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Some Generalized Forms of Compactness

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Abstract

The aim of this paper is to continue the study of a property $P_{\alpha\beta\gamma}$ of three variables which generalizes the notions of compactness, paracompactness, closeness and many of their corresponding weaker forms. This property enables us to study the former defined notions and connections between them. Moreover, it helps to construct many types of these forms. The notion bg-compactness, as a new case of the several cases of $P_{\alpha\beta\gamma}$, is introduced and relations between bg-compactness and some types of compactness are discussed. The images of this sort of spaces under some non-continuous mappings are investigated. We define new notions of filters and nets to give more properties of our compact spaces.

Mathematics Subject Classification: Primary 54C10, 54D05; Secondary 54D30, 54D10

Keywords: compactness, paracompactness, closeness, α-open set, feebly open set, β-open set, weakly open set, bg-open set, bg- compactness, generalized open sets, generalized closed sets

1. Introduction

The notion of compactness is useful and fundamental notion of not only general topology but also of other advanced branches of mathematics.

Since 1906, when Frechet used for the first time the term compact, many sorts of compactness were introduced by different topologists. Asha Mathur [31] described compactness and its weaker forms through a table containing 72 properties. M. E. Abd El-Monsef and A. M. Kozae [2] introduced a property $P_{\alpha\beta\delta}$ for generalizing 1920 types of compactness and closeness. In this paper, we shall use the property $P_{\alpha\beta\delta}$ for generalizing 15456 types of compactness and closeness.

2. Preliminary Notes

Throughout this paper (X, τ) and (Y, σ) are topological spaces with no separation axioms assumed unless otherwise stated. Let A be a subset of X, the interior of A and the closure of A will be denoted by Int(A) and Cl(A), respectively.

In this section we shall recall some basic definitions of some types of near open sets and discuss the relations among these types. So we shall show that the two definitions β-open set [1] and weakly open set [16] are equivalent. Also, we shall introduce the properties ig-open set, gi-open set and ig*-open set, where $i \in$ {clo, co, regular, θ, δ, α, pre, semi, b, β, simply}, for generalizing 33 types of near g-open sets and study the relations among these types.

Definition 2.1: A subset A of a space X is called

1. Clo-open [43] if A is open and closed.
2. Co-open [29] if A and Cl(A) are open.
3. Regular open [43] if A= Int(Cl(A)).
4. θ-open [5], [45] if every point of A has an open neighbourhood whose closure is contained in A.
5. δ-open [45] if every point of A has an open neighbourhood whose interior closure is contained in A.
6. α-open [35] if $A \subseteq Int(Cl(Int(A)))$.
7. Pre-open [28] if $A \subseteq Int(Cl(A))$.
8. Semi-open [23] if $A \subseteq Cl(Int(A))$.
9. B-open [6] if $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$.

10. β-open [1] if $A \subseteq Cl(Int(Cl(A)))$.
11. Simply open (briefly smp-open) [34] if $Int(Cl(A)) \subseteq Cl(Int(A))$.

The complement of a clo-open (respectively co-open, regular open, θ-open, δ-open, α-open, pre-open, semi-open, b-open, β-open, simply open) set is called clo-closed (respectively co-closed, regular closed, θ-closed, δ-closed, α-closed, pre-closed, semi-closed, b-closed, β-closed, simply closed).

The largest clo-open (respectively co-open, regular open, θ-open, δ-open, α-open, pre-open, semi-open, b-open, β-open, simply open) set contained in A is called clo-interior (respectively co-interior, regular interior, θ-interior, δ-interior, α-interior, pre-interior, semi-interior, b-interior, β-interior, simply interior) of A and shall be denoted by $Int_{clo}(A)$ (respectively $Int_{co}(A)$, $Int_r(A)$, $Int_\theta(A)$, $Int_\delta(A)$, $Int_\alpha(A)$, $Int_p(A)$, $Int_s(A)$, $Int_b(A)$, $Int_\beta(A)$, $Int_{smp}(A)$).

The smallest clo-closed (respectively co-closed, regular closed, θ-closed, δ-closed, α-closed, pre-closed, semi-closed, b-closed, β-closed, simply closed) containing A is called clo-closure (respectively co-closure, regular closure, θ-closure, δ-closure, α-closure, pre-closure, semi-closure, b-closure, β-closure, simply closure) of A and shall be denoted by $Cl_{clo}(A)$ (respectively $Cl_{co}(A)$, $Cl_r(A)$, $Cl_\theta(A)$, $Cl_\delta(A)$, $Cl_\alpha(A)$, $Cl_p(A)$, $Cl_s(A)$, $Cl_b(A)$, $Cl_\beta(A)$, $Cl_{smp}(A)$).

It's easily to show that the complement of a clo-open subset of a space X is clo-open too.

Definition 2.2: A subset A of a space X is called
1. Generalized closed set (briefly g-closed) [22] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
2. Feebly open set [25] if there exists an open set U such that $U \subseteq A \subseteq Cl_s(U)$.
3. Weakly open set [16] if there exists an open set U such that $Cl(A)= Cl(U)$.

The complement of a generalized closed set in a space X is called generalized open set and denoted by g-open. So we can show that a subset A of a space X is g-open if $F \subseteq Int(A)$ whenever $F \subseteq A$ and F is closed set.

Jankovic and Reilly [20] proved the equivalence of the two notions α-open set and feebly open set in a space X. In the following, we shall give another proof of the equivalence of α-open set with feebly open set, and so we indicate the equivalence of β-open set with weakly open set.

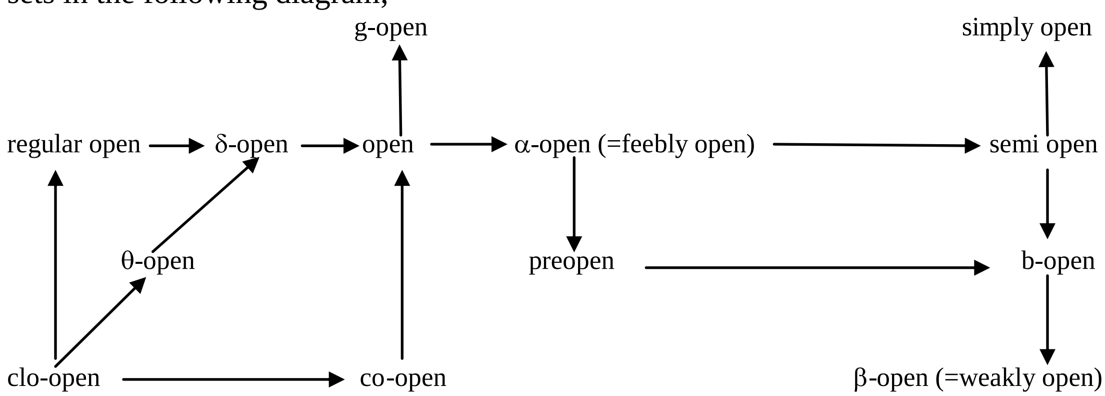
Lemma 2.3: If A is an open subset of a space X, then $Cl_s(A)= Int(Cl(A))$.
Proof: Since $Cl_s(A)= A \cup Int(Cl(A))$ [6], the proof is over.

Proposition 2.4 [20]: A subset A of a space X is α-open if and only if it is a feebly open.
Proof: Lemma 2.3 is applicable.

Proposition 2.5: A subset A of a space X is β-open if and only if it is weakly open.
Proof: Necessity, let A be a β-open subset of a space X. Implies, Cl(A)= Cl(Int(Cl(A))). Hence, A is weakly open.

Sufficiency, let A be a weakly open subset of a space X. Then there exists an open set U such that Cl(A)= Cl(U). Implies, Cl(Int(Cl(A)))= Cl(A). Hence A is β-open.

Now, we shall summarize the relationships between various types of near open sets in the following diagram;



The above implications are not reversed, in general.

In the following definition we shall introduce forms for generalizing 33 types of near g-closed sets and study the relations between these types.

Definition 2.6: A subset A of a space X is called

1. i-generalized closed set (briefly ig-closed) if $Cl_i(A) \subseteq U$ whenever $A \subseteq U$ and U is open set. The complement of ig-closed set is called ig-open.
2. generalized i-closed set (briefly gi-closed) if $Cl_i(A) \subseteq U$ whenever $A \subseteq U$ and U is i-open set. The complement of gi-closed set is called gi-open.
3. i-generalized star closed set (briefly ig*-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is i-open set. The complement of ig*-closed set is called ig*-open.

Where $i \in \{clo, co, r, \theta, \delta, \alpha, p, s, b, \beta, smp\}$.

More of these definitions are known aforetime, like sg-closed set (namely gs-closed set in [7]), gs-closed set (namely sg-closed set in [12]), sg*-closed set [32], αg-closed set [33], gα-closed set [27], αg*-closed set[32], pg-closed set (namely gp-closed set in [10]), gp-closed set (namely pg-closed set in [10]), pg*-closed set [32], bg-closed set (namely gb-closed set in [3]), gb-closed set [17], bg*-closed set [32], βg-closed set (namely gsp-closed set in [13]), gβ-closed set (namely spspg-closed set in [37]), βg*-closed set [32], rg*-closed set (namely rg-closed set in [37]), θg-closed set (namely gθ-closed set in [36]), gθ-closed set (namely θg-closed set in [33]) and δg-closed set [14]. Other definitions are new (to the best of our knowledge).

Definition 2.7: Let A be a subset of a space X. For i∈{clo, co, regular, θ, δ, α, pre, semi, b, β, simply},then

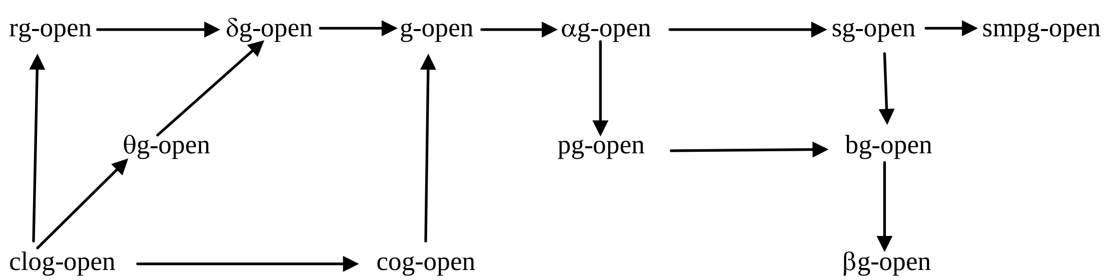
1. The largest ig-open (respectively gi-open, ig*-open) set contained in A is called the ig-interior (respectively gi-interior, ig*-interior) of A and shall be denoted by $Int_{ig}(A)$ (respectively $Int_{gi}(A)$, $Int_{ig*}(A)$).
2. The smallest ig-closed (respectively gi- closed, ig*- closed) set containing A is called the ig-closure (respectively gi-closure, ig*-closure) of A and shall be denoted by $Cl_{ig}(A)$ (respectively $Cl_{gi}(A)$, $Cl_{ig*}(A)$).

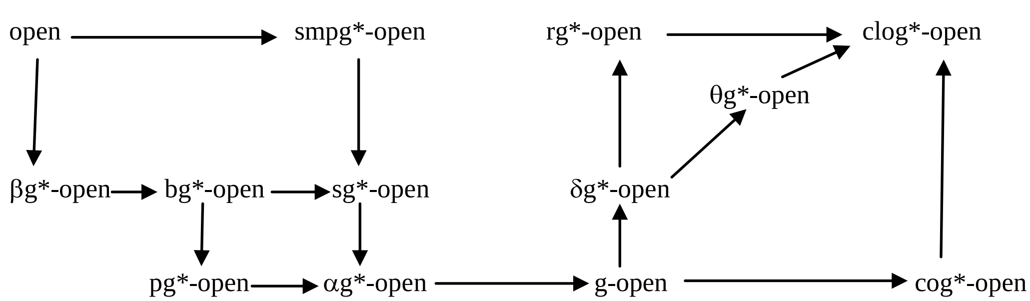
Proposition 2.8: A subset A of a space X is

1. ig-open set if $F \subseteq Int_i(A)$ whenever $F \subseteq A$ and F is closed set.
2. gi-open set if $F \subseteq Int_i(A)$ whenever $F \subseteq A$ and F is i-closed set.
3. ig*-open set if $F \subseteq Int(A)$ whenever $F \subseteq A$ and F is i-closed set.

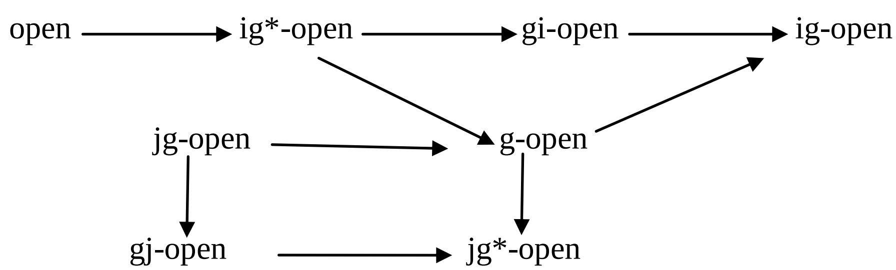
Where i∈{clo, co, r, θ, δ, α, p, s, b, β, smp}.

In the following diagrams, we shall summarize the relationships between various types of generalized open sets;





And for i∈{α, p, s, b, β, smp}, for j∈{clo, co, r, δ, θ} the following diagram holds;



3. **The Property $P_{\alpha\beta\delta}$**

Consider;

A. the following 552 types for a cover $\{U_i: i\in I\}$ of a space X,

1. Open cover 2. Clo-open cover 3. Co-open cover 4. θ-open cover
5. Regular open cover 6. δ-open cover 7. α-open cover 8. Preopen cover
9. Semi-open cover 10. B-open cover 11. β-open cover 12. Simply open cover
13. g-open cover 14. Clog-open cover 15. Cog-open cover 16. θg-open cover
17. Rg-open cover 18. δg-open cover 19. αg-open cover 20. Pg-open cover
21. Sg-open cover 22. Bg-open cover 23. βg-open cover 24. Smpg-open cover
25. gclo-open cover 26. gco-open cover 27. gθ-open cover 28. gr-open cover
29. gδ-open cover 30. gα-open cover 31. gp-open cover 32. gs-open cover
33. gb-open cover 34. gβ-open cover 35. gsmp-open cover 36. clog*-open cover
37. cog*-open cover 38. θg*-open cover 39. Rg*-open cover 40. δg*-open cover
41. αg*-open cover 42. pg*-open cover 43. sg*-open cover 44. bg*-open cover
45. βg*-open cover 46. Smpg*-open cover 47. Locally finite open cover
48. Locally finite clo-open cover 49. Locally finite co-open cover
…………………………………. 92. Locally finite smpg*-open cover
93. Point finite open cover 94. Point finite clo-open cover
95. Point finite co-open cover ……….. 138. Point finite smpg*-open cover
139. Star finite open cover 140. Star finite clo-open cover
141. Star finite co-open cover ……….. 184. Star finite smpg*-open cover

And c1 (respectively, c2, … , c184) denotes countable open cover (respectively, countable clo-open cover, …, countable star finite smpg*-open cover). Also, m1 (respectively, m2, … , m184) denotes open cover of cardinality ≤ m (respectively, clo-open cover of cardinality ≤ m, …, star finite smpg*-open cover of cardinality ≤ m).

B. the following 4 types of the subfamily $\{U_i: i\in I_0\}$, I_0 is a finite subset of I, the index set;
a. Finite subfamily.
b. Locally finite refinement.
d. Pointe finite refinement.
e. Star finite refinement.

C. the following 7 types for the union of members of the finite subfamily;
f. $X=\cup\{U_i : i\in I_0\}$.
h. $X=\cup\{Int(Cl(U_i)): i\in I_0\}$.
j. $X=\cup\{Cl(U_i) : i\in I_0\}$.
k. $X=\cup\{Cl(Int(U_i)) : i\in I_0\}$.
l. $X=\cup\{Int(Cl(Int(U_i))) : i\in I_0\}$.
q. $X=\cup\{Cl(Int(Cl(U_i))) : i\in I_0\}$.
n. $X=\cup\{ Cl(Int(U_i))\cup Int(Cl(U_i)): i\in I_0\}$.

Definition 3.1: By the property $P_{\alpha\beta\vartheta}$ we mean: given a cover $\{U_i: i\in I\}$ of type α, where $\alpha\in\{1, 2, \ldots, 184, c1, c2, \ldots, c184, m1, m2, \ldots, m184\}$, for a space X, there exists a subfamily $\{U_i: i\in I_0\}$ of type β, where $\beta\in\{a, b, d, e\}$, such that the property of type ϑ, where $\vartheta\in\{f, h, j, k, l, q, n\}$, is satisfied.

Thus, we have 15456 properties for a space X. Some of these properties are equivalent, others are distinct.

In the following, we clarify that many well known forms of compactness and closeness are special cases of $P_{\alpha\beta\vartheta}$. Moreover, we indicate the equivalence of some cases.

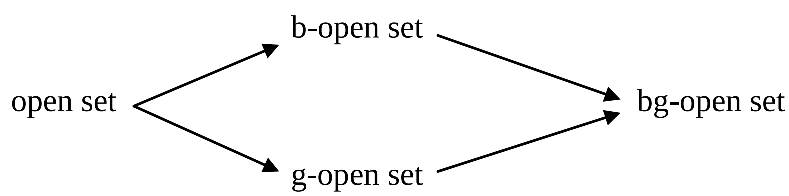
1. The property P_{1af}: Any open cover of X has a finite subcover, is nothing but compactness.
2. The property P_{1ah}: Any open cover of X has a finite subfamily, the interior of closures of whose members cover X, is the near compactness of M. K. Singal [40]. This property is equivalent to each of the following properties: P_{5af}, P_{1al}, P_{5ah}.
3. The property P_{c1ah}: Any countable open cover of X has a finite subfamily, the interior of closures of whose members cover X, is the mild compactness of M. K. Singal [41]. This property is equivalent to each of the following properties: P_{c5af}, P_{c1al}, P_{c5ah}. [2]
4. The property P_{m1ah}: Any open cover of X which cardinality less than or equal to m has a finite subfamily, the interior of closures of whose members cover X, is the m-near compactness of M. E. Abd El-Monsef [2]. This property is equivalent to each of the following properties: P_{m5af}, P_{m5ah}, P_{m5al}, P_{m1al}.
5. The property P_{8af} is the strong compactness of A. S. Mashhour [29].

6. The property P_{7af} is the α-compactness [15], [26].
7. The property P_{1bf} is the paracompactness of J. A. Dieudonn (1944).
8. The property P_{1df} is the metacompactness of R. Arns and J. Dugundji (1950).
9. The property P_{c1bf} is the countable paracompactness of C. H. Dowker (1951).
10. The property P_{1ef} is the hypocompactness [18] and strong paracompactness [42].
11. The property P_{c1ef} is the countable hypocompactness of S. T. Hu [19].
12. The property P_{c1df} is the countably metacompactness of Hayoshi [18].
13. The property P_{1bh} is called near paracompactness by M. K. Singal [39] which is equivalent to P_{1bl}.
14. The property P_{3af} is called co-compactness by A. S. Mashhour [30].
15. The property P_{3bf} is called co-paracompactness by A. S. Mashhour [30].
16. The property P_{3aj} is a almost co-compactness of A. S. Mashhour [30] which is equivalent to each of the following properties: P_{3ah}, P_{3ak}, P_{3al}, P_{3an}.
17. The property P_{c1bh} is the strongly paracopmactness of I. Kovacivic [21] which is equivalent to the property P_{c1bl}.
18. The property P_{1aj} is called H(i), (quasi H-closed), [38] which is equivalent to each of the following properties: P_{1ak}, P_{1an}.
19. The property P_{c1aj} is called light compactness [8] which is equivalent to each of the following properties: P_{c1ak}, P_{c1an}, P_{c1aq}.
20. The property P_{1aj} is equivalent to each of the following properties: P_{1ak}, P_{1aq}, P_{5aj}, P_{5ak}, P_{5aq}.
21. The property P_{c1aj} is equivalent to each of the following properties: P_{c1ak}, P_{c1aq}, P_{c5aj}, P_{c5ak}, P_{c5aq}.
22. The property P_{m1aj} is equivalent to each of the following properties: P_{m1ak}, P_{m1aq}, P_{m5aj}, P_{m5ak}, P_{m5aq}.
23. The property P_{9aj} is equivalent to each of the following properties: P_{9ak}, P_{9aq}, P_{9an}.
24. The property P_{c9aj} is equivalent to each of the following properties: P_{c9ak}, P_{c9aq}, P_{c9an}.
25. The property P_{m9aj} is equivalent to each of the following properties: P_{m9ak}, P_{m9aq}, P_{m9an}.
26. The property P_{8aj} is equivalent to the property P_{8aq}.
27. The property P_{c8aj} is equivalent to the property P_{c8aq}.
28. The property P_{m8aj} is equivalent to the property P_{m8aq}.
29. The property P_{11aj} is equivalent to the property P_{11aq}.
30. The property P_{c11aj} is equivalent to the property P_{c11aq}.
31. The property P_{m11aj} is equivalent to the property P_{m11aq}.

4. Bg-Compact Spaces

In this section we introduce the notion bg-compactness in the topological spaces. So we shall study some properties of this type of compactness. The relations between bg-compactness, b-compactness, g-compactness and compactness are discussed, also.

First, we summarize the relations between open set, b-open set, g-open set and bg-open set in a topological spaces by the following diagram;



The converses of the above implications are not hold, in general. As the following examples shows:

Example 4.1: Let X= {1, 2, 3, 4} with τ = {X, ϕ, {1}, {1, 2,3}}, then {2} is g-open set and bg-open which is neither b-open set nor open.

Example 4.2: Let X= {1, 2, 3, 4} with τ = {X, ϕ, {1}, {2}, {1, 2}, {1, 2,3}}, then {1, 2, 4} is b-open set and bg-open which is neither g-open set nor open.

Next, we give a necessary and sufficient condition for a bg-open set to be b-open.

Proposition 4.3: Let A be a bg-open subset of a space X, then the unique open set containing $Int_b(A) \cup (X\text{-}A)$ is X.

Proof: Let U be an open subset of a space X such that $Int_b(A) \cup (X\text{-}A) \subseteq U$. Implies X-U is a closed subset of A, it follows that $X\text{-}Int_b(A) \subseteq U$. Hence U= X.

Corollary 4.4: Let A be a bg-open subset of a space X. Then A is b-open if and only if $Int_b(A) \cup (X\text{-}A)$ is open.

Proof: Let A be a bg-open subset of a space X. If A is b-open, then we have $Int_b(A) \cup (X\text{-}A) = A \cup (X\text{-}A) = X$, which is open set. Conversely, let $Int_b(A) \cup (X\text{-}A)$ be open. Then, by Proposition 4.3, the unique open set containing $Int_b(A) \cup (X\text{-}A)$ is X, then $Int_b(A) \cup (X\text{-}A) = X$. Implies, $A = Int_b(A)$ and so A is b-open set.

Definition 4.5: A collection $\{A_i: i\in I\}$ of bg-open sets in a topological space X is called a bg-open cover of a subset B of X if $B \subseteq \cup\{A_i: i\in I\}$ holds.

Definition 4.6: A topological space X is called bg-compact if it has the property P_{22af}. i.e., every bg-open cover of X has a finite subcover.

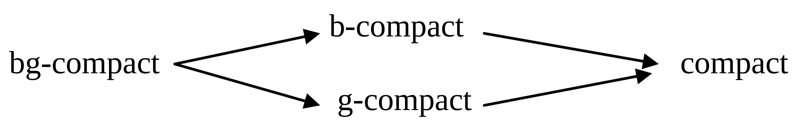
Definition 4.7: A subset B of a space X is said to be bg-compact relative to X if for every collection $\{A_i: i\in I\}$ of bg-open subset of X such that $B \subseteq \cup\{A_i: i\in I\}$ there exists a finite subset I_0 of I such that $B \subseteq \cup\{A_i: i\in I_0\}$.

Definition 4.8: A subset B of a space X is said to be bg-compact if B is bg-compact as a subspace of X.

Definition 4.9: [4] A topological space X is called b-compact if it has the property P_{10af}. i.e., every b-open cover of X has a finite subcover.

Definition 4.10: [9] A topological space X is called g-compact if it has the property P_{13af}. i.e., every g-open cover of X has a finite subcover.

Remark 4.11: The following diagram shows the relations between compactness, b-compactness, g-compactness and bg-compactness.



The converses of the above implications are not hold, in general. As the following examples shows;

Example 4.12: Let N be the set of all natural numbers, and let τ_I is indiscrete topology on N.

Evidently, N is compact space. However, it is not b-compact (respectively, not g-compact, not bg-compact) space, since $\{\{n\}: n\in N\}$ is b-open cover (respectively, g-open cover, bg-open cover) of N which has no finite subcover.

Example 4.13: Let $X= N\cup \{0\}$ with $\tau= \wp(N)\cup \{H\subseteq X: 0\in H$ and X-H is finite$\}$ be a topological space, where $\wp(N)$ is the power set of the natural numbers.

Then, any g-open cover of X must contains a g-open set A, such that $0\in A$. Implies, X-A is finite. Hence, X is g-compact space. But X is neither b-compact space nor bg-compact. Since, $(E^+\cup \{0\})\cup\{\{n\}: n\in O^+\}$ is b-open cover, also bg-open cover, of X which has no finite subcover.

Proposition 4.14: Let X be a b-compact space and for each bg-open set $A\subseteq X$, $Int_b(A) \cup (X-A)$ is open. Then X is bg-compact.
Proof: Follows from Corollary 4.4.

Proposition 4.15: A bg-closed subset of a bg-compact space is bg-compact relative to X.
Proof: Let A be a bg-closed subset of a bg-compact space X. Let $\{G_\alpha: \alpha\in I\}$ be a cover of A by bg-open sets in X. Then $C= \{G_\alpha: \alpha\in I\}\cup (X-A)$ is bg-open cover of X. Since X is bg-compact space, C is reducible to a finite subcover of X, and $X= G_{\alpha 1}\cup$

$G_{\alpha 2} \cup \ldots \cup G_{\alpha n} \cup$ (X-A). Hence, $A \subseteq G_{\alpha 1} \cup G_{\alpha 2} \cup \ldots \cup G_{\alpha n}$. Therefore A is bg-compact relative to X.

Thus every bg-closed subset of a bg-compact space is bg-compact.

Definition 4.16: [3] A function f: X→Y is said to be bg-continuous (respectively, bg-irresolute) if f^{-1}(V) is bg-closed in X for every closed (respectively, bg-closed) set V of Y.

Proposition 4.17: A function f: X→Y is said to be bg-continuous (respectively, bg-irresolute) if f^{-1}(V) is bg-open in X for every open (respectively, bg-open) set V of Y.

Proposition 4.18: A bg-continuous image of a bg-compact space is compact.
Proof: Let f: X→Y is a bg-continuous function from a bg-compact space X onto a space Y. Let $\{A_\alpha: \alpha \in I\}$ be an open cover of Y. Then $\{f^{-1}(A_\alpha): \alpha \in I\}$ is a bg-open cover of X. Since X is bg-compact, it has a finite subcover say $\{f^{-1}(A_{\alpha i})$: i= 1, 2, …, n}. Since f is onto, $\{A_{\alpha i}$: i= 1, 2, …, n} is a cover of Y, which is finite. Therefore Y is compact.

Proposition 4.19: If a map f: X→Y is bg-irresolute and a subset B of X is bg-compact relative to X, then f(B) is bg-compact relative to Y.

Corollary 4.20: A bg-irresolute image of a bg-compact space is b-compact (respectively, g-compact, compact).

Lemma 4.21: The continuous and open mapping from a space X into a space Y is bg-irresolute.
Proof: Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be a continuous and open function, let V be a bg-open subset of Y. Let F be a closed subset of f^{-1}(V) in X. Implies, f(F) is closed subset of Y, and $f(F) \subseteq f(f^{-1}(V))$= V. Then $f(F) \subseteq \sigma\text{-Int}_b(V)$, so $f^{-1}(f(F)) \subseteq f^{-1}(\sigma\text{-Int}_b(V))$. Therefore $F \subseteq \tau\text{-Int}_b(f^{-1}(V))$. Hence, f^{-1}(V) is bg-open subset of X. Thus f is bg-irresolute.

Proposition 4.22: If f is a continuous and open mapping from a space X onto a space Y and X is bg-compact space, then Y is bg-compact.
Proof: Follows from Lemma 4.21 and Proposition 4.19.

Corollary 4.23: The homeomorphic image of a bg-compact space is bg-compact.

Definition 4.24: A function f: X→Y is bg-open if f(V) is bg-open subset of Y whenever V is bg-open subset of X.

Proposition 4.25: If f: $X \to Y$ is bg-open bijective mapping and Y is bg-compact space, then X is bg-compact.
Proof: Obvious.

In what follows, we give some properties of such spaces.

Proposition 4.26: For a space X, the following statements are equivalent.
1. X is bg-compact.
2. Any family of bg-closed subsets of X satisfying the finite intersection property has a non-empty intersection.
3. Any family of bg-closed subsets of X with empty intersection has a finite subfamily with empty intersection.

Proof: 1 $\longrightarrow$ 2:

Let X be a bg-compact space and $\{F_i: i \in I\}$ be a family of bg-closed subsets of X which satisfying the finite intersection property. To prove that $\cap \{F_i: i \in I\} \neq \phi$.

Suppose the inverse, i.e., $\cap \{F_i: i \in I\} = \phi$. Then $X = \cup \{X\text{-}F_i: i \in I\}$, so $\{X\text{-}F_i: i \in I\}$ is bg-open cover of X which is bg-compact space. Implies, there exists a finite subset I_0 of I such that $X = \cup \{X\text{-}F_i: i \in I_0\}$ and so $\cap \{F_i: i \in I\} = \phi$, which is a contradiction. Thus, $\cap \{F_i: i \in I\} \neq \phi$.

2 $\longrightarrow$ 1:

Suppose 2 hold and X is not bg-compact space, then there exists a bg-open cover $\{V_i: i \in I\}$ of X has no finite subcover. Thus, for any finite subset I_0 of I, we have $X \neq \cup \{V_i: i \in I_0\}$. So, $\cap\{(X\text{-}V_i): i \in I_0\} \neq \phi$. Therefore the family $\{(X\text{-}V_i): i \in I\}$ satisfies the finite intersection property, then it has a non-empty intersection, i.e., $\cap\{(X\text{-}V_i): i \in I\} \neq \phi$. Implies, $X \neq \cup \{V_i: i \in I\}$, which is a contradiction. Hence, X is bg-compact space.

2 $\longleftrightarrow$ 3: Obvious.

The notions of a filter and a net play an important role in all compact spaces. Therefore, we introduce the following notions which will be used in this paper to give some characterizations of bg-compact spaces in terms nets and filter bases.

Definition 4.27: Let A be a subset of a space X. A point $x \in X$ is said to be bg-limit point of A if for each bg-open set U containing x, we have $(U\text{-} \{x\}) \cap A \neq \phi$. The set of all bg-limit points of A is called a bg-derived set of A and denoted by $D_{bg}(A)$.

Definition 4.28: Let I be a directed set. A net $\varphi = \{x_\alpha: \alpha \in I\}$ is bg-accumulates at a point x of a space X if for each U bg-open set containing x and for each $\alpha_0 \in I$, there is some $\alpha \geq \alpha_0$ such that $x_\alpha \in U$.

The net φ is bg-converges to a point x of X if for each U bg-open set containing x, there is $\alpha_0 \in$ I such that $x_\alpha \in$ U for all $\alpha \geq \alpha_0$.

Definition 4.29: A filter base Ɓ= $\{A_\alpha: \alpha \in I\}$ bg-accumulates at a point $x \in X$ if for each U bg-open set containing x and for each $\alpha \in$ I, then $U \cap A_\alpha$ is non-empty.

A filter base Ɓ bg-converges to a point $x \in X$ if for each U bg-open set containing x, there exists an A_α in Ɓ such that $A_\alpha \subseteq U$.

Definition 4.30: In a space X, a point x is said to be bg-adherent point of a filter base Ɓ on X if it lies in the bg-closure of all sets of Ɓ.

Proposition 4.31: For a space X, the following statements are equivalent.
1. X is bg-compact.
2. Each filter base in X has at least one bg-adherent point.
3. Every net in X with a well ordered directed set as its domain bg-accumulates to some point of X.

Proof: 1 $\longrightarrow$ 2:

Let X be a bg-compact space and Ɓ= $\{A_\alpha: \alpha \in I\}$ be a filter base on X. Since all finite intersection of members of Ɓ is non-empty, it follows all finite intersection of bg-closure members of Ɓ is non-empty. In view of Proposition 4.26, $D_{bg} \cap \{Cl_{bg}(A): \alpha \in I\}$ is non-empty. This means Ɓ has at least one bg-adherent point.

2 $\longrightarrow$ 1:

Let C= $\{U_\alpha: \alpha \in I\}$ be a bg-open cover of X which has no finite subcover. Implies, for any finite subfamily $\{U_\alpha: \alpha \in I_0\}$ of C, I_0 is a finite subset of I, then $X \neq \cup\{U_\alpha: \alpha \in I_0\}$. So $\cap\{(X-U_\alpha): \alpha \in I_0\} \neq \phi$. Thus the family Ɓ= $\{(X-U_\alpha): \alpha \in I\}$ of bg-closed sets has a finite intersection property. So, Ɓ is a filter base on X, then Ɓ has a bg-accumulation point x in X. It follows, $x \in D_{bg} \cap \{(X- U_\alpha): \alpha \in I\}$. Therefore, by Proposition 4.26, X is bg-compact space.

2 $\longleftrightarrow$ 3: Obvious.

Proposition 4.32: X is bg-compact space if and only if each filter base on X with at most one bg-adherent point is bg-convergent.

Proof: Suppose X is bg-compact space and Ɓ= $\{A_\alpha: \alpha \in I\}$ be a filter base on X with at most one bg-adherent point $x \in$ X. By Proposition 4.31, the bg-adherence set of Ɓ is equal to $\{x\}$.

Now, we claim that a filter base Ɓ is bg-converge at x. If not, then there exists a bg-open set V containing x, such that for each A_α in Ɓ, $A_\alpha \cap$ (X-V) is non-empty. Then $\omega = \{(A_\alpha - V): \alpha \in I\}$ is a filter base on X. By Proposition 4.31, the bg-adherence set of ω is non-empty.

However, $D_{bg}\cap$ $\{Cl_{bg}(A_{\alpha}- V): \alpha\in I\}\subseteq(D_{bg}\cap$ $\{Cl_{bg}(A_{\alpha}): \alpha\in I\})\cap$ $(X-V)= \{x\} \cap (X-V)=\phi$. But this is a contradiction. Hence, for each bg-open set V containing x, there exists an $A_{\alpha}\in$ Ɓ such that $A_{\alpha}\subseteq$ V. So Ɓ is bg-converge at x.

Conversely, it is enough to show that each filter base on X has at least one bg-accumulation point.

Assume that Ɓ= $\{A_{\alpha}: \alpha\in I\}$ is a filter base on X with no bg-adherent point. This means $D_{bg}\cap$ $\{Cl_{bg}(A_{\alpha}): \alpha\in I\}= \phi$. By hypothesis, Ɓ is bg-converge to some point x in X. Let A_{α} be any element in Ɓ, for each V bg-open set containing x there exists A_i $\in$Ɓ such that $A_i \subseteq$V. Since Ɓ is a filter base, there exists $A_{8}\in$ Ɓ such that $A_{8}\subseteq A_{\alpha}\cap A_i\subseteq A_{\alpha}\cap V$. This means $A_{\alpha}\cap$ V is non-empty for each V bg-open set containing x. Then x is a point of $Cl_{bg}(A_{\alpha})$ and $x\in D_{bg}\cap$ $\{Cl_{bg}(A_{\alpha}): \alpha\in I\}$. which is a contradiction. Thus, X is bg-compact space.

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