International Mathematical Forum, Vol. 7, 2012, no. 5, 209 - 214

Analysis of a Kind of Time-Limited Pest Control of a Predator-Prey System with Impulsive Harvest

Yongliang Li , Yiyi Zhang and Zhongyi Xiang

Department of Mathematics
Hubei University for Nationalities
Enshi, Hubei Province, 445000, P. R. China

Abstract

In this paper, the applications of boundary value problem to the system of two species with impulsive control in finite time are investigated. This paper presents a kind of time-limited pest control of a predator-prey model with impulsive harvest. By the comparison principle, the conditions under which the model has a solution are found by a series of the upper solutions.

Mathematics Subject Classification: 34C05, 92D25

Keywords: Impulsive differential equations; Boundary value problem; Pest management; Permanence; Sate impulse

1. Introduction

In order to consider the consequences of spraying pesticide and introducing additional predators into a natural pest-predator system, many authors have suggested impulsive differential equations to investigate the dynamics of pest

---

The Youth Science Foundation of Educational Department of Hubei Province in China, the Science Foundation(Q20101903).

Corresponding author. e-mail: jwliyongliang@163.com

control model [1-3], and some results were obtained. Which the basic model this paper is considering is the predator-prey system as following:

$$\begin{cases} \dot{x}(t) = ax(t) - bx^2(t) - \frac{cx^2(t)}{d+ex(t)}y(t), \\ \dot{y}(t) = y(t)(fx(t) - g), \end{cases} \tag{1.1}$$

Where $x(t)$ and $y(t)$ are population density of prey species and predator species. $a > 0$ is an intrinsic rate constant of prey species. $b > 0$ is the restrict of population density. $\frac{cx^2(t)}{d+ex(t)}$ is a function which is increased monotonously. $f > 0$ is the rate of the transformation form prey to predator species, $g > 0$ is the death rate of predator species. Assume that the number of insect pests by impulsive harvest in fixed time, and consider the time-limited control problem. The model which has the initial boundary value problem and impulsive control can be written as following:

$$\begin{cases} \left.\begin{array}{l} \dot{x}(t) = ax(t) - bx^2(t) - \frac{cx^2(t)}{d+ex(t)}y(t), \\ \dot{y}(t) = y(t)(fx(t) - g), \end{array}\right\} \quad t \neq k\tau, \\ \left.\begin{array}{l} \Delta x(t) = x(t^+) - x(t) = -px(t), \\ \Delta y(t) = y(t^+) - y(t) = 0, \end{array}\right\} \quad t = k\tau, \\ x(0) = x(0^+) = A, x(T) \leq B < A, \\ y(0) = y(0^+) = y_0 > 0, k = 1, 2, \cdots, n, \end{cases} \tag{1.2}$$

where $x(t)$ and $y(t)$ indicate the population density of pest and natural enemy, p is the death proportion of insect pest owing to carrying out artificial measures such as spraying pesticides, $0 < p < 1$. T is a finite time. τ is the period of the impulsive harvest. The other parameters and biological significance are the same as (1.1). Our aim is to control the pest population under B after n times of the impulsive harvest. We assume that the control methods will have no direst effect on the population of natural enemies in this text. For example, we can use some pesticides or control methods which have excellent selectivity. Then we also assume that when $x(0) = A < \frac{f}{g}$ insect pests have occurred, and we must take some effective measures.

2. main result

Let $k_1 = f(1-p)B - g, r = \frac{c}{d+\frac{ae}{b}}, q = \frac{c}{d}, k_2 = \frac{fe^{a\tau}}{\frac{qy_0e^{k_1\tau}+b}{a}(e^{a\tau}-1)+\frac{1}{A}} - g.$

Theorem 2.1. If

$$1-\frac{1}{b}ry_0e^{k_1\tau} > 0, \frac{rm_1^y}{a}(1-e^{-a\tau})+\frac{e^{-a\tau}}{A} > \frac{f}{g}, A(ry_0e^{k_1\tau}+b)(1-e^{-a\tau})+a(e^{-a\tau}+p-1) > 1,$$

and one of the following conditions holds:

a) when $T = n\tau$, $\frac{1-e^{-a\tau}}{a}(ry_0e^{nk_1\tau} + b)\frac{(1-p)^n-e^{-na\tau}}{(1-p)^n-(1-p)^{n-1}e^{-a\tau}} + \frac{e^{-na\tau}}{A(1-p)^{n-1}} > \frac{1}{B}$. b) when $T \geq n\tau$, $\frac{ry_0e^{k_1T}+b}{a}(1 - e^{-a(T-n\tau)}) + \frac{e^{-a(T-n\tau)}(1-e^{-a\tau})}{a}(ry_0e^{nk_1\tau} + b)\frac{(1-p)^n-e^{-na\tau}}{(1-p)^{n+1}-(1-p)^ne^{-a\tau}} + \frac{e^{-aT}}{A(1-p)^n} \geq \frac{1}{B}$. Then the solution of (1.2) which satisfied initial -boundary value conditions is in the existence.

Proof: $\dot{x}(t) > 0$, $\dot{x}(t) \leq ax(t)(1-\frac{b}{a}x(t))$, then $1-\frac{b}{a}x(t) > 0$, namely $x(t) < \frac{a}{b}$, we hold that $\frac{c}{d+ex(t)} > \frac{c}{d+\frac{ae}{b}} = r$, $x(t)$ is increased monotonously, so there must be exist a series of the upper solutions $\bar{x}_1(t), \bar{x}_2(t), \cdots, \bar{x}_n(t)$ so as to $x_1(t) \leq \bar{x}_1(t)$, for $t \in (0,\tau]$, $x_2(t) \leq \bar{x}_2(t)$, for $t \in (\tau^+, 2\tau]$, where $\bar{x}_1(t)$ satisfies:

$$\begin{cases} \dot{\bar{x}}_1(t) = a\bar{x}_1(t) - b\bar{x}_1^2(t) - rm_1^y\bar{x}_1^2(t) = a\bar{x}_1(t) - r_1\bar{x}_1^2(t), \\ \bar{x}_1(0) = x_0 = A, \end{cases} \tag{2.1}$$

we get $\bar{x}_1(t) = \frac{e^{at}}{\frac{r_1}{a}(e^{at}-1)+\frac{1}{A}}$, $t \in (0,\tau]$ and $\bar{m}_1^x = A \leq \bar{x}_1(t) \leq \frac{e^{a\tau}}{\frac{r_1}{a}(e^{a\tau}-1)+\frac{1}{A}} \leq \bar{M}_1^x$

When $t \in (\tau^+, 2\tau]$, $x_2(t) \leq \bar{x}_2(t)$, where $\bar{x}_2(t)$ satisfies:

$$\begin{cases} \dot{\bar{x}}_2(t) = a\bar{x}_2(t) - r_2\bar{x}_2^2(t), \\ \bar{x}_2(\tau^+) = (1-p)\bar{M}_1^x, \end{cases} \tag{2.2}$$

We obtain $\bar{x}_2(t) = \frac{e^{a(t-\tau)}}{\frac{r_2}{a}(e^{a(t-\tau)}-1)+\frac{1}{(1-p)\bar{M}_1^x}} = \bar{M}_2^x$, then $\bar{m}_2^x = 1-p)\bar{M}_1^x \leq \bar{x}_2(t) \leq \frac{e^{a\tau}}{\frac{r_2}{a}(e^{a\tau}-1)+\frac{1}{1-p)\bar{M}_1^x}} = \bar{M}_2^x$. If $(1-p)\frac{e^{a\tau}}{\frac{r_1}{a}(e^{a\tau}-1)+\frac{1}{A}} \leq A$, then $\bar{x}(\tau^+) = (1-p)x(\tau) < (1-p)\bar{M}_1^x < A$. Because $(1-p)\bar{M}_1^x < A$, $\bar{M}_2^x = \frac{e^{a\tau}}{\frac{r_2}{a}(e^{a(t-\tau)}-1)+\frac{1}{(1-p)M_1^x}} < \bar{M}_1^x$, namely, $\frac{1}{\bar{M}_2^x} > \frac{1}{\bar{M}_1^x}$, we can also get $\frac{r_2}{a}(1-e^{-a\tau})+\frac{1}{(1-p)\bar{M}_1^x}e^{-a\tau} > \frac{1}{\bar{M}_1^x}$, $\bar{M}_1^x < \frac{A}{1-p}$. Farther, $\frac{Ar_2}{(1-p)a}(1 - e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1$, then the inequality above is tenable, consequently, $(1-p)x(2\tau) < (1-p)x(\tau) < A$.

When $t \in ((n-1)\tau^+, n\tau]$, $x_n(t) \leq \bar{x}_n(t)$, where $\bar{x}_n(t)$ satisfies:

$$\begin{cases} \dot{\bar{x}}_n(t) = a\bar{x}_n(t) - r_n\bar{x}_n^2(t), \\ \bar{x}_n((n-1)\tau^+) = (1-p)\bar{M}_{n-1}^x, \end{cases} \tag{2.3}$$

we obtain $\bar{x}_n(t) = \frac{e^{a(t-(n-1)\tau)}}{\frac{r_n}{a}(e^{a(t-(n-1)\tau)}-1)+\frac{1}{(1-p)\bar{M}^x_{n-1}}}, t \in ((n-1)\tau^+, n\tau]$, then $\bar{m}^x_n = (1-p)\bar{M}^x_{n-1} \leq \bar{x}_n(t) \leq \frac{e^{a\tau}}{\frac{r_n}{a}(e^{a\tau}-1)+\frac{1}{1-p)\bar{M}^x_{n-1}}} = \bar{M}^x_n$. If $\frac{Ar_1}{(1-p)a}(1-e^{-a\tau})+$

$$\frac{e^{-a\tau}}{1-p} > 1, \frac{Ar_2}{(1-p)a}(1-e^{-a\tau})+\frac{e^{-a\tau}}{1-p} > 1, \cdots, \frac{Ar_n}{(1-p)a}(1-e^{-a\tau})+\frac{e^{-a\tau}}{1-p} > 1,$$

by the discussion above, we know

$$\frac{1}{\bar{M}^x_n} = \frac{1-e^{-a\tau}}{(1-p)}[r_n + \frac{e^{-a\tau}}{(1-p)}r_{n-1} + \cdots + \frac{e^{-(n-2)a\tau}}{(1-p)^{n-2}}r_2 + \frac{e^{-(n-1)a\tau}}{(1-p)^{n-1}}r_1] + \frac{e^{-na\tau}}{A(1-p)^{n-1}}. \tag{2.4}$$

When $T = n\tau$,

$$\frac{1}{\bar{M}^x_n} = \frac{1-e^{-a\tau}}{(1-p)}[r_n + \frac{e^{-a\tau}}{(1-p)}r_{n-1} + \cdots + \frac{e^{-(n-2)a\tau}}{(1-p)^{n-2}}r_2 + \frac{e^{-(n-1)a\tau}}{(1-p)^{n-1}}r_1] + \frac{e^{-na\tau}}{A(1-p)^{n-1}} \ . \tag{2.5}$$

When $T > n\tau$, $x_T(t) \leq \bar{x}_T(t)$, where $x_T(t)$ satisfies:

$$\begin{cases} \dot{\bar{x}}_T(t) = a\bar{x}_T(t) - r_T\bar{x}^2_T(t), \\ \bar{x}_T(n\tau^+) = (1-p)\bar{M}^x_n, \end{cases} \tag{2.6}$$

hence,

$$\frac{1}{\bar{M}^x_T} = \frac{r_n}{a}(1-e^{-a(T-n\tau)}) + \frac{e^{-a(T-n\tau)}(1-e^{-a\tau})}{a(1-p)}[r_n + \cdots + \frac{e^{-a(n-1)\tau}}{(1-p)^{n-1}}r_1] + \frac{e^{-a\tau}}{A(1-p)^n}. \tag{2.7}$$

Similarly, if

$$\begin{array}{l} \frac{Ar_1}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1, \frac{Ar_2}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1, \cdots, \\ \frac{Ar_n}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1, \frac{Ar_T}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1, \\ (1-p)\bar{x}(T) < (1-p)\bar{x}(n\tau) < \cdots < (1-p)\bar{x}(\tau) < A, \end{array}$$

we will find some conditions so as to

$$\begin{array}{l} \frac{Ar_1}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1, \frac{Ar_2}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1, \cdots, \\ \frac{Ar_n}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1, \frac{Ar_T}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1. \end{array}$$

As $m^y_1 > m^y_2 > \cdots > m^y_n > (m^y_T)$, and $r_1 = b + rm^y_1$, $r_2 = b + rm^y_2, \cdots$, $r_n = b + rm^y_n$,$r_T = b + rm^y_T$, inequations $r_1 > r_2 > \cdots > r_n(> r_T)$ are tenable, and when $\frac{Ar_1}{(1-p)a}(1-e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1$ holds, all of the inequations above are tenable. On the other hand, by the reasons of $\dot{x} > 0$, for $t \in [0, T]$,$x(t) \geq$

$(1-p)B$. Farther more, $\dot{y}(t) = (fx(t) - g)y(t) \geq (f(1-p)B - g)y(t)$ and $y(t)$ are continuous. We obtain $y(t) \geq \underline{y}(t)$, where $\underline{y}(t)$ satisfies:

$$\begin{cases} \dot{\underline{y}}(t) = (f(1-p)B - g)\underline{y}(t) = k_1\underline{y}(t), \\ \underline{y}(0) = y_0, \end{cases} \tag{2.8}$$

$B < A < \frac{g}{f}$, so $k_1 = (f(1-p)B - g) < 0$. Due to $y(t) \geq \underline{y}(t)$, $m_1^y \geq y_0 e^{k_1\tau}$, $m_2^y \geq y_0 e^{2k_1\tau}$,$\cdots$,$m_n^y \geq y_0 e^{nk_1\tau}$,($m_T^y \geq y_0 e^{k_1 T}$), namely $r_1 \geq r y_0 e^{k_1\tau} + b$, $r_2 \geq r y_0 e^{2k_1\tau} + b$, $\cdots$, $r_n \geq r y_0 e^{nk_1\tau} + b$, $r_T \geq r y_0 e^{k_1 T} + b$, then, $\frac{Ar_i}{(1-p)a}(1 - e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1$, if $\frac{Ar_1}{(1-p)a}(1 - e^{-a\tau}) + \frac{e^{-a\tau}}{1-p} > 1$ or $A(ry_0 e^{k_1\tau} + b)(1 - e^{-a\tau}) + a(e^{-a\tau} + p - 1) > 1$, $i = 1, 2, \cdots, n$.

When $T = n\tau$, $\frac{1}{\bar{M}_n^x} =$

$$\tfrac{1-e^{-a\tau}}{(1-p)}[r_n + \tfrac{e^{-a\tau}}{(1-p)} r_{n-1} + \cdots + \tfrac{e^{-(n-2)a\tau}}{(1-p)^{n-2}} r_2 + \tfrac{e^{-(n-1)a\tau}}{(1-p)^{n-1}} r_1] + \tfrac{e^{-na\tau}}{A(1-p)^{n-1}} \geq \tfrac{1}{B} \tag{2.9}$$

When the condition a) of theorem (2.1) is tenable, the following equation is obtained

$$x(T) \leq \bar{M}_n^x \leq B, T = n\tau. \tag{2.10}$$

When $T \geq n\tau$,

$$\tfrac{1}{\bar{M}_T^x} = \tfrac{r_n}{a}(1 - e^{-a(T-n\tau)}) + \tfrac{e^{-a(T-n\tau)}}{(1-p)\bar{M}_n^x} \geq \tfrac{1}{B}. \tag{2.11}$$

When the condition b) of theorem (2.1) is tenable, the following equation is obtained

$$x(T) \leq \bar{M}_T^x \leq B, T > n\tau. \tag{2.12}$$

By the comparison principle, the conditions under which the model has a solution are found by a series of the upper solutions form theorem (2.1). Now we want to obtain the conditions under which the model has no solution by a series of the lower solutions.

References

[1] S.Y Tang, Y.N Xiao, L.S Chen, Integrated pest management models, and their dynamical behavior [J]. Bulletin of Mathematical Biology, 2005, 67(1): 115-135.

[2] X. Song and Z. Xiang, The prey-dependent consumption two-prey one-predator models with stage structure for the predator and impulsive effects, *J. Theor. Biol.* **242**(3) (2006) 683-698.

[3] X. Song and Y. Li, Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect, *Nonlinear Analysis: Real World Applications* **9**(1) (2008) 64-79.

Received: July, 2011