Bohr’s Inequality and its Extensions
in Banach ∗-Algebras

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Abstract
A classical Bohr inequality states that for complex numbers \(a, b\) and real numbers \(p, q > 1\) such that \(1/p + 1/q = 1\),
\[
|a + b|^2 \leq p|a|^2 + q|b|^2
\]
with equality if and only if \(b = (p - 1)a\). Over the years, various generalizations of Bohr’s inequality are established in the context of complex numbers, matrices and operators on a Hilbert space. In this paper, we propose a technique of reducing problems in operator algebra to problems in matrix theory for obtaining absolute value inequalities related to Bohr’s inequality in the framework of Banach ∗-algebras. The analogues of Bohr’s inequality and its extensions in the previous results are discovered. Moreover, we get some related absolute value inequalities for multiple elements.

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1 Introduction
In 1924, H. Bohr established the classical Bohr’s inequality [3] which asserts that
\[
|a + b|^2 \leq p|a|^2 + q|b|^2
\]
for complex numbers \(a, b\) and real numbers \(p, q > 1\) such that \(1/p + 1/q = 1\). Such \(p\) and \(q\) are termed conjugate exponents. The equality in (1) occurs if and only if \(pa = qb\) (i.e. \(b = (p - 1)a\) or \(a = (q - 1)b\)).
Over the years, various numbers of extensions and variations of Bohr’s inequality have been established. The results for complex numbers are obtained in [8, 9, 14]. The case of matrices is discussed in [2]. Hirzallah [7] first established this inequality in the context of operators acting on a Hilbert space by using direct computations. Later, Cheung and Pečarić [6] used the same technique to extend Hirzallah’s results. Then many authors [1, 5, 10, 15] further discuss various generalizations of operator Bohr’s inequality.

In this paper an extension of Bohr inequality in an abstract setting of Banach ∗-algebra (which includes the Hilbert space operators) and the best possibility of constant in the inequality are obtained. We obtain the analogue results of [5, 6, 7] with a very hand-free technique–reducing inequalities in Banach ∗-algebra to the positivity of associated matrices. For each inequality we also determine a necessary and sufficient condition for equality case. The technique presented here can be applied widely in order to obtain absolute value inequalities related to Bohr’s inequality.

An element \( a \) in a Banach ∗-algebra is called self-adjoint if \( a^* = a \). An element which has real spectrum is said to be hermitian. A Banach ∗-algebra is called hermitian if each self-adjoint element is hermitian. A class of hermitian Banach ∗-algebra includes any \( C^* \)-algebra, in particular, the \( C^* \)-algebra of continuous linear operators on a Hilbert space and the \( C^* \)-algebra of continuous complex-valued functions on a compact Hausdorff space. Moreover, any measure algebra of discrete group, any group algebra of an abelian group and any group algebra of a compact group are hermitian Banach ∗-algebras. Throughout this paper, \( \mathcal{A} \) denotes a hermitian Banach ∗-algebra.

Every hermitian Banach ∗-algebra is equipped with a natural order structure as follows. Given self-adjoint elements \( a, b \in \mathcal{A} \), the relation \( a \leq b \) means that \( b - a \) is self-adjoint and the spectrum of \( b - a \) is contained in the nonnegative real numbers. Then the relation “\( \leq \)” forms a partial order on the real vector space of self-adjoint elements in \( \mathcal{A} \). The set of \( a \in \mathcal{A} \) such that \( a \geq 0 \) forms a positive cone in \( \mathcal{A} \) (see [4, Lemma 41.4]).

The Shirali-Ford Theorem [12, Theorem 1] asserts that \( a^* a \geq 0 \) for any \( a \in \mathcal{A} \). Then we can define the absolute value of each \( a \in \mathcal{A} \) to be \( (a^* a)^{1/2} \). Then by the spectral mapping theorem, \( \sigma(|a|) \subseteq [0, \infty) \) and hence \( |a| \geq 0 \) for every \( a \in \mathcal{A} \). Note that \( |a| = 0 \) if and only if \( a = 0 \). Indeed, if \( |a| = 0 \), then \( a^* a = 0 \) which implies \( \|a^* a\| = 0 \) and by [13, Lemma 3], we obtain

\[
\|a^*\| |a| \leq 4\|a^* a\| = 0,
\]

i.e. \( a = 0 \).
2 Inequalities of Bohr’s type

Lemma 2.1. (i) Let \( a, b \in A \) and \( \alpha, \beta, \gamma \in \mathbb{C} \) such that \( \alpha \gamma \geq |\beta|^2 \). If \( \alpha, \gamma \geq 0 \), then \( \alpha |a|^2 + \beta a^* b + \bar{\beta} b^* a + \gamma |b|^2 \geq 0 \).

(ii) If \( \alpha |a|^2 + \beta a^* b + \bar{\beta} b^* a + \gamma |b|^2 \geq 0 \) for all \( a, b \in A \) and \( \alpha, \beta, \gamma \in \mathbb{C} \), then \( \alpha, \gamma \geq 0 \) and \( \alpha \gamma \geq |\beta|^2 \), i.e. the matrix

\[
\begin{pmatrix}
\alpha & \beta \\
\bar{\beta} & \gamma
\end{pmatrix}
\]

is positive semidefinite.

Proof. (i) Suppose that \( \alpha, \gamma \geq 0 \). If \( \beta = 0 \), we are done. If \( \beta \neq 0 \), then \( \alpha > 0 \) and \( \gamma > 0 \). Set \( \lambda = \alpha \gamma - |\beta|^2 \). Since \( \alpha = (\lambda + |\beta|^2)/\gamma \), it follows that

\[
\alpha |a|^2 + \beta a^* b + \bar{\beta} b^* a + \gamma |b|^2 = \frac{\lambda + |\beta|^2}{\gamma} |a|^2 + \beta a^* b + \bar{\beta} b^* a + \gamma |b|^2
\]

\[
= \frac{\lambda}{\gamma} |a|^2 + \left| \frac{\beta}{\sqrt{\gamma}} a + \sqrt{\gamma} b \right|^2
\]

\[\geq 0.\]

(ii) Suppose that matrix is positive semidefinite for all \( a, b \in A \). The special case \( a = 0 \) implies \( \gamma \geq 0 \). Similarly, \( b = 0 \) yields \( \alpha \geq 0 \). Consider four cases: (i) \( \alpha = \gamma = 0 \), (ii) \( \alpha = 0, \gamma > 0 \), (iii) \( \alpha > 0, \gamma = 0 \) and (iv) \( \alpha, \gamma > 0 \). Write \( \beta = x + iy \) for some \( x, y \in \mathbb{R} \). The case \( \alpha = \gamma = 0 \) implies that \( \beta a^* b + \bar{\beta} b^* a \geq 0 \) for all \( a, b \in A \). Putting \( a = b \neq 0 \) yields \( x \geq 0 \). Similarly, setting \( a = -b \neq 0 \) implies \( x \leq 0 \). Then putting \( b = ia \neq 0 \) and \( b = -ia \neq 0 \) yield \( y \leq 0 \) and \( y \geq 0 \), respectively. Hence, \( \beta = 0 \) and \( \alpha \gamma \geq |\beta|^2 \). Consider the case \( \alpha = 0, \gamma > 0 \). We have that for all \( a, b \in A \),

\[
\beta a^* b + \bar{\beta} b^* a + \gamma |b|^2 \geq 0.
\]

For each \( \mu > 0 \), by replacing \( a \) with \( (1/\sqrt{\mu}) a \) and \( b \) with \( \sqrt{\mu} b \) in (2), we get

\[
\beta a^* b + \bar{\beta} b^* a + \mu \gamma |b|^2 \geq 0.
\]

Since \( \mu \) is arbitrary, (2) holds for all \( \gamma > 0 \). Setting \( a = b \neq 0 \) in (2) now yields \( -2x \leq \gamma \) for all \( \gamma > 0 \) and so \( x > 0 \). Putting \( a = -b \neq 0 \) in (2) yields \( 2x \leq \gamma \) for all \( \gamma > 0 \) and so \( x \leq 0 \). Similarly, putting \( a = ib \neq 0 \) and \( a = -ib \neq 0 \) yield \( y \geq 0 \) and \( y \leq 0 \). So \( \beta = 0 \) and hence \( \alpha \gamma \geq |\beta|^2 \). The case (iii) is similar to (ii). For \( \alpha, \gamma > 0 \), putting \( b = -4(\sqrt{\alpha/\gamma})a \) and \( b = 4i(\sqrt{\alpha/\gamma})a \) yield \( x \leq \frac{5}{8} \sqrt{\alpha \gamma} \) and \( y \leq \frac{5}{8} \sqrt{\alpha \gamma} \), respectively and hence \( |\beta|^2 = x^2 + y^2 \leq \alpha \gamma \). \qed

This suggests us to transform the problem of determining the positive definiteness of quadratic form of elements in \( A \) to the problem of determining the positive definiteness of the associated matrices.
Lemma 2.2. Let \( a, b \in A \). For \( \alpha, \beta, \gamma \in \mathbb{C} \) such that \( \alpha, \gamma \geq 0 \) and \( \alpha \gamma \geq |\beta|^2 \), the equality
\[
\alpha |a|^2 + \beta a^* b + \overline{\beta}^* a + \gamma |b|^2 = 0 \tag{3}
\]
occurs if and only if one of the following conditions holds:

(i) \( \alpha = \gamma = 0 \),

(ii) \( a = b = 0 \),

(iii) \( a = 0 \) and \( \gamma = 0 \),

(iv) \( b = 0 \) and \( \alpha = 0 \),

(v) \( \alpha a + \beta b = 0 \) and \( \alpha \gamma = |\beta|^2 \neq 0 \) (i.e. \( \beta a + \gamma b = 0 \) and \( \alpha \gamma = |\beta|^2 \neq 0 \)).

Proof. The sufficiency is routine. For the necessity, there are 4 choices of \( \alpha, \gamma \).
If \( \alpha = 0 \) or \( \gamma = 0 \), we are done. If \( \alpha > 0, \gamma = 0 \), we get \( \beta = 0 \) and then \( a = 0 \).
If \( \alpha = 0, \gamma > 0 \), we get \( \beta = 0 \) and then \( b = 0 \). Now for the case \( \alpha > 0, \gamma > 0 \), let \( \lambda = \alpha \gamma - |\beta|^2 \).
We have \( \lambda \geq 0 \) and \( \gamma = (\lambda + |\beta|^2)/\alpha \). Hence
\[
\alpha |a|^2 + \beta a^* b + \overline{\beta}^* a + \gamma |b|^2 = \alpha |a|^2 + \beta a^* b + \overline{\beta}^* a + \frac{\lambda + |\beta|^2}{\alpha} |b|^2
\]
\[
= \left| \sqrt{\alpha} a + \frac{\beta}{\sqrt{\alpha}} b \right|^2 + \frac{\lambda}{\alpha} |b|^2.
\]
If \( \alpha \gamma = |\beta|^2 \), then \( \left| \sqrt{\alpha} a + (\beta/\sqrt{\alpha}) b \right| = 0 \), i.e., \( \alpha a + \beta b = 0 \). So we have \( \alpha a + \beta b = 0, \alpha \gamma = |\beta|^2 \) and \( \alpha \neq 0 \). But if \( \gamma = 0 \), we have \( \beta = 0 \) and \( a = 0 \) which is included in (iii). Therefore, we can simplify the condition \( \alpha a + \beta b = 0, \alpha \gamma = |\beta|^2 \) and \( \alpha \neq 0 \) to the condition \( \alpha a + \beta b = 0 \) and \( \alpha \gamma = |\beta|^2 \neq 0 \) which is equivalent to the condition \( \beta a + \gamma b = 0 \) and \( \alpha \gamma = |\beta|^2 \neq 0 \). If \( \alpha \gamma > |\beta|^2 \), then \( \left| \sqrt{\alpha} a + (\beta/\sqrt{\alpha}) b \right| = |b| = 0 \) and hence \( a = b = 0 \).

The Bohr’s inequality is generalized to the context of hermitian Banach *-algebras, where the condition on conjugate exponents \( p, q > 1 \) is replaced by \( pq > 0 \), in the next theorem.

Theorem 2.3. Let \( a, b \in A \) and \( p, q \in \mathbb{R} \) such that \( 1/p + 1/q = 1 \).

(i) If \( pq > 0 \), then
\[
|a + b|^2 \leq p |a|^2 + q |b|^2 \tag{4}
\]
with equality if and only if \( pa = qb \) (i.e. \( b = (p-1)a \) or \( a = (q-1)b \)).
(ii) If \( pq < 0 \), then
\[
|a + b|^2 \geq p|a|^2 + q|b|^2
\](5)

with equality if and only if \( pa = qb \).

**Proof.** (i) Assume \( pq > 0 \). We have the identity
\[
|a + b|^2 = |a|^2 + (a^*b + b^*a) + |b|^2.
\]
So, we get
\[
p|a|^2 + q|b|^2 - |a+b|^2 = (p-1)|a|^2 - (a^*b + b^*a) + (q-1)|b|^2.
\]

In the view of Lemma 2.1, it suffices to show that
\[
X := \begin{pmatrix} p-1 & -1 \\ -1 & q-1 \end{pmatrix} \geq 0.
\]

If \( 0 < p < 1 \), then \( q < 0 \) and \( pq < 0 \), a contradiction. If \( p < 0 \), then \( q > 0 \)
which is impossible. Then \( p > 1 \) and \( q > 1 \). Since \( (p-1)(q-1) = 1 \), the
matrix \( X \) is positive semidefinite, i.e., (4) holds. If follows from Lemma 2.2
that the equality in (4) occurs if and only if one of the following holds:

(i) \( a = b = 0 \),

(ii) \( (p-1)a - b = 0 \) and \( (p-1)(q-1) = (-1)^2 \neq 0 \).

From the hypothesis, this is equivalent to \( a = (q-1)b \) or \( pa = qb \). The proof
of (ii) is similar to (i).

The Bohr inequality (4) can be stated equivalently
\[
|a + b|^2 \leq (1 + t)|a|^2 + (1 + \frac{1}{t})|b|^2
\](6)

for any \( t > 0 \). In this case, the equality holds if and only if \( b = ta \).

We would like to find the maximum value of a real constant \( c \) for which (4)
holds for all \( a, b \in \mathcal{A} \). The best possibility of the constant is discussed in the
next proposition.

**Proposition 2.4.** (i) For each \( p, q \in \mathbb{R} \) such that \( p+q > 0 \), the maximum
value of \( c \in \mathbb{R} \) for which the inequality
\[
c|a + b|^2 \leq p|a|^2 + q|b|^2.
\](7)

holds for all \( a, b \in \mathcal{A} \) is equal to \( pq/(p+q) \). Moreover, if \( c = pq/(p+q) \),
then the equality holds if and only if \( pa = qb \).
(ii) For each \( p, q \in \mathbb{R} \) such that \( p + q < 0 \), the minimum value of \( c \in \mathbb{R} \) for which the inequality
\[
  c|a + b|^2 \geq p|a|^2 + q|b|^2.
\]
holds for all \( a, b \in A \) is equal to \( pq/(p + q) \).

Proof. (i) Suppose \( c \) satisfies \( (7) \) for all \( a, b \in A \). Define \( X \) to be the matrix
\[
\begin{pmatrix}
  c - p & c \\
  c & c - q
\end{pmatrix}.
\]
By the second part of Lemma 2.2, the matrix \( X \) is necessarily negative semidefinite, i.e., \( c - p \leq 0 \), \( c - q \leq 0 \) and \((c - p)(c - q) \geq c^2 \). Hence, we have
\[
c \leq \min (p, q, pq/(p + q)) = pq/(p + q).
\]
and therefore the maximum value of \( c \) is equal to \( pq/(p + q) \). If \( c = pq/(p + q) \), then by direct computation, the equality in \( (7) \) is valid if and only if \( pa = qb \). The proof of (ii) is similar to (i). \( \square \)

So, the constant 1 in (4) and (5) is the optimal constant. This shows the sharpness of the inequalities. As a consequence result of Theorem 2.3, we get an analogue result of [6, Corollary 3].

Corollary 2.5. Let \( a, b \in A \) and \( t > 0 \). Then
\[
(i) \quad a^*b + b^*a \leq t|a|^2 + \frac{1}{t}|b|^2 \text{ with equality if and only if } b = ta,
\]
\[
(ii) \quad -(a^*b + b^*a) \leq t|a|^2 + \frac{1}{t}|b|^2 \text{ with equality if and only if } b = -ta.
\]

Proof. (i) Set \( t = p - 1 \) in Theorem 2.3 (i) and use the fact that \((p - 1)(q - 1) = 1 \).
To prove (ii), replace \( a \) with \( -a \) in (i). \( \square \)

The Bohr’s inequality is extended to all possible cases of conjugate exponents in the following theorems. The analogue results for the case of operators on a Hilbert space are obtained in [6] (cf. Theorem 2, Theorem 1 and Corollary 1 in [6], respectively).

Theorem 2.6. Let \( a, b \in A \) and \( p, q \) real numbers such that \( 1/p + 1/q = 1 \).

(i) If \( p < 1 \), then
\[
|a - b|^2 + |(p - 1)a + b|^2 \geq p|a|^2 + q|b|^2,
\]
\[
|a - b|^2 + |a + (q - 1)b|^2 \geq p|a|^2 + q|b|^2,
\]
with equality if and only if \( b = (1 - p)a \).
(ii) If $1 < p \leq 2$, then
\[ |a - b|^2 + |(p - 1)a + b|^2 \leq p|a|^2 + q|b|^2, \]  
\[ |a - b|^2 + |a + (q - 1)b|^2 \geq p|a|^2 + q|b|^2, \]  
with equality if and only if $p = q = 2$ or $b = (1 - p)a$.

(iii) If $p > 2$, then
\[ |a - b|^2 + |(p - 1)a + b|^2 \geq p|a|^2 + q|b|^2, \]  
\[ |a - b|^2 + |a + (q - 1)b|^2 \leq p|a|^2 + q|b|^2, \]  
with equality if and only if $b = (1 - p)a$.

Proof. The proofs of (i)-(iii) are similar and so we prove (i) only. By expanding, we have
\[
|a - b|^2 + |(p - 1)a + b|^2 = |a|^2 - (a^*b + b^*a) + |b|^2 + (p - 1)^2|a|^2 + (p - 1)(a^*b + b^*a) + |b|^2 - p|a|^2 - q|b|^2
\]
\[= (p^2 - 3p + 2)|a|^2 + (p - 2)(a^*b + b^*a) + (2 - q)|b|^2.\]

We can check that
\[
\left( \begin{array}{cc} p^2 - 3p + 2 & p - 2 \\ p - 2 & 2 - q \end{array} \right) \leq 0.
\]

By Lemma 2.1, (9) holds. In the view of Lemma 2.2, the equality in (9) holds if and only if one of the following holds:

(i) $a = b = 0$,

(ii) $(p^2 - 3p + 2)a + (p - 2)b = 0$ and $(p^2 - 3p + 2)(2 - q) = (p - 2)^2 \neq 0$.

which can be simplified to the single condition $b = (1 - p)a$.

A proof of (10) is similar to that of (9). In this case, we get
\[
|a - b|^2 + |a + (q - 1)b|^2 - p|a|^2 - q|b|^2
\]
\[= (2 - p)|a|^2 + (q - 2)(a^*b + b^*a) + (q^2 - 3q + 2)|b|^2.
\]

We can check that
\[
\left( \begin{array}{cc} 2 - p & q - 2 \\ q - 2 & q^2 - 3q + 2 \end{array} \right) \geq 0,
\]
which means (10) holds. The equality case in (10) is obtained via 2.2. Repeating the above procedure yields that the equality holds if and only if $(q - 2)a + (q^2 - 3q + 2)b = 0$, which is $a = (1 - q)b$, i.e., $b = (1 - p)a$. \qed
Remark 2.7. The exactly Bohr inequality can be obtained from (11) and (14).

Next we consider inequalities in a more general form, namely, $|sa + tb|^2 \leq p|a|^2 + q|b|^2$ where $s, t, p, q$ are constants.

Theorem 2.8. Let $a, b \in \mathcal{A}$ and $s, t \in \mathbb{C}$, $p, q \in \mathbb{R}\{0\}$ such that $\frac{|a|^2}{p} + \frac{|b|^2}{q} \leq 1$.

1. If $|s|^2 \leq p$ and $|t|^2 \leq q$, then

$$|sa + tb|^2 \leq p|a|^2 + q|b|^2. \quad (15)$$

2. If $|s|^2 \geq p$ and $|t|^2 \geq q$, then

$$|sa + tb|^2 \geq p|a|^2 + q|b|^2. \quad (16)$$

In both cases, equalities if and only if one of the following occurs:

(i) $a = b = 0$,
(ii) $a = 0$ and $q = |t|^2$,
(iii) $b = 0$ and $p = |s|^2$,
(iv) $(p - |s|^2)a = \bar{st}b$ and $|st| = \sqrt{(p - |s|^2)(q - |t|^2)}$ (equivalently, $\bar{s}ta = (q - |t|^2)b$ and $|st| = \sqrt{(p - |s|^2)(q - |t|^2)}$).

Proof. The proofs of (15) and (16) are similar. For the first part, by expanding $|sa + tb|^2$, we get

$$p|a|^2 + q|b|^2 - |sa + tb|^2 = (p - |s|^2)|a|^2 - \bar{st}a^*b - \bar{s}t^*a + (q - |t|^2)|b|^2.$$

Since $|s|^2/p + |t|^2/q \leq 1$, we have $(p - |s|^2)(q - |t|^2) - \bar{st}t \geq 0$. By Lemma 2.1 we arrive (15). For the case of equality, the sufficiency is routine. In the necessity, there are only 3 possible cases: (i) $p > |s|^2, q = |t|^2$, (ii) $p = |s|^2, q > |t|^2$ and (iii) $p > |s|^2, q > |t|^2$. For $p > |s|^2, q = |t|^2$, by Lemma 2.2, we get $a = 0$ and $q = |t|^2$. For $p = |s|^2, q > |t|^2$, by Lemma 2.2, we get $b = 0$ and $p = |s|^2$. Now for $p > |s|^2, q > |t|^2$, it can occurs in 2 ways; $st = 0$ and $|st| > 0$. If $st = 0$, it can be referred to part (ii) in Lemma 2.2 that $a = b = 0$. If $|st| > 0$, we obtain via Lemma 2.2 that $(p - |s|^2)a = \bar{st}b$ and $|st| = \sqrt{(p - |s|^2)(q - |t|^2)}$.

Corollary 2.9. The map $a \mapsto |a|^2$ is convex on $\mathcal{A}$, that is, for each $a, b \in \mathcal{A}$ and $t \in (0, 1)$,

$$|ta + (1-t)b|^2 \leq t|a|^2 + (1-t)|b|^2.$$
Remark 2.10. The map $a \mapsto |a|$ is not necessarily convex on $A$, even if $A$ is the set of 3-by-3 complex matrices.

Next we consider an inequality in more general form; compare $\sum_{i=1}^{n} |\alpha_i a + \beta_i b|^2$ with $p|a|^2 + q|b|^2$.

Theorem 2.11. Let $a, b \in A$, $\lambda_i, \mu_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$ and $p, q \in \mathbb{R}$. Set

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

where

$$x_{11} = \sum_{i=1}^{n} |\lambda_i|^2, \quad x_{12} = \sum_{i=1}^{n} \lambda_i \mu_i, \quad x_{22} = \sum_{i=1}^{n} |\mu_i|^2.$$ 

(a) If $X \succeq Y$, then $\sum_{i=1}^{n} |\lambda_i a + \mu_i b|^2 \geq p|a|^2 + q|b|^2$.

(b) If $X \preceq Y$, then $\sum_{i=1}^{n} |\lambda_i a + \mu_i b|^2 \leq p|a|^2 + q|b|^2$.

In both cases, the equality holds if and only if one of the following occurs:

(i) $x_{11} = p$ and $x_{22} = q$,

(ii) $a = b = 0$,

(iii) $a = 0$ and $x_{22} = q$,

(iv) $b = 0$ and $x_{11} = p$,

(v) $(x_{11} - p)a + (x_{12} - q)b = 0$ and $(x_{11} - p)(x_{12} - q) = |x_{12}|^2 \neq 0$.

Proof. (a) Expanding $\sum_{i=1}^{n} |\lambda_i a + \mu_i b|^2$, we get

$$\sum_{i=1}^{n} |\lambda_i a + \mu_i b|^2 - p|a|^2 - q|b|^2$$

$$= (|\lambda_1|^2 + \cdots + |\lambda_n|^2 - p)|a|^2 + (\lambda_1 \mu_1 + \cdots + \lambda_n \mu_n) a^* b$$

$$+ (\lambda_1 \mu_1^* + \cdots + \lambda_n \mu_n^*) b^* a + (|\mu_1|^2 + \cdots + |\mu_n|^2 - q)|b|^2$$

$$= (x_{11} - p)|a|^2 + x_{12} a^* b + \overline{x_{12}} b^* a + (x_{22} - q)|b|^2.$$ 

Since $X \succeq Y$, we have $x_{11} - p \geq 0, x_{22} - q \geq 0$ and $(x_{11} - p)(x_{22} - q) \geq |x_{12}|^2$.

It follows from Lemma 2.1 that (a) holds. The case of equality can be obtained from Lemma 2.2. Part (b) can be proved in the similar way. $\square$
This idea of this theorem is very useful. In order to obtain absolute value inequalities, it suffices to consider the inequalities between two representative matrices, namely, \( X \) and \( Y \) in this theorem. So, with appropriate values of the parameters \( \lambda_i, \mu_i, p, q \), we get beautiful inequalities.

\textbf{Theorem 2.12.} Let \( a, b \in A, \alpha_i, \beta_i \in \mathbb{C} \) for \( i = 1, 2, \ldots, n \) and \( \lambda_j, \mu_j \in \mathbb{C} \) for \( j = 1, 2, \ldots, m \). Set

\[
\Delta_1 = \sum_{i=1}^{n} |\alpha_i|^2, \quad \Omega_1 = \sum_{i=1}^{n} |\beta_i|^2, \quad \Theta_1 = \sum_{i=1}^{n} \alpha_i \beta_i,
\]

\[
\Delta_2 = \sum_{j=1}^{m} |\lambda_j|^2, \quad \Omega_2 = \sum_{j=1}^{m} |\mu_j|^2, \quad \Theta_2 = \sum_{j=1}^{m} \lambda_j \mu_j.
\]

If \( \Delta_1 \geq \Delta_2 \), \( \Omega_1 \geq \Omega_2 \) and \( \Theta_1 = \Theta_2 \), then

\[
\sum_{i=1}^{n} |\alpha_i a + \beta_i b|^2 \geq \sum_{j=1}^{m} |\lambda_j a + \mu_j b|^2 \tag{17}
\]

with equality if and only if one of the following holds:

(i) \( \Delta_1 = \Delta_2 \) and \( \Omega_1 = \Omega_2 \),
(ii) \( a = b = 0 \),
(iii) \( a = 0 \) and \( \Omega_1 = \Omega_2 \),
(iv) \( b = 0 \) and \( \Delta_1 = \Delta_2 \).

\textbf{Proof.}

\[
\sum_{i=1}^{n} |\alpha_i a + \beta_i b|^2 - \sum_{j=1}^{m} |\lambda_j a + \mu_j b|^2
\]

\[
= \left( \sum_{i=1}^{n} |\alpha_i|^2 - \sum_{j=1}^{m} |\lambda_j|^2 \right) |a|^2 + \left( \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{j=1}^{m} \lambda_j \mu_j \right) a^* b
\]

\[
+ \left( \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{j=1}^{m} \lambda_j \mu_j \right) b^* a + \left( \sum_{i=1}^{n} |\beta_i|^2 - \sum_{j=1}^{m} |\mu_j|^2 \right) |b|^2
\]

\[
= (\Delta_1 - \Delta_2) |a|^2 + (\Theta_1 - \Theta_2) a^* b + (\Theta_1 - \Theta_2) b^* a + (\Omega_1 - \Omega_2) |b|^2.
\]

Since \( \Delta_1 - \Delta_2 \geq 0, \Omega_1 - \Omega_2 \geq 0 \) and \( (\Delta_1 - \Delta_2)(\Omega_1 - \Omega_2) \geq 0 = |\Theta_1 - \Theta_2|^2 \), by making use of Lemma 2.1, we obtain (17). The case of equality follows from Lemma 2.2.

For the case \( n = m = 1 \), we obtain:
Corollary 2.13. Let $a, b \in \mathcal{A}$ and $\alpha, \beta, \lambda, \mu \in \mathbb{C}$. If $|\alpha| \geq |\lambda|, |\beta| \geq |\mu|$ and $\overline{\alpha} \mu = \overline{\beta} \lambda$, then
\[|\alpha a + \beta b|^2 \geq |\lambda a + \mu b|^2\]with equality if and only if one of the following holds:
(i) $|\alpha| = |\lambda|$ and $|\beta| = |\mu|$.
(ii) $a = b = 0$.
(iii) $a = 0$ and $|\beta| = |\mu|$.
(iv) $b = 0$ and $|\alpha| = |\lambda|$.

For each $a$ in a hermitian Banach $*$-algebra, the relation $a > 0$ means that $a \geq 0$ and $0 \notin \sigma(a)$. Recall that if a hermitian Banach $*$-algebra $\mathcal{A}$ has a unit, we define $a^z$ for each $a \in \mathcal{A}$ such that $\sigma(a) \subset (0, \infty)$ and $z \in \mathbb{C}$ by $a^z = \exp(z \log a)$ where log is the principal value of the complex logarithm. The Löwner-Heinz inequality [11, Theorem 2] asserts the monotonicity of the map $a \mapsto a^r$, i.e. $0 < a \leq b$ implies $a^r \leq b^r$, for any $0 \leq r \leq 1$ when the involution on $\mathcal{A}$ is continuous with respect to the norm topology.

Corollary 2.14. Let $\mathcal{A}$ be a unital hermitian Banach $*$-algebra with norm-continuous involution and $a, b \in \mathcal{A}$. Let $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ be such that $|\alpha| \geq |\lambda|, |\beta| \geq |\mu|$ and $\overline{\alpha} \mu = \overline{\beta} \lambda$. If $0 \notin \sigma(|\alpha a + \beta b|) \cap \sigma(|\lambda a + \mu b|)$, then for any $r \in [0, 2]$
\[|\alpha a + \beta b|^r \geq |\lambda a + \mu b|^r.\]

Proof. By spectral mapping theorem, $\sigma(|\alpha a + \beta b|) = \{k \in \mathbb{C} : k \in \sigma(\alpha a + \beta b)\} \subset (0, \infty)$ and hence $|\alpha a + \beta b| > 0$ and, similarly, $|\lambda a + \mu b| > 0$. The desired result follows via applying the Löwner-Heinz inequality to (18). ☐

3 Related absolute value inequalities for multiple elements

Lemma 3.1. For $a_i \in \mathcal{A}$ ($i = 1, 2, \ldots, n$), we have the following identities
\[\left| \sum_{i=1}^{n} a_i \right|^2 = \sum_{i=1}^{n} |a_i|^2 + \sum_{1 \leq i < j \leq n} a_i^* a_j + a_j^* a_i\] \tag{19}\]
\[\sum_{1 \leq i < j \leq n} |a_i - a_j|^2 = (n - 1) \sum_{i=1}^{n} |a_i|^2 - \sum_{1 \leq i < j \leq n} a_i^* a_j + a_j^* a_i.\] \tag{20}
Proof. Expand.

**Corollary 3.2.** For $a_i \in A$ ($i = 1, 2, \ldots, n$), the inequality

$$\left| \sum_{i=1}^{n} a_i \right|^2 \leq n \sum_{i=1}^{n} |a_i|^2$$

(21)

holds with equality if and only if all $a_i$'s are equal.

**Proof.** The identities (19) and (20) imply

$$n \sum_{i=1}^{n} |a_i|^2 - \sum_{i=1}^{n} |a_i|^2 = n \sum_{i=1}^{n} |a_i|^2 - \sum_{1 \leq i < j \leq n} a_i a_j + a_j a_i$$

$$= (n - 1) \sum_{i=1}^{n} |a_i|^2 - \sum_{1 \leq i < j \leq n} a_i a_j + a_j a_i$$

$$= \sum_{1 \leq i < j \leq n} |a_i - a_j|^2$$

$$\geq 0.$$ 

Hence, we arrive at (21). Moreover, the equality in (21) occurs if and only if $a_i - a_j = 0$ for $i \neq j$ which is $a_i = a_j$ for $i \neq j$. 

**Remark 3.3.** For each $i = 1, 2, \ldots, n$, let $a_i, b_i \in A$ and let $p_i, q_i \in \mathbb{R}$ be such that $1/p_i + 1/q_i = 1$. It is easy to see that if $p_i q_i > 0$ for all $i$, then

$$\sum_{i=1}^{n} |a_i + b_i|^2 \leq \sum_{i=1}^{n} \left( p_i |a_i|^2 + q_i |b_i|^2 \right).$$

(22)

and the inequality is reverse if $p_i q_i < 0$ for all $i$. The computation shows that equalities hold if and only if $b_i = (p_i - 1)a_i$ for all $i$.

The next theorem is an extension of Bohr’s inequality for multiple elements. This result generalizes [6, Theorem 4].

**Theorem 3.4.** For any integer $n > 2$, let $a_i \in A$ ($i = 1, 2, \ldots, n$) and $p_{ij}, q_{ij} \in \mathbb{R}$ such that $1/p_{ij} + 1/q_{ij} = 1$ for $1 \leq i < j \leq n$.

(i) If $p_{ij} > 1$ for all $1 \leq i < j \leq n$, then

$$\left| \sum_{i=1}^{n} a_i \right|^2 \leq \left( \sum_{j=2}^{n} p_{1j} + 2 - n \right) |a_1|^2 + \sum_{k=2}^{n-1} \left( \sum_{j=k+1}^{n} p_{kj} + \sum_{j=1}^{k-1} q_{jk} + 2 - n \right) |a_k|^2$$

$$+ \left( \sum_{j=1}^{n-1} q_{jn} + 2 - n \right) |a_n|^2.$$

(23)
(ii) If \( p_{ij} < 1 \) for all \( 1 \leq i < j \leq n \), then the reverse of (23) is obtained.

Moreover, all equalities hold if and only if \( a_j = (p_{ij} - 1)a_i \) for all \( 1 \leq i < j \leq n \).

**Proof.** We shall prove only (i) since the proof of (ii) is similar to that of (i). From (19) in Lemma 3.1, we have

\[
\left| \sum_{i=1}^{n} a_i \right|^2 - \sum_{i=1}^{n} |a_i|^2 = \sum_{1 \leq i < j \leq n} (a_i^* a_j + a_j^* a_i)
\]

\[
= \sum_{1 \leq i < j \leq n} \left[ |a_i + a_j|^2 - (|a_i|^2 + |a_j|^2) \right].
\]

The Bohr’s inequality and Remark 3.3 yield

\[
\sum_{1 \leq i < j \leq n} \left[ |a_i + a_j|^2 - (|a_i|^2 + |a_j|^2) \right] \leq \sum_{1 \leq i < j \leq n} \left[ (p_{ij} - 1)|a_i|^2 + (q_{ij} - 1)|a_j|^2 \right]
\]

with equality if and only if \( (p_{ij} - 1)a_i = a_j \) for all \( 1 \leq i < j \leq n \). Denote \( \tilde{p}_{ij} = p_{ij} - 1 \) and \( \tilde{q}_{ij} = q_{ij} - 1 \) for each \( i, j \). Hence

\[
\left| \sum_{i=1}^{n} a_i \right|^2 \leq \sum_{i=1}^{n} |a_i|^2 + \sum_{1 \leq i < j \leq n} [\tilde{p}_{ij}|a_i|^2 + \tilde{q}_{ij}|a_j|^2]
\]

\[
= |a_1|^2 + \sum_{j=2}^{n} \tilde{p}_{1j}|a_1|^2 + \sum_{j=1}^{n-1} \tilde{q}_{jn}|a_n|^2 + |a_n|^2 + \sum_{k=2}^{n-1} \left( 1 + \sum_{j=k+1}^{n} \tilde{p}_{kj} + \sum_{j=1}^{k-1} \tilde{q}_{jk} \right)|a_k|^2
\]

\[
= \left( 1 + \sum_{j=2}^{n} \tilde{p}_{1j} \right)|a_1|^2 + \left( 1 + \sum_{j=1}^{n-1} (q_{jn} - 1) \right)|a_n|^2
\]

\[
+ \sum_{k=2}^{n-1} \left( \sum_{j=k+1}^{n} p_{kj} + \sum_{j=1}^{k-1} q_{jk} + 2 - n \right)|a_k|^2
\]

\[
= \left( \sum_{j=2}^{n} p_{1j} + 2 - n \right)|a_1|^2 + \left( \sum_{j=1}^{n-1} q_{jn} + 2 - n \right)|a_n|^2
\]

\[
+ \sum_{k=2}^{n-1} \left( \sum_{j=k+1}^{n} p_{kj} + \sum_{j=1}^{k-1} q_{jk} + 2 - n \right)|a_k|^2,
\]

with equality if and only if \( (p_{ij} - 1)a_i = a_j \) for all \( 1 \leq i < j \leq n \). \( \square \)

This result has a pattern. An easy way to remember is to use the rectangular array as follows. For the case \( n = 4 \), write the 4-by-4 rectangular array of \( p_{ij} \) without the main diagonal entry. For \( n = 4 \), there are \( n - 1 = 3 \) coefficients...
of \( p_{ij} \) and \( q_{ij} \) to sum and we have to subtract them with \( n-2 = 2 \). For each entry below the main diagonal, interchange it from \( p_{ij} \) to \( q_{ji} \). So, we obtain

\[
\begin{bmatrix}
p_{12} & p_{13} & p_{14} \\
q_{12} & p_{23} & p_{24} \\
q_{13} & q_{23} & p_{34} \\
q_{14} & q_{24} & q_{34}
\end{bmatrix}.
\]

Here, the coefficients \( p_{ij} \) and \( q_{ij} \) of \(|a_i|^2\) appear in the \( i \)th row of resulting array. The resulting inequality is for \( a_1, a_2, a_3, a_4 \in \mathcal{A} \) and \( p_{ij}, q_{ij} > 1 \) such that \( 1/p_{ij} + 1/q_{ij} = 1 \) for \( 1 \leq i < j \leq 4 \), we have

\[
|a_1 + a_2 + a_3 + a_4|^2 \leq (p_{12} + p_{13} + p_{14} - 2)|a_1|^2 + (p_{23} + p_{24} + q_{12} - 2)|a_2|^2
\]
\[
\quad + (p_{34} + q_{13} + q_{23} - 2)|a_3|^2 + (q_{14} + q_{24} + q_{34} - 2)|a_4|^2
\]

with equality if and only if \( (p_{12} - 1)a_1 = a_2, (p_{13} - 1)a_1 = a_3, (p_{14} - 1)a_1 = a_4, (p_{23} - 1)a_2 = a_3, (p_{24} - 1)a_2 = a_4 \) and \( (p_{34} - 1)a_3 = a_4 \). The method can be used for arbitrary \( n > 2 \).

**Remark 3.5.** Corollary 3.2 can be obtained from Theorem 3.4 by setting \( p_{ij} = q_{ij} = 2 \) for all \( 1 \leq i < j \leq n \).

**References**


Bohr’s inequality and its extensions in Banach \(*\)-algebras


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