International Mathematical Forum, Vol. 7, 2012, no. 51, 2545 - 2548

Eigenfunctions of Certain Weighted Composition Operators on Hilbert Function Spaces

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Abstract

In this paper we characterize the eigenfunctions of weighted composition operators acting on Hilbert spaces of analytic functions.

Mathematics Subject Classification: 47B37, 47A25

Keywords: Hilbert spaces of analytic functions, reproducing kernels, weighted composition operator, Farrell-Rubel-Shields Theorem

1 Introduction

Let H be a Hilbert space of analytic functions on the open unit disk U such that for each $z \in U$, the evaluation function $e_\lambda : H \to C$ defined by $e_\lambda(f) = f(\lambda)$ is bounded on H. By the Riesz Representation Theorem there is a vector $k_z \in H$ such that $f(z) =< f, k_z >$ for every $z \in U$. Furthermore, we assume that H contains the constant functions and multiplication by the independent variable z defines a bounded linear operator M_z on H. The operator M_z is called *polynomially bounded* on H if there exists a constant $c > 0$ such that $||M_p|| \le c||p||_U$ for every polynomial p. Here $||p||_U$ denotes the supremum norm of p on U. It is well-known that any operator which is similar to a contraction is polynomially bounded. A complex valued function φ on U for which $\varphi H \subseteq H$ is called multiplier of H. The set of all multipliers of H is denoted by $M(H)$ and it is well-known that $M(H) \subseteq H^\infty(U)$ ([7]). Moreover, we suppose that ψ is a holomorphic self-map of U that is not an elliptic automorphism, and φ is a nonzero multiplier of H which is defined as radial limit at the Denjoy-Wolff point of ψ. The weighted composition operator $C_{\varphi,\psi}$ acting on H is defined by $C_{\varphi,\psi} = M_\varphi C_\psi$. The adjoint of a composition operator and a multiplication operator has not been yet well characterized on holomorphic functions. Nevertheless their action on reproducing kernels is determined. In fact $C_\psi^*(k_z) = k_{\psi(z)}$ and $M_\varphi^*(k_z) = \overline{\varphi(z)}k_z$ for every $z \in U$. Thus for each f in H and $z \in U$, we have

$$C_{\varphi,\psi}^* k_z = C_\psi^* M_\varphi^* k_z = \overline{\varphi(z)} C_\psi^* k_z = \overline{\varphi(z)} k_{\psi(z)}.$$

For some sources related to the topics of this paper one can see [1–8].

2 Main Result

In the main theorem of this paper we characterize the eigenfunctions of weighted composition operators acting on Hilbert spaces of analytic functions. The holomorphic self maps of the open unit disc U are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in U. The maps of that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem ([6]). By ψ_n we denote the nth iterate of ψ and by $\psi'(w)$ we denote the angular derivative of ψ at $w \in \partial U$. Note that if $w \in U$, then $\psi'(w)$ has the natural meaning of derivative.

Theorem 2.1 *(Denjoy-Wolff Iteration Theorem) Suppose ψ is a holomorphic self-map of U that is not an elliptic automorphism.*
(i) If ψ has a fixed point $w \in U$, then $\psi_n \to w$ uniformly on compact subsets of U, and $|\psi'(w)| < 1$.
(ii) If ψ has no fixed point in U, then there is a point $w \in \partial U$ such that $\psi_n \to w$ uniformly on compact subsets of U, and the angular derivative of ψ exists at w, with $0 < \psi'(w) \leq 1$.

We call the unique attracting point w, the Denjoy-Wolff point of ψ.

Theorem 2.2 *Let $||\psi||_U < 1$ and φ be nonzero on the Denjoy-Wolff point of ψ. If M_z is polynomially bounded, then*

$$\prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n)$$

is an eigenfunction of the weighted composition operator $C_{\varphi,\psi}$ acting on H.

Proof. First suppose that $\psi(0) = 0$. Since the closure of $\psi(U)$ is contained in U, there exists $0 < \lambda < 1$ such that $\psi(U) \subseteq \lambda\, U$. By the Schwarz's Lemma we have $|\psi(z)| \leq \lambda |z|$ for all z in U. On the other hand, since φ is bounded, an application of Schwartz's Lemma shows that there exist some constant $M > 0$ such that

$$|\varphi(0) - \varphi(z)| < M|z|$$

and consequently

$$|\varphi(z)| < M|z| + |\varphi(0)|$$

for every $z \in U$. It follows that if $z \in U$, then

$$|\varphi(\psi_n(z))| < M|\psi_n(z)| + |\varphi(0)| \leq M\lambda^n|z| + |\varphi(0)|.$$

But $\varphi(0) \neq 0$, thus

$$\begin{aligned} \frac{|\varphi(\psi_n(z))|}{|\varphi(0)|} &< M\lambda^n \frac{|z|}{|\varphi(0)|} + 1 \\ &\leq \exp(M\lambda^n \frac{|z|}{|\varphi(0)|}) \\ &\leq \exp(\frac{M}{|\varphi(0)|}\lambda^n). \end{aligned}$$

Thus

$$\begin{aligned} \prod_{n=0}^{\infty} |\frac{1}{\varphi(0)}\varphi(\psi_n(z)| &\leq \exp(\sum_{n=0}^{\infty}(\frac{M}{|\varphi(0)|}\lambda^n) \\ &= \exp(\frac{M}{|\varphi(0)|}\frac{1}{1-\lambda}) \end{aligned}$$

for every $z \in U$. Set

$$g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\psi_n(z)),$$

then g is nonzero and belongs to $H^\infty(U)$. Now by the Farrell-Rubel-Shields Theorem [3, Theorem 5.1, p.151] there is a sequence $\{p_n\}_n$ of polynomials converging to g such that for all n, $\|p_n\|_U \leq c_0$ for some $c_0 > 0$. So we obtain

$$\|M_{p_n}\| \leq c\|p_n\|_U \leq cc_0$$

for all n. But ball $B(H)$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in B(H)$, $M_{p_n} \longrightarrow A$ in the weak operator topology. Using the fact that $M^*_{p_n} \longrightarrow A^*$ in the weak operator topology and acting these operators on e_λ we obtain that

$$\overline{p_n(\lambda)}e_\lambda = M^*_{p_n}e_\lambda \longrightarrow A^*e_\lambda$$

weakly. Since $p_n(\lambda) \longrightarrow g(\lambda)$ we see that $A^*e_\lambda = \overline{g(\lambda)}e_\lambda$. Because the closed linear span of $\{k_\lambda : \lambda \in U\}$ is dense in H, we conclude that $A = M_g$ and this implies that $g \in M(H)$. Hence indeed $g \in H$, since H contains the constant functions. But, one can see that $\varphi \cdot g \circ \psi = \varphi(0)g$ and so g is an eigenfunction of the weighted composition operator $C_{\varphi,\psi}$. If $w \neq 0$, consider the self maps $\Psi = \alpha_w \circ \psi \circ \alpha_w$ and $\Phi = \varphi \circ \alpha_w$ where

$$\alpha_w(z) = \frac{z - w}{1 - \overline{w}z}.$$

Clearly we can see that $\Psi(0) = 0$ and $\Phi(0) \neq 0$, so the first step of theorem shows that there exists a nonzero holomorphic map G in $H^\infty(U)$ such that, $\Phi \cdot G \circ \Psi = \Phi(0)G$. Thus

$$\begin{aligned} \varphi \circ \alpha_w \cdot G \circ (\alpha_w \circ \psi \circ \alpha_w)(z) &= (\varphi \circ \alpha_w(0))G(z) \\ &= \varphi(w)G(z). \end{aligned}$$

Note that $\alpha_w \circ \alpha_w(z) = z$ and substitute $\alpha_w(z)$ instead of z in the above equality. So we get $\varphi \cdot g \circ \psi = \varphi(w)g$ where $g = G \circ \alpha_w$ belongs to $H^\infty(U)$. Now by the same method as we used earlier, we can see that $g \in H$. This completes the proof. □

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Received: January, 2012