\delta\) is the Dirac delta function. The inverse Hartley transform is therefore given by
\[
f(x) = \int_{-\infty}^{\infty} (Hf)(u) \cas(2\pi ux) \, du.
\]

The relationship between Fourier and Hartley transforms can be derived from the above expressions, giving
\[
(Hf)(u) = \frac{1 + i}{2} \hat{f}(u) + \frac{1 - i}{2} \hat{f}(-u),
\]
where \(\hat{f}(u)\) is the Fourier transform of \(f\).

The following are properties of Hartley transforms

(i) Linearity condition: If \(f(x)\) and \(g(x)\) are real functions then
\[
H(\alpha f(x) + \beta g(x))(u) = \alpha H(f(x))(u) + \beta H(g(x))(u).
\]

(ii) Scalling condition: If \(f(x)\) is a real function, then
\[
\int_{-\infty}^{\infty} f(\alpha x) \cas(2\pi ux) \, dx = \frac{1}{\alpha} (Hf) \left( \frac{u}{\alpha} \right).
\]

(iii) Condition of convolution: If \(f(x)\) and \(g(x)\) are real functions then
\[
H(f(x) \ast g(x))(u) = \frac{1}{2} \left[ (Hf)(u)(Hg)(u) + (Hf)(u)(Hg)(-u) + (Hf)(-u)(Hg)(u) - (Hf)(-u)(Hg)(-u) \right].
\]

Unfortunately, advantages may be offset by other disadvantages of the Hartley transform such as the above complicated convolution theorem. In addition, no relationship may exist between \((Hf)(u)\) and \((Hf)(-u)\) in sense of Hartley transforms.
(iv) Parseval relation: If \( f(x) \) and \( g(x) \) are real functions then
\[
\int_{-\infty}^{\infty} f(x) g(x) \, dx = \int_{-\infty}^{\infty} (Hf)(u) (Hg)(u) \, du.
\]

2. General Construction of Boehmian Spaces

The idea of construction of Boehmians was initiated by the concept of regular operators introduced by Boehme [1]. Regular operators form a subalgebra of the field of Mikusinski operators and they include only such functions whose support is bounded from the left.

The construction of Boehmians is similar to the construction of the field of quotients and in some cases, it is just the field of quotients. On the other hand, the construction is possible where there are zero divisors, such as space \( C \) of continuous functions with the operations of pointwise additions and convolution.

Let \( G \) be a linear space and \( S \) be a subspace of \( G \). We assume that to each pair of elements \( f \in G \) and \( \phi \in S \), is assigned the product \( f \ast \phi \) such that the following conditions are satisfied:

(i) If \( \phi, \psi \in S \), then \( \phi \ast \psi \in S \) and \( \phi \ast \psi = \psi \ast \phi \).

(ii) If \( f \in G \) and \( \phi, \psi \in S \), then \( (f \ast \phi) \ast \psi = f \ast (\phi \ast \psi) \).

(iii) If \( f, g \in G, \phi \in S \) and \( \lambda \in \mathbb{R} \), then
\[
(f + g) \ast \phi = f \ast \phi + g \ast \phi \quad \text{and} \quad \lambda (f \ast \phi) = (\lambda f) \ast \phi.
\]

Let \( \Delta \) be a family of sequences from \( S \), such that

(iv) If \( f, g \in G, (\delta_n) \in \Delta \) and \( f \ast \delta_n = g \ast \delta_n \) \( (n = 1, 2, \ldots) \), then \( f = g \).

(v) If \((\phi_n), (\delta_n) \in \Delta \), then \((\phi_n \ast \psi_n) \in \Delta \).

Elements of \( \Delta \) will be called delta sequences. Consider the class \( U \) of pair of sequences defined by
\[
U = \left\{ ((f_n), (\phi_n)) : (f_n) \subseteq G^N, (\phi_n) \in \Delta \right\},
\]
for each \( n \in \mathbb{N} \).

An element \((f_n), (\phi_n)) \in U \) is called a quotient of sequences, denoted by \( f_n/\phi_n \), if
\[
f_i \ast \phi_j = f_j \ast \phi_i, \forall i, j \in \mathbb{N}.
\]

Two quotients of sequences \( f_n/\phi_n \) and \( g_n/\psi_n \) are said to be equivalent, \( f_n/\phi_n \sim g_n/\psi_n \), if
\[
f_i \ast \psi_j = g_j \ast \phi_i, \forall i, j \in \mathbb{N}.
\]

The relation \( \sim \) is an equivalent relation on \( U \) and hence, splits \( U \) into equivalence classes. The equivalence class containing \( f_n/\phi_n \) is denoted by \([f_n/\phi_n]\). These equivalence classes are called Boehmians and the space of all Boehmians is denoted by \( B \).
The sum of two Boehmians and multiplication by a scalar can be defined in a natural way
\[ \left[ f_n/\phi_n \right] + \left[ g_n/\psi_n \right] = \left[ \left( (f_n * \psi_n) + (g_n * \phi_n) \right) / \phi_n * \psi_n \right] \]
and
\[ \alpha \left[ f_n/\phi_n \right] = \left[ \alpha f_n/\phi_n \right] , \alpha \in \mathbb{C} . \]
The operation * and the differentiation are defined by
\[ \left[ f_n/\phi_n \right] * \left[ g_n/\psi_n \right] = \left[ (f_n * g_n) / (\phi_n * \psi_n) \right] \]
and
\[ D^\alpha \left[ f_n/\phi_n \right] = \left[ D^\alpha f_n/\phi_n \right] . \]

Many a time, \( \mathcal{G} \) is equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product * are given by:

\((vi)\) If \( f_n \to f \) as \( n \to \infty \) in \( \mathcal{G} \) and, \( \phi \in \mathcal{S} \) is any fixed element, then
\[ f_n * \phi \to f * \phi \) in \( \mathcal{G} \) (as \( n \to \infty \)).

\((vii)\) If \( f_n \to f \) as \( n \to \infty \) in \( \mathcal{G} \) and \((\delta_n) \in \Delta \), then
\[ f_n * \delta_n \to f \text{ in } \mathcal{G} \text{ (as } n \to \infty) . \]

The operation * can be extended to \( \mathcal{B} \times \mathcal{S} \) by:

\((viii)\) If \( [f_n/\delta_n] \in \mathcal{B} \) and \( \phi \in \mathcal{S} \), then \( [f_n/\delta_n] * \phi = [(f_n * \phi) / \delta_n] \).

In \( \mathcal{B} \), two types of convergence, \( \delta \)-convergence and \( \Delta \)-convergence, are defined as follows:

\(\delta \)-convergence A sequence of Boehmians \((\beta_n)\) in \( \mathcal{B} \) is said to be \( \delta \)-convergent to a Boehmian \( \beta \) in \( \mathcal{B} \), denoted by \( \beta_n \delta \to \beta \), if there exists a delta sequence \((\delta_n)\) such that
\[ (\beta_n * \delta_n), (\beta * \delta_n) \in \mathcal{G} \text{, } \forall k, n \in \mathbb{N} , \]
and
\[ (\beta_n * \delta_k) \to (\beta * \delta_k) \text{ as } n \to \infty , \text{ in } \mathcal{G} , \text{ for every } k \in \mathbb{N} . \]

The following is equivalent for the statement of \( \delta \)-convergence:
\[ \beta_n \delta \to \beta \text{ (n → \infty) in } \mathcal{B} \text{ if and only if there is } f_{n,k}, f_k \in \mathcal{G} \text{ and } \delta_k \in \Delta \text{ such that } \beta_n = [f_{n,k}/\delta_k] , \beta = [f_k/\delta_k] \text{ and for each } k \in \mathbb{N} , \]
(2.1)
\[ f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } \mathcal{G} . \]

\(\Delta \)-convergence A sequence of Boehmians \((\beta_n)\) in \( \mathcal{B} \) is said to be \( \Delta \)-convergent to a Boehmian \( \beta \) in \( \mathcal{B} \), denoted by \( \beta_n \Delta \to \beta \), if there exists a \((\delta_n) \in \Delta \) such that \( (\beta_n - \beta) * \delta_n \in \mathcal{G} \), \( \forall n \in \mathbb{N} \), and \( (\beta_n - \beta) * \delta_n \to 0 \text{ as } n \to \infty \text{ in } \mathcal{G} \).
3. The Space of Lebesgue Integrable Boehmains

For our main results, we recall definitions and notations from [7]. Let

\[ S = \{ \phi \in D : \phi \geq 0 \text{ and } \int_{\mathbb{R}} \phi = 1 \} . \]

A sequence \((\phi_n) \in S\) is said to be a delta sequence if

\[ \text{supp} \phi_n \to 0 \text{ as } n \to \infty. \]

Equivalently, a sequence \((\phi_n) (n \to \infty)\) is a delta sequence if and only if

\((i)\) \(\int \phi_n = 1, \text{ for all } n \in \mathbb{N};\)
\((ii)\) \(\phi_n \geq 0, \text{ for all } n \in \mathbb{N};\)
\((iii)\) \(\inf \{ \varepsilon > 0 : \text{supp } \phi \subseteq (-\varepsilon, \varepsilon) \} \to 0 \text{ as } n \to \infty.\)

The set of all delta sequences is denoted by \(\Delta\).

If \(f \in L^p\) and \(\phi \in S\), the convolution product of \(f\) and \(\phi\) is given by

\[ (f * \phi)(y) = \int_{\mathbb{R}} f(y - x) \phi(x) \, dx, \]

where the integral exists by the Hölder’s inequality.

With the same delta sequence and same set \(S\) of functions, the following are established in [7] to construct the space \(\mathcal{B}_{L^p}\) of Lebesgue integrable Boehmians which is, in turn, employed in extending the Hilbert transform to Boehmian spaces.

Lemma 3.1 If \(f \in L^p\) and \(\phi \in S\) then \(f * \phi \in L^p\).

Proof see [7, Lemma 2.1].

Lemma 3.2 Let \(\delta - \lim f_n = f\) and \(\delta \in \Delta\), then \(\delta - \lim f_n * \delta = f * \delta\), in \(L^p\).

Lemma 3.3 If \(\delta - \lim f_n = f\) and \((\delta_n) \in \Delta\), then \(\delta - \lim f_n * \delta_n = f\), in \(L^p\). For proofs and further discussion, see [7].

4. The Extended Hartley Transform of Lebesgue Boehmians

Let \(f_1\) and \(f_2\) be in \(L^p\). Since the Hartley transform is a symmetrical representation of the Fourier transform, the convergence condition for the Hartley transform satisfies for \(L^p\)–spaces. The Hartley transform of the convolution product of \(f_1\) and \(f_2\) is defined by

\[ H(f_1 * f_2)(u) = \frac{1}{2} \left[ (Hf_1)(u) (Hf_2)(u) + (Hf_1)(u) (Hf_2)(-u) + \right. \]
\[ \left. (Hf_1)(-u) (Hf_2)(u) - (Hf_1)(-u) (Hf_2)(-u) \right]. \]

In particular, if \(f \in L^p\) and \(\phi \in S\), then

\[ H(f * \phi)(u) = \frac{1}{2} \left[ (Hf)(u) (H\phi)(u) + (Hf)(u) (H\phi)(-u) + \right. \]
\[ \left. (Hf)(-u) (H\phi)(u) - (Hf)(-u) (H\phi)(-u) \right], \]

in the space \(C\) of continuous functions.
Now, if \((\delta_n)\) is a delta sequence then \((H\delta_n)(u)\) and \((H\delta_n)(-u)\) converges uniformly to the constant function 1 on compact subsets of \(\mathbb{R}\). Invoking this fact in Equation (4.1) and considering the limit on compact sets yield

\[
H(f_n \ast \delta_n)(u) = \lim Hf_n
\]

in the space \(C\) of continuous functions.

Hence, the generalized Hartley transform of a Lebesgue Boehmian \([f_n/\delta_n]\) in \(B_{L^p}\) is a continuous function which we define as

\[
\hat{H}[f_n/\delta_n] = \lim Hf_n
\]

when the limit ranges over compact subsets of \(\mathbb{R}\).

**Theorem 4.1.** The definition in \((4.2)\) is well-defined.

**Proof** Let \([f_n/\delta_n] = [g_n/\psi_n]\) in the sense of \(B_{L^p}\). Using the concept of equivalence classes in \(B_{L^p}\) leads to

\[
f_n \ast \psi_m = g_m \ast \delta_n = g_n \ast \delta_m.
\]

Applying the Hartley transform on both sides yields

\[
Hf_n = Hg_n,
\]

in the sense of \(\delta\). Hence,

\[
\hat{H}[f_n/\delta_n] = \hat{H}[g_n/\psi_n].
\]

This asserts the theorem.

**Theorem 4.2.** The generalized Hartley transform \(\hat{H} : B_{L^p} \rightarrow C\) is Linear.

**Proof** Let \(\beta_1, \beta_2 \in B_{L^p}\) be such that \(\beta_1 = [f_n/\delta_n]\) and \(\beta_2 = [g_n/\psi_n]\), then

\[
\beta_1 + \beta_2 = \left[\left((f_n \ast \psi_n) + (g_n \ast \phi_n)\right)/\phi_n \ast \psi_n\right].
\]

Hence,

\[
\hat{H}(\beta_1 + \beta_2) = \lim H(f_n \ast \psi_n) + \lim H(g_n \ast \phi_n) = \lim Hf_n + \lim Hg_n = \hat{H}\beta_1 + \hat{H}\beta_2,\text{ on compact sets.}
\]

Also, \(\hat{H}(\alpha \beta) = \hat{H}[\alpha h_n/\delta_n] = \alpha \lim Hh_n = \alpha \hat{H}\beta\), where \(\alpha \in C, \beta = [h_n/\delta_n]\). Hence the theorem.

**Lemma 4.3** If \(\beta = [h_n/\delta_n]\) in \(B_{L^p}\) and \(\hat{H}\beta = 0\) in \(C\), then \(\beta = 0\) in \(B_{L^p}\).

**Proof** Assume \(\beta = [h_n/\delta_n]\) and \(\hat{H}\beta = 0\), then using definition of \(\hat{H}\), we get \(\lim Hh_n = 0\) on compact sets. With aid of the definition of Hartley transforms, \(h_n \rightarrow 0\) almost everywhere in \(L^p\) and \(h_n/\delta_n\) is a zero quotient of functions or, equivalently, \(\beta = [h_n/\delta_n]\) is the zero equivalence class in \(B_{L^p}\). This completes the proof of the theorem.

**Theorem 4.4** The generalized Hartley transform \(\hat{H}\) is a one-one mapping from \(B_{L^p}\) into the space \(C\) of continuous functions.

**Proof.** Let \(\beta_1 = \beta_2, \beta_1, \beta_2 \in B_{L^p}\), then using Theorem 4.2 we find \(\hat{H}(\beta_1 - \beta_2) = 0\) in \(C\). By virtue of Lemma 4.3 we get \(\beta_1 = \beta_2\). This proves the theorem.
Theorem 4.5. The generalized Hartley transform $\hat{H} : B_{L^p} \to C$ is continuous with respect to the $\delta$-convergence.

Proof. Let $\beta_n \xrightarrow{\delta} \beta (n \to \infty)$ in $B_{L^p}$. (2.1) shows that there is $f_{n,k}, f_k \in L^p$ and $\delta_k \in \Delta$ such that $\beta_n = [f_{n,k}/\delta_k]$, $\beta = [f_k/\delta_k]$ and for each $k \in \mathbb{N}$,

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } L^p.$$ 

Continuity condition of the Hartley transform justifies that $H(f_{n,k}) \to H(f_k)$ as $n \to \infty$ in $C$. Hence $\hat{H}[f_{n,k}/\delta_k] \xrightarrow{\delta} \hat{H}[f_k/\delta_k]$ in $C$ as $\to \infty$. This completes the proof of the theorem.

Lemma 4.6 If $\beta = [f_k/\delta_k] \in B_{L^p}$ and $\phi \in S$ then

$$\hat{H}(\beta * \phi) = \frac{1}{2} \left[ \hat{H} \beta (H\phi) (u) + \hat{H} \beta (H\phi) (-u) \right],$$

where $\hat{H} \beta = \lim (Hf_k)(-u)$.

Proof of this theorem follows from (4.1). Detailed proof, thus, avoided.

Theorem 4.7 The generalized Hartley transform $\hat{H} : B_{L^p} \to C$ is continuous with respect to the $\Delta$-convergence.

Proof. Let $\beta_n \xrightarrow{\Delta} \beta (n \to \infty)$ in $B_{L^p}$. Then, the $\Delta$-convergence implies $(\beta_n - \beta) \ast \delta_n = [f_n \ast \delta_n/\delta_n] \to 0$ for some $f_n \in L^p, \delta_n \in \Delta$ and $f_n \to 0$ as $n \to \infty$. Now,

$$\hat{H}((\beta_n - \beta) \ast \delta_n) = \hat{H} [f_n \ast \delta_n/\delta_n] = \lim \hat{H} (f_n \ast \delta_n) = \lim \frac{1}{2} \left[ (Hf_n)(u)(H\delta_n)(u) + (Hf_n)(u)(H\delta_n)(-u) + (Hf_n)(-u)(H\delta_n)(u) - (Hf_n)(-u)(H\delta_n)(-u) \right] = \lim \hat{H} (f_n) \to 0, (4.3)$$

as $n \to \infty$, in $C$, since $f_n \to 0$ and $(H\delta_n)(u) = (H\delta_n)(-u) \to 0$ as $n \to \infty$. Applying Theorem 4.6 for lefthand side of (4.3), we get $\hat{H}(\beta_n - \beta) \to 0$ as $n \to \infty$.

Therefore, from Theorem 4.2 we obtain $\hat{H}(\beta_n) - \hat{H}(\beta) \to 0$ (as $n \to \infty$). It follows that,

$$\hat{H}(\beta_n) \to \hat{H}(\beta) \text{ as } n \to \infty.$$ 

The proof of the theorem is therefore completed.

Theorem 4.8 (Convolution Theorem). Let $[f_n/\phi_n], [g_n/\psi_n] \in B_{L^p}$, then we have

$$\hat{H}([f_n/\phi_n] \ast [g_n/\psi_n]) = \frac{1}{2} \left[ \hat{H} [f_n/\phi_n] \hat{H} [g_n/\psi_n] + \hat{H} [f_n/\phi_n] \hat{H} [g_n/\psi_n] \right],$$

where $\hat{H} [g_n/\psi_n] = \lim (Hg_n)(-u)$. 

Proof. With aid of Theorem 4.1, we have
\[
\hat{H} \left[ \frac{f_n}{\phi_n} \ast \frac{g_n}{\psi_n} \right] = \lim_{n \to \infty} H \left( \frac{f_n}{\phi_n} \ast \frac{g_n}{\psi_n} \right) \\
= \frac{1}{2} \lim_{n \to \infty} \left[ \left( \hat{H}f_n \right) \left( u \right) \left( \hat{H}g_n \right) \left( u \right) + \left( \hat{H}f_n \right) \left( -u \right) \left( \hat{H}g_n \right) \left( -u \right) \right] \\
= \frac{1}{2} \left[ \hat{H} \left[ \frac{f_n}{\phi_n} \right] \hat{H} \left[ \frac{g_n}{\psi_n} \right] + \hat{H} \left[ \frac{f_n}{\phi_n} \right] \hat{H} \left[ \frac{g_n}{\psi_n} \right] - \hat{H} \left[ \frac{f_n}{\phi_n} \right] \hat{H} \left[ \frac{g_n}{\psi_n} \right] \right],
\]
where \( \hat{H} \) has the usual meaning. This completes the proof of the theorem.

5. HARTLEY TRANSFORM FOR ULTRADISTRIBUTION SPACES

The Hartley transform of \( f(x) \) is continued to the complex plane as [6]
\[
(Hf)(w) = \int_{-\infty}^{\infty} f(x) \cos(2\pi wx) \, dx,
\]
where \( w = u + iv \) is a complex number.

Let \( Z \) be the set of entire functions \( \psi \) of a complex variable \( w \) such that for some constants \( a \) and \( c_k \) we have
\[
|w|^k |\psi(w)| \leq c_k e^{av},
\]
where \( v = \text{Im} w, k \) is a nonnegative integer. Let \( Z' \) be the dual space of all continuous linear functionals on \( Z \). Its elements are called ultradistributions. The space \( D' \) (The Schwartz space of test functions of bounded support) is dense in \( Z \) and \( D \cap Z = \emptyset \), see [8].

Let \( \phi \) be in \( D \). We show that the Hartley transform of \( \phi \) traverses the space \( Z \) of test functions and the Hartley transform of a function in \( Z \) is a function in \( D \). The Hartley transform of a distribution is therefore a kind of continuous linear functionals in the space \( Z' \) of ultradistributions. Making use of the relation between Hartley and Fourier transforms and their application to even and odd functions we denote by \( Z_e(Z_o) \) the subsets of entire even (odd) functions \( \psi \) from \( Z \) and \( Z_e'(Z_o') \) their respective strong duals. Similarly, \( D_e(D_o) \) and \( D_e'(D_o') \) are defined in analogous manner.

The Hartley transform of a function \( f \) can be written as the sum of its even and odd parts \( (H_e f)(w) \) and \( (H_o f)(w) \), respectively. That is
\[
(Hf)(w) = (H_e f)(w) + (H_o f)(w),
\]
where
\[
(H_e f)(w) = \frac{1}{2} [(Hf)(w) + (Hf)(-w)]
\]
and
\[
(H_o f)(w) = \frac{1}{2} [(Hf)(w) - (Hf)(-w)].
\]
It is interesting to know that $H_e$ and $H_o$ of a signal $f$ are related by a Hilbert transform pair,

$$(H_0 f)(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(H_e f)(\xi)}{w - \xi} d\xi.$$  

We have the following:

**Theorem 5.1** (i) If $f \in D_e$, then $(H_0 f)(w) \in Z_e$.

(ii) If $(f_n)$ is a convergent sequence in $D_e$, then $(H_0 f)(w)$ is a convergent sequence in $Z_e$.

**Proof:** Let $f \in D_e$ such that $\text{supp} f \subset \{ t : |t| \leq a \}$ then

$$(H_e f)(w) = \frac{1}{2} [(H f)(w) + (H)(-w)]$$

$$\leq \frac{1}{2} \left[ \int_{-a}^{a} f(t) \text{cas}(wt) \, dt + \int_{-a}^{a} f(t) \text{cas}(-wt) \, dt \right].$$

Hence, equating even and odd parts

$$|(H_e f)(w)| \leq \int_{-a}^{a} |f(t)| |e^{wt}| \, dt$$

$$\leq ce^{a|v|},$$

where $v = \text{Im} w$ and $c = \int_{-a}^{a} |f(t)| \, dt$. This proves Part (i) of the theorem.

Now, let $f_n \to f$ in $D_e$ such that $|\text{supp} f_n| \leq a$, for some $a$. Since $D_e$ is closed under convergence, $f \in D_e$.

Therefore both of $(H_e f_n)(w)$ and $(H_0 f)(w)$ are in $Z_e$, $\forall n \in \mathbb{N}$, and

$$(5.1) \quad |w^k| |(H_0 f_n)(w) - (H_0 f)(w)| \leq C_p e^{a|v|},$$

for some constant $C_k$. The theorem is thus completely proved.

**Theorem 5.2** (i) If $f \in D_o$, then $(H_0 f)(w) \in Z_o$.

(ii) If $(f_n)$ is a convergent sequence in $D_o$, then $(H_0 f)(w)$ is a convergent sequence in $Z_o$.

**Proof** of above theorem is similar to that of the previous theorem and thus we avoid the same.

In view of Theorem 5.1 and 5.2 we have the following.

**Theorem 5.3.** Let $T \in \tilde{Z}_e \left( \tilde{Z}_o \right)$ then the ultradistributional Hartley transforms $\tilde{H}_e$ and $\tilde{H}_o$ of $T$ is defined as

$$(5.2) \quad \langle \tilde{H}_e T, f \rangle = \langle T, H_e f \rangle, \forall f \in D_e.$$
\[ \left\langle \tilde{\mathbf{H}}_o T, f \right\rangle = \left\langle T, \mathbf{H}_o f \right\rangle, \forall f \in D_o. \]

Definitions presented in Eqn. (5.2) and Eqn. (5.3) are respectively well-defined by Theorem 5.1 and 5.2. Moreover, it is easy to observe that the ultradistributional Hartley transforms \( \tilde{\mathbf{H}}_e \) and \( \tilde{\mathbf{H}}_o \) of an ultradistribution \( T \) in \( Z_e \) and \( Z_o \) are distributions in \( D'_e \) and \( D'_o \), respectively.

**Theorem 5.4** The ultradistributional Hartley transforms \( \tilde{\mathbf{H}}_e \) and \( \tilde{\mathbf{H}}_o \) are linear.

**Proof** This theorem follows from the linearity of Hartley transform.

Denote by \( \mathbb{N}_e = \{2, 4, 6, 8, \ldots\} \) and \( \mathbb{N}_o = \{1, 3, 5, 7, \ldots\} \) then if \( f \in L^1 \), then its Fourier cosine and sine transforms \( F_c(F_s) \) exist, and

\[ F_c \left[ f^{(m)}(x), \xi \right] = \begin{cases} -\xi^m & F_c \left[ f(x), \xi \right], m \in \mathbb{N}_e. \\ \xi^m & F_s \left[ f(x), \xi \right], m \in \mathbb{N}_o. \end{cases} \]

(5.4)

\[ F_s \left[ f^{(m)}(x), \xi \right] = \begin{cases} \xi^m & F_s \left[ f(x), \xi \right], m \in \mathbb{N}_e. \\ -\xi^m & F_c \left[ f(x), \xi \right], m \in \mathbb{N}_o. \end{cases} \]

(5.5)

If \( x^m f \in L^1 \) and \( f \in L^1 \), then the Fourier cosine and sine transforms \( \hat{f_c} \) and \( \hat{f_s} \) are \( m \)-times differentiable and

\[ \hat{f_c}^{(m)}(\xi) = \begin{cases} (1 - \frac{m+1}{2})^{m/2} & F_s \left[ x^m f(x), \xi \right], m \in \mathbb{N}_o. \\ (1 - m/2)^{m/2} & F_c \left[ x^m f(x), \xi \right], m \in \mathbb{N}_e. \end{cases} \]

(5.6)

\[ \hat{f_s}^{(m)}(\xi) = \begin{cases} (1 - \frac{m+1}{2})^{m/2} & F_c \left[ x^m f(x), \xi \right], m \in \mathbb{N}_o. \\ (1 - m/2)^{m/2} & F_s \left[ x^m f(x), \xi \right], m \in \mathbb{N}_e. \end{cases} \]

(5.7)

From (5.4) - (5.7), the following are a straightforward corollaries

(i) If \( \phi \in D_e(Z_o) \), then

\[ \left( \mathbf{H}_e \phi^{(m)} \right)(w) = -w^m \left( \mathbf{H}_e \phi \right)(w), m \in \mathbb{N}_e. \]

(5.8)

and

\[ \frac{d^m}{dw^m} \left( \mathbf{H}_e \phi \right)(w) = (-1)^{m/2} \left( \mathbf{H}_e \left( x^m \phi(x) \right) \right)(w), m \in \mathbb{N}_e. \]

(5.9)

(ii) If \( \phi \in D_o(Z_o) \), then

\[ \left( \mathbf{H}_o \phi^{(m)} \right)(w) = -w^m \left( \mathbf{H}_o \phi \right)(w), m \in \mathbb{N}_e. \]

(5.10)

and

\[ \frac{d^m}{dw^m} \left( \mathbf{H}_o \phi \right)(w) = (-1)^{m/2} \left( \mathbf{H}_o \left( x^m \phi(x) \right) \right)(w), m \in \mathbb{N}_e. \]

(5.11)

**Theorem 5.5** (i) If \( \phi \in D_e \) with \( \text{supp} \phi \subset [-a, a] \) then

\[ \frac{d^m}{dw^m} \left( \mathbf{H}_e \phi \right)(w) \in Z_e, m \in \mathbb{N}_e. \]

(5.12)
(ii) If $\phi \in D$ with supp $\phi \subset [-a, a]$ then
\[
\frac{d^m}{dw^m} (H \phi) (w) \in Z, m \in \mathbb{N}.
\]

**Proof** Let $\phi \in D$ with supp $\phi \subset [-a, a]$ and $k$ be a nonnegative integer then $x^m \phi (x)$ is uniformly bounded with supp $x^m \phi \subset [-a, a]$ . Also, using (5.8)-(5.11) we get
\[
|w|^k \left| \frac{d^m}{dw^m} (H \phi) (w) \right| \leq |w|^k \left| (H \phi) (w) \right|,
\]
where $v = \text{Im} \ w, A_{k,m} = \int_{-a}^{a} |\Phi(k)(x)| \ dx$.

This proves Part (i). Part(ii) can be proved similarly.

**References**


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