Hybrid Iteration Method for Fixed Points of Nonself Nonexpansive Mapping in Real Banach Spaces and its Applications

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Abstract

In this paper, a Hybrid iteration method is studied and the strong convergence of the iteration scheme to a fixed point of nonself nonexpansive mapping is obtained in Banach spaces. In section 4, we apply these results to solve the equilibrium problems. Our results improve and extend the corresponding results of Jung [6], Wang [13], Song [8], Aoyama [1].

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1. Introduction

Let $E$ be a real Banach space and $C$ be a nonempty closed convex subset of $E$. A mapping $T : C \to C$ is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in C$. The fixed point set of $T$ is denoted by $F(T) = \{x \in C; Tx = x\}$.

In order to find a fixed point of nonexpansive mapping $T$, Halpern [5] firstly introduced the following iteration scheme in a Hilbert space: $u, x_0 \in C, \lambda_n \in [0, 1],$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0 \quad (1.1)$$

He pointed out that the conditions $(C_1), \lim_{n \to \infty} \alpha_n = 0$ and $(C_2), \sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the convergence of the iteration scheme (1.1) to a fixed point of nonexpansive mapping $T$. After that, Jung [7] and Aoyama [2] et. al. extended Halpern’s result, respectively.

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For approximating the fixed points of nonexpansive mappings, Zeng and Yao [14] introduced the following implicit hybrid iteration method.

For an arbitrary given \( x_0 \in H \), the sequence \( \{x_n\} \) is generated as follows:

\[
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T_\alpha x_n, \quad n \geq 0
\]

where \( T_n = T_{n \mod N} \{\alpha_n\} \) is a sequence in \((0,1)\). By using the iteration scheme (1.2), they obtained the weak and strong convergence theorems in real Hilbert space. Recently, Wang [13] introduced an explicit hybrid iteration method for nonexpansive mapping \( T \) in Hilbert space: for arbitrary \( u \in H \) and \( x_0 \in H \)

\[
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T_\alpha x_n, \quad n \geq 0
\]

where \( T_\alpha x_n = T x_n - \lambda_n \mu F(T x_n) \). Under some appropriate conditions, they proved the weak and strong convergence of the iteration scheme (1.3) to a fixed point of \( T \).

It is well known that the sequence \( \{x_n\} \) which generated by above iteration scheme is well defined. However, \( T \) maps \( X \) into \( C \), then the sequence \( \{x_n\} \) may not be well defined. One method that has been used to overcome this in the case of the operator \( T \) is to generalize the iteration scheme by introducing a retraction \( P : C \to E \).

Motivated by above works, in this paper, we consider the the strong convergence of the iteration scheme \( \{x_n\} \) defined by the following scheme (1.4) for nonself nonexpansive mapping in a real Banach spaces with uniformly Gâteaux differentiable norm.

Let \( T : E \to C \) is nonself nonexpansive mapping, and \( F : C \to C \) an \( L \)-Lipschitzian mapping. Then an iterative scheme is the sequences of mappings \( \{x_n\} \) defined by, for given \( u, x_0 \in C \), \( \alpha_n, \beta_n, \gamma_n \) are real sequences in \([0,1]\) and \( \alpha_n + \beta_n + \gamma_n = 1 \) for all \( n \geq 1 \),

\[
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T_\alpha x_n, \quad n \geq 0
\]

where \( T_\alpha x_n = (PT)x_n - \lambda_n \mu F((PT)x_n), \mu > 0 \).

**Remark 1.1.** In (1.4), when \( \alpha_n = 0 \) for all nonnegative integer \( n \), the iteration (1.4) reduces to the iteration (1.2). When \( \alpha_n = 0 \) and \( \lambda_n = 0 \) for all nonnegative integer \( n \), the iteration (1.4) reduces to the Mann iteration. When \( \beta_n = 0 \) and \( \lambda_n = 0 \) for all nonnegative integer \( n \), the iteration (1.4) reduces to the Halpern type iteration.

2. Preliminaries

Let \( S(E) := \{x \in E; \|x\| = 1\} \) denote the unit sphere of a Banach space \( E \). A Banach space \( E \) is said to have a uniformly Gâteaux differentiable norm, if for each \( y \in S(E) \), the limit \( \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \) is uniformly attained for \( x \in S(E) \).

If we define a map \( \varphi : E \to \mathcal{R} \) by \( \varphi(y) = \mu_n \|x_n - y\|^2 \), then \( \varphi(y) \) is convex and continuous, and \( \varphi(y) \to \infty \) as \( \|y\| \to \infty \). If \( C \) be a weakly compact convex subset of a real Banach spaces \( E \), there exists \( x \in C \) such that \( \varphi(x) = \inf_{y \in C} \varphi(y) \) (see [9]). So the set \( K_{\min} = \{x \in C : \varphi(x) = \inf_{y \in C} \varphi(y)\} \neq \emptyset \). Clearly, \( K_{\min} \) is closed convex by the convexity and continuity of \( \varphi(y) \).
We restate the following lemmas which play important roles in our proofs.

**Remark 2.1.** [4] Dugundji pointed out that any nonempty closed convex subset \( C \) of a real Banach space \( E \), \( C \) is retract of \( E \). And there exists a retraction \( P \) of \( E \) onto \( C \) such that \( \| x - Px \| \leq (1 + \alpha) \rho(x, X), \forall x \in X \), where \( \rho(x, X) \) is the distance of \( x \) onto \( X \).

**Lemma 2.2.** [10] Let \( E \) be a real smooth Banach space, and \( C \) be a nonempty closed convex subset of \( E \) with \( P \) as a sunny nonexpansive retraction, and let \( T : C \to E \) be a mapping satisfying weakly inward condition. Then \( F(T) = F(PT) \).

Let \( \mu \) be a continuous linear functional on \( l^\infty \) satisfying \( \| \mu \| = 1 = \mu(1) \). For every \( a = (a_1, a_2, \cdots ) \in l^\infty \), we denote by \( \mu(n)(a) \) the value of \( \mu(a) \). If a linear continuous functional \( \mu \) on \( l^\infty \) such that \( \| \mu \| = \mu(1) = 1 \) and \( \mu(n)(a) = \mu_n(a_{n+1}) \) for each \( a = (a_1, a_2, \cdots ) \in l^\infty \), then \( \mu \) is called a Banach limit. Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit, see [10, 11] for more details.

**Lemma 2.3.** [7]. Let \( \alpha \) be a real number and \( (x_0, x_1, \cdots ) \in l^\infty \) such that \( \mu_n x_n \leq \alpha \) for all Banach limits. If \( \limsup_{n \to \infty} (x_{n+1} - x_n) \leq 0 \), then \( \limsup_{n \to \infty} x_n \leq \alpha \).

**Lemma 2.4.** [1] Let \( \{ s_n \} \) be a sequence of nonnegative real numbers satisfying the condition \( s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \gamma_n + \beta_n, \forall n \geq 1 \), where \( \{ \alpha_n \}, \{ \beta_n \} \) and \( \{ \gamma_n \} \) are sequences of real numbers such that (i) \( \{ \alpha_n \} \subset [0,1] \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \); (ii) \( \{ \beta_n \} \) be nonnegative and \( \sum_{n=1}^{\infty} \beta_n < \infty \); (iii) \( \limsup_{n \to \infty} \gamma_n \leq 0 \). Then, \( \lim_{n \to \infty} s_n = 0 \).

**Lemma 2.5.** [9] Let \( \{ x_n \} \) and \( \{ x_n \} \) be bounded sequences in a Banach space \( X \) and let \( \{ \beta_n \} \) be a sequence in \( [0,1] \) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Suppose \( x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \) for all integers \( n \geq 0 \) and \( \lim_{n \to \infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \leq 0 \). Then, \( \lim_{n \to \infty} \| y_n - x_n \| = 0 \).

**Lemma 2.6.** [11] Let \( C \) be a nonempty closed convex subset of a Banach space \( E \) with a uniformly Gâteaux differentiable norm. Let \( \{ x_n \} \) be a bounded sequence of \( E \) and let \( \mu_n \) be a Banach limit and \( x \in C \). If we define a map \( \varphi : E \to \mathcal{R} \) by \( \varphi(y) = \mu_n \| x_n - y \|^2 \). Then \( \varphi(x) = \inf_{y \in C} \varphi(y) \) if and only if \( \mu_n \langle y - x, J(x_n - x) \rangle \leq 0, \forall y \in C \).

### 3. Main results

In this section, we shall state the strong convergence of the Hybrid iteration scheme (1.4) to a fixed point for nonself nonexpansive mapping in real Banach spaces with uniformly Gâteaux differentiable norm.

**Theorem 3.1.** Let \( E \) be a real Banach space with a uniformly Gâteaux differentiable norm and \( C \) be a nonempty weakly compact convex subset of \( E \). Suppose that \( T \) is a nonself nonexpansive mapping. \( \{ x_n \} \) defined by (1.4), \( \{ \alpha_n \}, \{ \lambda_n \} \) are real number sequences in \( [0,1] \) satisfying the following conditions: (i) \( (C_1) \) and \( (C_2) \); (ii) \( \sum_{n=2}^{\infty} \lambda_n < \infty \). If \( F = K_{min} \cap F(PT) \neq \emptyset \), then \( \{ x_n \} \) strongly converges to some point of \( F \).

**Proof.** We prove the theorem in stages.

**step 1:** First we show that \( \{ x_n \} \) is bounded. Take \( x \in F \). Then, from (1.4), we
estimate as follows:
\[
\|x_{n+1} - x\| \leq \alpha_n \|u - x\| + \beta_n \|x_n - x\| + \gamma_n \|(PT)x_n - x\| + \gamma_n \|\lambda_{n+1}\mu F((PT)x_n)\|
\]
\[
\leq \alpha_n \|u - x\| + (\beta_n + \gamma_n) \|x_n - x\| + \gamma_n \lambda_{n+1}\mu L \|x_n - x\| + \gamma_n \lambda_{n+1}\mu \|F(x)\|
\]
\[
\leq (1 + \lambda_{n+1}\mu L)\|x_n - x\| + \|u - x\| + \mu \|F(x)\|
\]
\[
\leq (1 + \lambda_{n+1}\mu L) \max \{\|x_n - x\|, \|u - x\|, \mu \|F(x)\|\}
\]
\[
\leq \cdots \leq k \cdot \max \{\|x_0 - x\|, \|u - x\|, \mu \|F(x)\|\},
\]
where \(k = \Pi_{n=1}^{\infty} (1 + \lambda_{n+1}\mu L)\).

\[
\|PTx_n\| \leq \|PTx_n - x\| + \|x\| \leq \|x_n - x\| + \|x\| \leq k \cdot \max \{\|x_0 - x\|, \|u - x\|, \mu \|F(x)\|\} + \|x\|.
\]

This proves the boundedness of the sequences \(\{x_n\}, \{(PT)x_n\}\), so \(\{F((PT)x_n)\}\) is bounded also.

**step2** : Define \(x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n\), then by (1.4) we have
\[
y_{n+1} - y_n = \frac{\alpha_{n+1}u + \beta_{n+1}x_{n+1} + \gamma_{n+1}((PT)x_{n+1} - \lambda_{n+2}\mu F((PT)x_{n+1})) - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}}
\]
\[
= \frac{\alpha_{n}u + \beta_{n}x_n + \gamma_n((PT)x_n - \lambda_{n+1}\mu F((PT)x_n)) - \beta_{n}x_n}{1 - \beta_{n}}
\]
\[
= \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_{n}}{1 - \beta_{n}}\right)(u - (PT)x_n) + \gamma_{n+1}\lambda_{n+2}\mu F((PT)x_{n+1}) - \beta_{n+1}x_{n+1}
\]
\[
+ \frac{\gamma_{n+1}\lambda_{n+2}\mu F((PT)x_{n+1}) - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}}
\]

So
\[
\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_{n}}{1 - \beta_{n}}\|(u\| + \|(PT)x_n\|) - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\|
\]
\[
+ \frac{\gamma_{n+1}\lambda_{n+2}\mu F((PT)x_{n+1}) - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}}\|F((PT)x_{n+1})\| + \frac{\gamma_{n}\lambda_{n+1}\mu F((PT)x_{n+1}) - \beta_{n}x_n}{1 - \beta_{n}}\|F((PT)x_{n+1})\|.
\]

Since \((C_1), \Sigma_{n=2}^\infty \lambda_n < \infty\) and \(\{x_n\}, \{(PT)x_n\}, \{F((PT)x_n)\}\) are bounded, we have \(\limsup_{n \to \infty} \|(y_{n+1} - y_n) - \|x_{n+1} - x_n\|\) \leq 0. Thus it follows from Lemma 2.5 that \(\lim_{n \to \infty} \|y_n - x_n\| = 0\). In addition, since \(\|x_{n+1} - x_n\| = (1 - \beta_n)\|y_n - x_n\|\), we have that \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\).

**step3** : Since \(C\) be a nonempty weakly compact convex subset of \(E\), there exists \(x \in F\), such that \(\mu_n \|x_n - x\|^2 = \inf_{y \in C} \mu_n \|x_n - y\|^2\). It follows from Lemma 2.6 that for \(y \in C\), \(\mu_n \langle y - x, J(x_n - x) \rangle \leq 0\). Since \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\), then it follows from the norm-weak* uniformly continuous of the duality mapping \(J\) that \(\lim_{n \to \infty} \langle y - x, J(x_{n+1} - x) \rangle = 0\). Hence, the sequence \(\langle y - x, J(x_{n+1} - x) \rangle\) satisfies the conditions of Lemma 2.3. As a result, we have
\[
\limsup_{n \to \infty} \langle y - x, J(x_{n+1} - x) \rangle \leq 0.
\]

(3.1)
Using (1.4), we get that
\[
\|x_{n+1} - x\|^2 = \alpha_n \langle u - x, J(x_{n+1} - x) \rangle + \beta_n \langle x_n - x, J(x_{n+1} - x) \rangle + \gamma_n \langle (PT)x_n - x, J(x_{n+1} - x) \rangle \\
+ \gamma_n \langle \lambda_{n+1} F((PT)x_n), J(x_{n+1} - x) \rangle \\
\leq \alpha_n \langle u - x, J(x_{n+1} - x) \rangle + (\beta_n + \gamma_n) \|x_n - x\| \|x_{n+1} - x\| \\
+ \gamma_n \lambda_{n+1} \|F((PT)x_n)\| \|x_{n+1} - x\| \\
\leq (1 - \alpha_n) \frac{\|x_n - x\|^2 + \|x_{n+1} - x\|^2}{2} + \alpha_n \langle u - x, J(x_{n+1} - x) \rangle \\
+ \gamma_n \lambda_{n+1} \|F((PT)x_n)\| \|x_{n+1} - x\|.
\]

Thus, \[\|x_{n+1} - x\|^2 \leq (1 - \sigma_n) \|x_n - x\|^2 + \sigma_n \rho_n + \zeta_n,\] where \[\sigma_n = \frac{2 \alpha_n}{1 + \alpha_n} \geq \alpha_n, \quad \rho_n = \langle u - x, J(x_{n+1} - x) \rangle\] and \[\zeta_n = \frac{\gamma_n \lambda_n \|F((PT)x_n)\| \|x_{n+1} - x\|}{1 + \alpha_n}.\] Since \((C_2), \sum_{n=2}^{\infty} \lambda_n < \infty, (3.1)\) and Lemma 2.4, we have that \(\lim_{n \to \infty} \|x_{n+1} - x\| = 0,\) the proof is completed.

4. Some applications for the equilibrium problem

Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(E.\) Let \(\Phi\) be a bifunction of \(C \times C\) into \(\mathbb{R},\) where \(\mathbb{R}\) is the set of real numbers. The equilibrium problem for the bifunction \(\Phi\) is to find \(x \in C\) such that
\[
\Phi(x, y) \geq 0, \forall y \in C
\] (4.1)
The set of solutions of the above inequality is denoted by \(EP(\Phi).\) Many problems arising from physics, optimization, and economics can reduce to finding an equilibrium problem.
In 2005, Combettes and Hirstoaga [3] pointed out that if $C$ be a nonempty closed convex subset of $H$ and let $\Phi$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that $\Phi(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \forall y \in C$. Furthermore, We restate the following lemmas.

**Lemma 4.1.** [2] Let $C$ be a nonempty closed and convex subset of $H$. Let $\Phi : C \times C \rightarrow \mathbb{R}$ be a bifunction which satisfies the following conditions: (A1) $\Phi(x, x) = 0$ for all $x \in C$; (A2) $\Phi$ is monotone, that is, $\Phi(x, y) + \Phi(y, x) \leq 0$ for all $x, y \in C$; (A3) For all $x, y, z \in C$, $\lim_{t \to 0} \Phi(tx + (1 - t)x, y) \leq \Phi(x, y)$; (A4) For each $x \in C$, $y \mapsto \Phi(x, y)$ is convex and lower semicontinuous.

**Lemma 4.2.** [12] Assume that $\Phi$ Satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows: $T_r(x) = \{ z \in C; \Phi(z, y) + \frac{1}{r}(y - z, y - x) \geq 0, \forall y \in C \}, \forall x \in H$.

Then, the following hold: (1) $T_r$ is single-valued; (2) $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$, $\| T_r x - T_r y \| \leq \langle T_r x - T_r y, x - y \rangle$; (3) $\Phi(T_r) = EP(\Phi)$; (4) $EP(\Phi)$ is closed and convex.

**Corollary 4.3.** Let $C$ be a nonempty closed convex subset of $H$ and let $\Phi$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and $EP(\Phi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1, u \in C$ and

$$
\begin{align*}
\begin{cases} 
\Phi(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, & \forall y \in C \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n u_n
\end{cases}
\end{align*}
$$

for all $n \in \mathbb{N}$. Assume that $\alpha_n \in (0, 1)$ and $r_n \in (0, +\infty)$ satisfy (C1), (C2) and (C3'). $\liminf_{n \to \infty} r_n > 0$ and $\sum |r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $x \in EP(\Phi)$.

**Proof.** Following the proof technique of Theorem 3.1, by $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$, we can use the same method to prove that $\lim_{n \to \infty} \| x_{n+1} - x \| = 0$. Hence, by $\| u_n - x_{n+1} \| \leq \frac{\alpha_n}{\gamma_n} \| x_{n+1} - u \| + \frac{\delta_n}{\gamma_n} \| x_{n+1} - x_n \|$, we have that $\lim_{n \to \infty} \| u_n - x \| = 0$.

Now we show that $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$. Let $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, then we have

$$
y_{n+1} - y_n = \frac{x_{n+2} - \beta_n x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}
= \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) u + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (u_{n+1} - u_n) + \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) u_n
= \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) (u - u_n) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (u_{n+1} - u_n)
= \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) (u - u_n)
+ \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left( [T_{r_{n+1}} x_{n+1} - T_{r_{n+1}} x_n] + [T_{r_{n+1}} x_n - T_n x_n] \right). \tag{4.3}
$$
On the other hand, from the definition of $T_r$, we have

$$\Phi(T_r, x_n, y) + \frac{1}{r_n} \langle y - T_r x_n, T_r x_n - x_n \rangle \geq 0, \quad \forall y \in C \quad (4.4)$$

and

$$\Phi(T_{r+1}, x_n, y) + \frac{1}{r_{n+1}} \langle y - T_{r+1} x_n, T_{r+1} x_n - x_n \rangle \geq 0, \quad \forall y \in C \quad (4.5)$$

Putting $y = T_{r+1} x_n$ in (4.4) and $y = T_r x_n$ in (4.5), we have $\Phi(T_r, x_n, T_{r+1} x_n) + \frac{1}{r_n} \langle T_{r+1} x_n - T_r x_n, T_r x_n - x_n \rangle \geq 0$, and $\Phi(T_{r+1}, x_n, T_r x_n) + \frac{1}{r_{n+1}} \langle T_{r+1} x_n - T_r x_n, T_{r+1} x_n - x_n \rangle \geq 0$. So, from (A2) we have $\langle T_{r+1} x_n - T_r x_n, \frac{T_{r+1} x_n - T_r x_n}{r_{n+1}} - \frac{T_r x_n - x_n}{r_n} \rangle \geq 0$, and hence $\langle T_{r+1} x_n - T_r x_n, \frac{T_{r+1} x_n - T_r x_n}{r_{n+1}} \rangle \geq 0$. Then, we have

$$\frac{\|T_{r+1} x_n - T_r x_n\|^2}{r_{n+1}} \leq \langle T_{r+1} x_n - T_r x_n, \frac{1}{r_{n+1}} - \frac{1}{r_n} \rangle \langle T_r x_n - x_n \rangle \leq \|T_{r+1} x_n - T_r x_n\| \cdot \left| \frac{1}{r_{n+1}} - \frac{1}{r_n} \right| 2M.$$

where $M = \text{Sup} \{\|u\|, \|u_n\|, \|x_n\|, \|T_r x_n\|\}$. Thus, we have

$$\|T_r x_n - T_{r+1} x_n\| \leq \left| 1 - \frac{r_{n+1}}{r_n} \right| 2M. \quad (4.6)$$

Substituting (4.6) into (4.3), we have $\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| + \frac{\gamma_{n+1}}{R(1-\beta_{n+1})} \left| \frac{1}{r_{n+1}} - \frac{1}{r_n} \right| 2M$, for some $R$ with $r_n > R > 0$. Since $(C_1)$ and $(C_3')$, we have $\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Thus it follows from Lemma 2.5 that

$$\lim_{n \to \infty} \|y_n - x_n\| = 0. \quad (4.7)$$

In addition, since $\|x_{n+1} - x_n\| = (1 - \beta_n) \|y_n - x_n\|$, we have that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$. The proof is completed.

References


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