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Global Vertex-Edge Domination Sets in Graph

S. Chitra

Manonmaniam Sundaranar University
Valliammai Engineering College, Chennai, India
chitrapremkumaar@yahoo.co.in

R. Sattanathan

Research and PG Dept, Dept. of Mathematics
D.G. Vaishnav College, Chennai, India
rsattanathan@gmail.com

Abstract

In a graph G, a subset S of V is Global vertex-edge dominating set if S is vertex-edge dominating set in both G and $\bar{G}$. In this paper we have introduced new concepts such as Global vertex-edge dominating set, Global vertex-edge irredundant set, Global independent vertex-edge dominating set. We have identified the Global vertex-edge domination number of some family of graphs such as K_n, $K_{1,n-1}$, $K_{n,m}$, P_n, and C_n. An attempt is made to identify the Global vertex-edge domination chain,

$$ir_{gve}(G) \leq \gamma_{gve}(G) \leq i_{gve}(G) \leq IR_{gve}(G) \leq \gamma_{gve}(G).$$

Keywords: Global vertex-edge dominating set, Global vertex-edge irredundant Set, Global independent vertex-edge domination set, Global vertex-edge domination chain, and Global vertex-edge domination number

1 Introduction

In this paper we shall present vertex-edge domination, vertex-edge irredundant, and vertex-edge independent parameters. We have introduced global vertex-edge domination, global vertex-edge irredundant, global vertex-edge independent sets in graphs.

[4] Let $G = (V, E)$ be a graph with a vertex set $V(G)$ and edge set $E(G)$. The set D is a dominating set if every vertex $v \in V$ is either an element of D or is adjacent to an element of D.

The minimum cardinality of a dominating set of G is said to be domination number and is denoted by $\gamma(G)$. The maximum cardinality of a minimal dominating set of a graph G is called the upper domination number and is denoted by $\Gamma(G)$. [4]

A set D is called independent if no two vertices in D are adjacent, if the set D is both independent and dominating then it is said to be independent dominating set of G. The minimum cardinality of an independent dominating set is called the independent domination number and is denoted by $i(G)$. [5]

The open neighborhood $N(x)$ of a vertex x is the set of vertices adjacent to x and the closed neighborhood of x is the set $N[x] = \{x\} \cup N(x)$. For a set $D \subseteq V(G)$, an element $x \in D$ if $N[x] - N[D - \{x\}] = \phi$ then x is said to be redundant in D. A set D of vertices is called irredundant if $N[x]-N[D-\{x\}] \neq \phi$ for every $x \in D$. If D is irredundant and $x \in D$ the set $N[x] - N[D - \{x\}]$ is called the set of private neighbors of x and is denoted by $P_n[x, D]$. [2]

K.W Peters introduced two new graph theory concepts, Vertex-edge domination and Edge-vertex domination. We can informally define vertex-edge domination by saying that a vertex v dominates the edges incident to v as well as the edges adjacent to those incident edges. Vertex-edge domination set is simply called as ve-dominating set. [6]

[6] A set $S \subseteq V(G)$ is a vertex-edge dominating set if for all edges $e \in E(G)$, there exist a vertex $v \in S$ such that v dominates e. Otherwise for a graph $G = (V, E)$ a vertex $uV(G)$ ve-dominates an edge $vw \in E(G)$ if
(i) $u = v$ or $u = w$ (u is incident to vw), or
(ii) uv or uw is an edge in G (u is incident to an edge is adjacent to vw).

Global vertex-edge domination:

[3] A subset S of V is global vertex-edge dominating set if S is vertex-edge dominating set in both G and $\bar{G}$. A subset S of V is said to be minimal global vertex-edge dominating set if no proper subset of S is a global vertex-edge dominating set of G.

The minimum cardinality of a global vertex-edge dominating set is called global vertex-edge domination number of a graph G and is denoted by $\gamma_{gve}(G)$. The maximum cardinality of a global vertex-edge dominating set is called as upper global vertex-edge domination number and is denoted by $\Gamma_{gve}(G)$.

Note : For any connected graph G the ve-domination number is $\gamma_{ve}(G)$ and for ve-domination number $\gamma_{ve}(\bar{G})$ then $\gamma_{gve}(G) = max\{\gamma_{ve}(G), \gamma_{ve}(\bar{G})\}$. [9]

Vertex-edge irredundant set:

[7] A vertex $v \in S \subseteq V(G)$ has a Private edge $e = uw \in E(G)$ (with respect to set S) if:
(i) v is incident to e or v is adjacent to either u or w.
(ii) For all vertices $x \in S - \{v\}$, x is not incident to e and x is not adjacent to either u or w, that is v dominates the edge e and no other vertex in S dominates e. A set S is a vertex-edge irredundant set (or simply ve-irredundant set) if every vertex $v \in S$ has a private edge.

The minimum cardinality of a maximal vertex-edge irredundant set is called vertex-edge Irredundant number of a graph G and is denoted by $ir_{ve}(G)$. The maximum cardinality of a vertex-edge irredundant set is called upper vertex-edge irredundant number and is denoted by $IR_{ve}(G)$.

Global vertex-edge irredundant Set:

A set $S \subseteq V(G)$ is global vertex-edge Irredundant set, (or simply gve-irredundant set) if S is ve-irredundant set both in G and $\bar{G}$. Simply for every $v \in S$ has a private edge in both G and $\bar{G}$ then S is said to be gve-irredundant set. [7]

The minimum cardinality of a maximal global vertex-edge Irredundant set is called global vertex-edge Irredundant number of a graph G and is denoted by $ir_{gve}(G)$. The maximum cardinality of a global vertex-edge Irredundant set is called as upper global vertex-edge irredundant number and is denoted by $IR_{gve}(G)$.

Independent vertex-edge domination set:

A set S is an independent vertex-edge dominating set if S both an independent set and minimal Ve-dominating set. The minimum cardinality of an independent ve-dominating set is called independent ve-domination number of a graph G and is denoted by $i_{ve}(G)$. [8]

Global independent vertex-edge domination set:

A set S is independent ve-dominating set both in G and $\bar{G}$ then it is said to be Global independent vertex-edge domination set. [6]

The minimum cardinality of an global independent ve-dominating set is called global independent ve-domination number of a graph G and is denoted by $i_{gve}(G)$.

2 Bounds

Proposition 2.1 *[1] Global vertex-edge domination number of complete graph, is given by* $\gamma_{gve}(K_n) = 1$.

Proof In a complete graph all the edges are adjacent to each other, such that only one vertex is needed to dominate all the edges. Hence ve-domination number of K_n is 1.

But in $\overline{K_n}$ all the vertices are isolated such that ve-domination number of $\overline{K_n}$ is 0. By the note we have $\gamma_{gve}(K_n) = 1$. □

Proposition 2.2 *[1] Global vertex-edge domination number of Star graph is given by* $\gamma_{gve}(K_{1,n-1}) = 2$.

Proof Consider any one of the pendent vertex of star graph $K_{1,n-1}$ as dominated vertex, then it dominates all the edges, such that $\gamma_{ve}(K_{1,n-1}) = 2$.

We know that the compliment of star graph is a disconnected graph of a isolated vertex and a complete graph of $n-1$ vertices. From the above observation, ve-domination number of K_n is 1 and the pendent vertex is also dominating. Implies $\gamma_{ve}(\overline{K_{1,n-1}}) = 2$. Hence $\gamma_{gve}(K_{1,n-1}) = 2$. □

Proposition 2.3 *Global vertex-edge domination number of complete bi-partite graph is given by* $\gamma_{gve}(K_{n,m}) = 2$. *[1]*

Proof In a complete bi-partite graph $K_{n,m}$ only one vertex is needed to dominate all the edges. Therefore $\gamma_{ve}(K_{n,m}) = 1$.

Now, let us consider $\overline{K_{n,m}}$ such that which is a disconnected component of two complete graphs.

We know that ve-domination number of K_n is 1. Implies $\gamma_{ve}(\overline{K_{n,m}}) = 2$.

From the note $\gamma_{gve}(K_{n,m}) = max\{\gamma_{ve}(K_{n,m}), \gamma_{ve}(\overline{K_{n,m}})\}$. Therefore $\gamma_{gve}(K_{n,m}) = 2$ □

Proposition 2.4 *[1] Global vertex-edge domination number of Path is given by* $\gamma_{gve}(P_n) = \left\lfloor \frac{n+2}{4} \right\rfloor$, *for all* $n \geq 6$ *and* $n+2 \equiv 0, 1, 2, 3(mod\ 4)$.

Proof Let us prove the result by method of mathematical induction. When $n = 6$, the result is true and is trivial from the following graph,

Therefore $\gamma_{gve}(P_6) = 2$.
Also $\gamma_{gve}(P_6) = \left\lfloor \frac{6+2}{4} \right\rfloor = \left\lfloor \frac{8}{2} \right\rfloor = 2$, $n + 2 \equiv 0(mod\ 4)$.
By induction hypothesis, let us assume that it is true for all $n = k$.
i.e, $\gamma_{gve}(P_k) = \left\lfloor \frac{k+2}{4} \right\rfloor$, $(k+2) \equiv 0, 1, 2, 3(mod\ 4)$
By conditional hypothesis if global ve-domination number of P_6 and P_k are true, then it is true for P_{k+1}.
$\gamma_{gve}(P_{k+1}) = \left\lfloor \frac{(k+1)+2}{4} \right\rfloor = \left\lfloor \frac{k+3}{4} \right\rfloor$, $(k+3) \equiv 0, 1, 2, 3(mod\ 4)$
Here the result is true since $n = k$ is a path with k vertices and $k-1$ edges, $n = k+1$ is a path with $k+1$ vertices and k edges, also for every $u, v \in S$, $d(u, v) \leq 4$.

if $(k+2) \equiv 0(mod\ 4)$, then $(k+3) \equiv 1(mod\ 4)$
if $(k+2) \equiv 1(mod\ 4)$, then $(k+3) \equiv 2(mod\ 4)$
if $(k+2) \equiv 2(mod\ 4)$, then $(k+3) \equiv 3(mod\ 4)$
if $(k+2) \equiv 3(mod\ 4)$, then $(k+3) \equiv 0(mod\ 4)$

Thus by mathematical induction it is true for all positive n, such that $\gamma_{gve}(P_n) = \left\lfloor \frac{n+2}{4} \right\rfloor$, for all $n \geq 6$ and $n + 2 \equiv 0, 1, 2, 3(mod\ 4)$. □

Proposition 2.5 *[1] Global vertex-edge domination number of Cycle is given by*

$$\gamma_{gve}(C_n) = \left\lfloor \frac{n+3}{4} \right\rfloor, \text{ for all } n \geq 5 \text{ and } n+3 \equiv 0, 1, 2, 3(mod\ 4).$$

Proof This is true from the proof of Proposition 2.4. □

Theorem 2.1 *[1] For any Simple connected graph* G, $1 \leq \gamma_{gve}(G) \leq \frac{m}{2}$ *for all* $n, m \geq 4$. *(order & size of graph)*

Proof Let G be a graph with vertex set $V(G)$ and edge set $E(G)$.
To prove the lower bound, Let G be a complete graph, then only one vertex is needed to dominate all the edges and the compliment is a graph of isolated vertices such that Domination number is zero.

$$\begin{aligned} \gamma_{gve}(K_n) &= max\{\gamma_{ve}(K_n), \gamma_{ve}(\overline{K_n})\} \\ &= max\{1, 0\} \end{aligned}$$

Implies $1 \leq \gamma_{gve}(G)$. Lower bound is proved.
To prove the upper bound, Let G be a simple connected graph with order n and size m. By the definition of vertex-edge domination a vertex $v \in S$ will dominate an edge which is an incident edge of v and adjacent to incident edge and also for every $u, v \in S$ $d(u,v) \leq 4$. This implies one vertex will dominate at least 2 edges of G (since G is simple). Therefore at most $\frac{m}{2}$ number of vertices is needed to dominate all m edges, $\gamma_{gve}(G) \leq \frac{m}{2}$.

Thus $1 \leq \gamma_{gve}(G) \leq \frac{m}{2}$ for all $n, m \geq 4$. □

3 Global ve-Domination Chain

Theorem 3.1 *For any graph G, $\gamma_{ve}(G) \leq \gamma_{gve}(G)$. [6]*

Proof Let G be a graph, $\gamma_{ve}(G)$ be the ve-domination number of G.
Let $u \in S$ be a ve-dominating vertex of G then it dominates an edge $vw \in E(G)$ if $u = v$ or $u = w$ and uv or uw is an edge adjacent to vw.
We know that adjacent edges of G are non adjacent in $\bar{G}$.
Hence the ve-domination of $\bar{G}$ may increase or remains the same.
Also $\gamma_{gve}(G) = max\{\gamma_{ve}(G), \gamma_{ve}(\bar{G})\}$. Therefore $\gamma_{ve}(G) \leq \gamma_{ve}(\bar{G})$. □

Theorem 3.2 *For any graph G, $\gamma_{gve}(G) \leq \gamma(G)$. [6]*

Proof Let G be a graph, $\gamma_{ve}(G)$ be the vertex-edge domination number and with domination number $\gamma(G)$. By the definition of ve-domination, for every $u, v \in S$, $d(u,v) \leq 4$.
Also for every $u \in S$, such that u dominates the edges incident to it and adjacent to the incident edge. But if $u \in D$, then by definition u dominates only the adjacent vertex and the incident edge, also for every $u, v \in D$, $d(u,v) \leq 3$.
Implies more number of vertices are needed to dominate all the edges of G.

Therefore it is clear that, $\gamma_{gve}(G) \leq \gamma(G)$. □

Theorem 3.3 *For any graph G, $\gamma_{ve}(G) \leq \gamma_{gve}(G) \leq \gamma(G)$.*

Proof This is true from Theorem 3.1 and 3.2. □

Theorem 3.4 *For any graph G, $ir_{gve}(G) \leq \gamma_{gve}(G)$.*

Proof Let G be a graph with global vertex-edge irredundant set $ir_{gve}(G)$. By definition, for every, $v \in S$ has a private edge. Private edge we mean an edge $e = uw \in E(G)$ of $v \in S$ such that v is incident to e or v is adjacent to u or w, also for all $x \in S - \{V\}$, x is not adjacent to either u or w. Implies for every $x, v \in S$ a vertex-edge irredundant set, then $d(x, v) = 4$, if $d(x, v) \leq 4$ then private edge does not exist. Hence we have $ir_{gve}(G) \leq \gamma_{gve}(G)$. □

Theorem 3.5 *For any graph G, $\gamma_{gve}(G) \leq i_{gve}(G)$.*

Proof Let G be a graph with independent global ve-dominating set $i_{gve}(G)$. We know that if $u, v \in S$ is independent in G then u, v are adjacent in $\bar{G}$. Implies $\gamma_{gve}(G)$ should be a minimum set, such that all the vertices of S satisfies the independent condition. Also we know that if the set S is independent then it is ve-dominating set, but the converse need not be true. Hence $\gamma_{gve}(G) \leq i_{gve}(G)$. □

Theorem 3.6 *For any graph G $ir_{gve}(G) \leq \gamma_{gve}(G) \leq i_{gve}(G) \leq IR_{gve}(G) \leq \Gamma_{gve}(G)$. [6]*

Proof This is true from Theorem 3.4 and 3.5. □

Conclusion

In this paper we have identified global vertex-edge domination number of some family of graphs, and identified the lower and upper bound. We have investigated how this parameter is related with many other parameters such as independent and irredundant sets, and identified the global vertex-edge domination chain. We shall explore the above parameters on product graphs as a part of our future work.

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