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An Introduction to Intuitionistic Markov Chain

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Abstract

A Markov model is method of determining system behavior by using information about certain probabilities of events within the system. In this paper we define intuitionistic Markov chain on intuitionistic possibility space. Further we analyze the path transition and future behavior of the Intuitionistic Markov chain using composition of intuitionistic fuzzy relations.

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1 Introduction

A Markov model is a method of determining system behavior by using information about certain probabilities of events within the system. A classical model breaks the system into a number of states and each of these states is connected to the other states by a crisp transition rate. Fuzzy Markov model has been defined and is being widely used [4-6],[8-10]. Kai [6] has defined Possibility space using fuzzy sets. In this space he has discussed total possibility theorem, possibility variable, possibility distribution function.

It has been asserted by many authors that there are a large and large number of life problems for which IFS theory is a more suitable tool than fuzzy set theory for searching solution. For example, in decision making problems, particularly in the case of medical diagnosis, sales analysis, new product marketing, financial services, etc. there is a fair chance of the existence of a non-null hesitation part at each moment of evaluation of an unknown object. Intuitionistic fuzzy set theory (IFS theory) introduced by K.Atanassov in [1-3], is a significant extension of fuzzy set theory by Zadeh. Fuzzy sets can be viewed as Intuitionistic fuzzy sets but the converse is not true. Intuitionistic fuzzy relations are intuitionistic fuzzy sets in a cartesian product of universes. The composition of intuitionistic fuzzy relations is discussed in [7].

In this paper we define intuitionistic possibility space. In this space we define intuitionistic Markov chain (IMC) and analyze the path transition and future behavior of the IMC. IMC uses the composition of intuitionistic fuzzy relations (IFR) defined in [7].

2 Preliminary Notes

In this section we have given the necessary definitions.

Definition 2.1 *[1] Let X be an universal set. An intuitionistic fuzzy set (IFS) A assigns to each element $x \in X$ a membership degree $\mu_A(x) \in [0,1]$ and a non-membership degree $\nu_A(x) \in [0,1]$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. For all $x \in X$, the number $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation degree or the intuitionsitic index of x to A. An intuitionsitic fuzzy set on a universe X is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$.*

Definition 2.2 *[7] An intuitinistic fuzzy relation R (IFR) from a universe X to a universe Y is an intuitionsitic fuzzy set in $X \times Y$, i.e., $R = \{((x,y), \mu_R(x,y), \nu_R(x,y))\}$ where $\mu_R : X \times Y \to [0,1]$ and $\nu_R : X \times Y \to [0,1]$ satisty the condition $(\forall (x,y) \in X \times Y)(\mu_R(x,y) + \nu_R(x,y) \leq 1)$.*

Definition 2.3 *[7] Let R and S be intuitionistic fuzzy relations from universe X to Y and from universe Y to Z respectively. Then the composition of R and S is the IFR from X to Z defined as*

$$R \; {}^{sup}_{min} \circ^{inf}_{max} \; S = \left\{ \left((x,z), \mu_{{}^{sup}_{min} \circ^{inf}_{max}}(x,z), \nu_{{}^{sup}_{min} \circ^{inf}_{max}}(x,z) \right) \mid x \in X, z \in Z \right\},$$

where,

$\mu_{{}^{sup}_{min} \circ^{inf}_{max}}(x,z) = \sup_{y \in Y} [\min(\mu_R(x,y), \mu_S(y,z))]$

$\nu_{{}^{sup}_{min} \circ^{inf}_{max}}(x,z) = \inf_{y \in Y} [\max(\nu_R(x,y), \nu_S(y,z))]$

whenever $0 \leq \mu_{{}^{sup}_{min} \circ^{inf}_{max}}(x,z) + \nu_{{}^{sup}_{min} \circ^{inf}_{max}}(x,z) \leq 1, \forall (x,z) \in X \times Z$

3 Intuitionistic Possibility Space

In this section we define intuitionistic possibility space and in this space we define intuitionistic variable.

Definition 3.1 *Let Γ be the universe of discourse and $\Im$ be the power set of Γ. An intuitionistic possibility measure is the pair $\sigma = (\mu, \nu)$ where $\mu : \Im \to [0,1]$ and $\nu : \Im \to [0,1]$ such that $0 \leq \mu(x) + \nu(x) \leq 1$ with the following properties:*

1. $\mu(\phi) = 0; \nu(\phi) = 1.$
2. $\mu(\Gamma) = 1; \nu(\Gamma) = 0.$
3. $\mu(\cup A_\alpha) = \max_\alpha \{\mu(A_\alpha)\}; \ \nu(\cup A_\alpha) = \min_\alpha \{\nu(A_\alpha)\}.$

Then $(\Gamma, \Im, \sigma)$ is called intuitionistic possibility space.

Let $(\Gamma, \Im, \sigma)$ be a intuitionistic possibility space. Let $A, B \in \Im$. Given B occurring consider the intuitionistic possibility of occurrence of A ie., $\sigma(A|B)$. Suppose $\sigma(B) = (\mu(B), \nu(B))$ and $\sigma(AB) = (\mu(AB), \nu(AB))$ are known. If $\mu(B) = \mu(AB)$ and $\nu(B) = \nu(AB)$ i.e., $\sigma(B) = \sigma(AB)$ then it can be said that B achieves its realisation on AB. So $\sigma(A|B) = (1, 0)$. If $\mu(B) > \mu(AB)$ and $\nu(B) < \nu(AB)$ then it can be said that B achieves its realisation on $B - AB$ rather than on AB, so $\mu(A|B) = \mu(AB)$ and $\nu(A|B) = \nu(AB)$. The following definition gives the intuitionistic possibility of A conditional on B.

Definition 3.2 *Let $(\Gamma, \Im, \sigma)$ be a intuitionistic possibility space and $A, B \in \Im$. The intuitionistic possibility of A conditional B is defined as*

$$\sigma(A|B) = \begin{cases} (1,0), & \mu(AB) = \mu(B), \nu(AB) = \nu(B); \\ (\mu(AB), \nu(AB)), & \mu(AB) < \mu(B), \nu(AB) > \nu(B) \end{cases}$$

Note that $min(\mu(A|B), \mu(B)) = \mu(AB)$ and $max(\nu(A|B), \nu(B)) = \nu(AB)$ by the definition of $\sigma(A|B)$.

Theorem 3.3 *Suppose $(\Gamma, \Im, \sigma)$ is a intuitionistic possibility space and $\{B_\alpha\}$ is a collection of sets such that $\cup B_\alpha = \Gamma$ and let $A \in \Im$. Then*

$$\sigma(A) = (\sup_\alpha\{min[\mu(A|B_\alpha), \mu(B_\alpha)]\}, \inf_\alpha\{max[\nu(A|B_\alpha), \nu(B_\alpha)]\}) \qquad (1)$$

Proof: Suppose $(\Gamma, \Im, \sigma)$ is a intuitionistic possibility space and $\{B_\alpha\}$ is a collection of sets such that $\cup B_\alpha = \Gamma$ and let $A \in \Im$. Consider

$$\begin{aligned} \sigma(A) &= \sigma(A\Gamma) \\ &= \sigma(A(\cup B_\alpha) \\ &= (\mu(A(\cup B_\alpha)), \nu(A(\cup B_\alpha))), \text{by definition of } \sigma \\ &= (\sup_\alpha\{min[\mu(A|B_\alpha), \mu(B_\alpha)]\}, \inf_\alpha\{max[\nu(A|B_\alpha), \nu(B_\alpha)]\}) \end{aligned}$$

Definition 3.4 *Let $(\Gamma, \Im, \sigma)$ be a intuitionistic possibility space and U be an arbitrary universe. A intuitionistic variable is a mapping from Γ to $U \times U$. It is called a intuitionistic fuzzy variable if $U = (-\infty, \infty)$.*

Definition 3.5 *The point intuitionistic possibility distribution of a intuitionistic variable X is*

$$g(x) = \sigma(X = x) = (\mu(X = x), \nu(X = x)). \tag{2}$$

Definition 3.6 *The set intuitionistic possibility distribution of a intuitionistic variable X is*

$$\begin{aligned} G(A) &= \sigma(\bigcup_{x \in A} X = x) = (\mu(\bigcap_{x \in A} x), \nu(\bigcap_{x \in A} x)) \\ &= (\sup_{x \in A} \mu(x), \inf_{x \in A} \nu(x)). \end{aligned}$$

Definition 3.7 *Let $(\Gamma, \Im, \sigma)$ is a intuitionistic possibility space. Let $X_i, i = 1, 2, ..., n$ be intuitionistic variables defined on this space. Then the joint intuitionistic distribution of these n intuitionistic variables is given by $g(x_1, x_2, ..., x_n) = \sigma(X_1 = x_1, X_2 = x_2, ..., X_n = x_n)$*

Definition 3.8 *A collection of intuitionistic variables $\{X(t) : t \in T\}$ defined on $(\Gamma, \Im, \sigma)$ is called a intuitionistic random process on $(\Gamma, \Im, \sigma)$.*

Using the above definitions we define Intuitionistic Markov Chain in the following section.

4 Intuitionistic Markov Chain

In this section we define the Intuitionistic Markov Chain (IMC). and analyze the path transition and future behavior of the IMC. IMC uses the composition of intuitionistic fuzzy relations (IFR) given in Definition (2.3).

Definition 4.1 *A intuitionistic random process $\{X(n) : n \in N\}$ is said to an intuitionistic Markov chain if it satisfies the Markov property*

$$\sigma(X_{n+1} = j | X_n = i, X_{n-1} = k, ..., X_0 = m) = \sigma(X_{n+1} = j | X_n = i), \tag{3}$$

where $i, j, k, ...$ constitute the state space S of the process.

Here $\widetilde{P}_{ij} = \sigma(X_{n+1} = j | X_n = i)$ are called intuitionistic transition possibilities of moving from state i to state j in one step. So $\widetilde{P}_{ij} = (\mu_{\widetilde{P}_{ij}}, \nu_{\widetilde{P}_{ij}})$, where $\mu_{\widetilde{P}_{ij}}$ denotes the membership of transition from state i to state j and $\nu_{\widetilde{P}_{ij}}$ denotes the non-membership of transition from state i to state j. The matrix $\widetilde{P} = (\widetilde{P}_{ij})$ is called the intuitionistic transition possibility matrix.

Definition 4.2 *Let $\{X(n) : n \in N\}$ be a intuitionistic Markov chain. $\widetilde{P}_{ij}(m) = \sigma(X_{n+m} = j|X_n = i)$ denotes the intuitionistic transition possibility of moving from state i to state j in m steps. The matrix $\widetilde{P}(m) = (\widetilde{P}_{ij}(m))$ is called the m - step intuitionistic transition possibility matrix.*

Definition 4.3

$$\begin{aligned}\widetilde{p}_j(n) &= \sigma(X_n = j) \\ &= (\mu_{\widetilde{p}_j(n)}, \nu_{\widetilde{p}_j(n)})\end{aligned}$$

where $\mu_{\widetilde{p}_j(n)}$ denotes the membership of being in state j at step n and $\nu_{\widetilde{p}_j(n)}$ denotes the nonmembership of being in state j at step n. The vector $\widetilde{p}(n) = (\widetilde{p}_j(n))$ is called the n-step intuitionistic state designator. $\widetilde{p}(0)$ is called the initial intuitionistic state designator.

A IMC consists of a set of states, called the state space S and a intuitionistic transition possibility matrix whose entries are intuitionistic transition possibilities of moving from one state to another states. The intuitionistic transition possibility matrix can be viewed as an intuitionistic fuzzy relation on $S \times S$ and the n-step intuitionistic state designator $\widetilde{p}(n)$ can be viewed as an intuitionistic fuzzy relation on $N \times S$, where $N = \{0, 1, ...\}$. So we can use composition of intuitionistic fuzzy relations, given in Definition 2.3. Throughout this paper we use $\circ$ instead of ${}^{sup}_{min}\circ^{inf}_{max}$ to denote composition of intuitionistic fuzzy relations.

The following theorem gives the relation between m step intuitionistic transition possibility matrix and intuitionistic transition possibility matrix.

Theorem 4.4 *Let $\{X(n) : n \in N\}$ be a intuitionistic Markov chain with intuitionistic transition possibility matrix $\widetilde{P}$. Then $\widetilde{P}(m) = (\widetilde{P})^m$.*

Proof: The proof is by induction on m. We know that
$\widetilde{P}(1) = (\widetilde{P}_{ij}(1)) = \widetilde{P}$. So the result is true for $m = 1$.
Consider

$$\begin{aligned}\widetilde{P}_{ij}(2) &= \sigma(X_{n+2} = j|X_n = i) \\ &= (\mu_{\widetilde{P}_{ij}}(2), \nu_{\widetilde{P}_{ij}}(2))\end{aligned}$$

where$\mu_{\widetilde{P}_{ij}}(2)$ is the membership of moving from state i to state j in two steps and $\nu_{\widetilde{P}_{ij}}(2)$ is the nonmembership of moving from state i to state j in two steps.

For a fixed k,

$$\begin{aligned}\widetilde{P}_{ij}(2) &= \sigma(X_{n+2} = j, X_{n+1} = k|X_n = i) \\ &= \sigma(X_{n+2} = j|X_{n+1} = k) \circ \sigma(X_{n+1} = k|X_n = i) \\ &= (\min(\mu_{\widetilde{P}_{ik}}, \mu_{\widetilde{P}_{kj}}), \max(\nu_{\widetilde{P}_{ik}}, \nu_{\widetilde{P}_{kj}}))\end{aligned}$$

For any intermediate state k,
$\widetilde{P}_{ij}(2) = (\sup_k\{\min(\mu_{\widetilde{P}_{ik}}, \mu_{\widetilde{P}_{kj}})\}, \inf_k\{\max(\nu_{\widetilde{P}_{ik}}, \nu_{\widetilde{P}_{kj}})\})$.

Hence, $\widetilde{P}(2) = \widetilde{P} \circ \widetilde{P} = \widetilde{P}^2$.

Assume the result is true for m. Consider,

$$\begin{aligned}
\widetilde{P}_{ij}(m+1) &= \sigma(X_{n+m+1} = j | X_{n+m} = i) \\
&= \sigma(X_{n+m+1} = j | X_{n+m} = k) \circ \sigma(X_{n+m} = k | X_n = i) \\
&= (\sup_k\{\min(\mu_{\widetilde{P}_{ik}}(m), \mu_{\widetilde{P}_{kj}})\}, \inf_k\{\max(\nu_{\widetilde{P}_{ik}}(m), \nu_{\widetilde{P}_{kj}})\})
\end{aligned}$$

Hence, $\widetilde{P}(m+1) = \widetilde{P}^m \circ \widetilde{P}$. This means that,

$$\begin{aligned}
\widetilde{P}(m) &= \widetilde{P}^{m-1} \circ \widetilde{P} \\
&= \widetilde{P}^{m-2} \circ \widetilde{P} \circ \widetilde{P} \\
&= \widetilde{P}^m.
\end{aligned}$$

The following theorem gives the relation between the n-step intuitionistic state designator and n-step intuitionistic transition matrix.

Theorem 4.5 *Let $\{X(n) : n \in N\}$ be a intuitionistic Markov chain with intuitionistic transition possibility matrix $\widetilde{P}$, initial intuitionistic state designator $\widetilde{p}(0)$. Then the n-step intuitionistic state designator is given by $\widetilde{p}(n) = \widetilde{p}(0) \circ \widetilde{P}^n$.*

Proof: Consider the j-entry $\widetilde{p}_j(n)$ of the vector $\widetilde{p}(n)$, starting with any initial state i,

$$\begin{aligned}
\widetilde{p}_j(n) &= \sigma(X_n = j) \\
&= \sigma(X_0 = i) \circ \sigma(X_n = j | X_0 = i) \\
&= (\sup\{\min_i(\mu_{\widetilde{p}_i(0)}, \mu_{\widetilde{P}_{ij}(n)})\}, \inf\{\max_i(\nu_{\widetilde{p}_i(0)}, \nu_{\widetilde{P}_{ij}(n)})\})
\end{aligned}$$

Hence $\widetilde{p}(n) = \widetilde{p}(0) \circ \widetilde{P}(n) = \widetilde{p}(0) \circ \widetilde{P}^n$.

In the following section we illustrate these concepts through an example.

5 Illustration

Consider a intuitionistic Markov chain with the following intuitionistic transition matrix $\widetilde{P}$ and initial intuitionistic state designator $\widetilde{p}(0)$,

$$\widetilde{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc}
(0.8, 0.1) & (0.6, 0.2) & (0.0, 0.9) & (0.4, 0.3) \\
(0.6, 0.3) & (0.7, 0.3) & (0.8, 0.2) & (0.5, 0.3) \\
(0.7, 0.2) & (0.6, 0.2) & (0.4, 0.4) & (0.8, 0.1) \\
(0.9, 0.1) & (0.8, 0.2) & (0.6, 0.3) & (0.9, 0.0)
\end{array}\right] \end{array}.$$

and $\widetilde{p}(0) = \begin{bmatrix} (0.8, 0.1) & (0.4, 0.4) & (0.7, 0.2) & (0.9, 0.0) \end{bmatrix}$.

Then the 1-step intuitionistic state designator calculated using $\widetilde{p}(1) = \widetilde{p}(0) \circ \widetilde{P}$ gives $\widetilde{p}(1) = \begin{bmatrix} (0.9, 0.1) & (0.8, 0.2) & (0.6, 0.3) & (0.9, 0.0) \end{bmatrix}$.

The 2-step intuitionisitic possibility matrix $\widetilde{P}(2) = \widetilde{P} \circ \widetilde{P}$ is

$$\widetilde{P}(2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\ \begin{bmatrix} (0.8, 0.1) & (0.6, 0.2) & (0.6, 0.2) & (0.5, 0.3) \\ (0.7, 0.2) & (0.7, 0.2) & (0.7, 0.3) & (0.8, 0.2) \\ (0.8, 0.1) & (0.8, 0.1) & (0.6, 0.2) & (0.8, 0.1) \\ (0.9, 0.1) & (0.8, 0.2) & (0.8, 0.2) & (0.9, 0.0) \end{bmatrix} \end{array}.$$

Then the 2-step state designator obtained using $\widetilde{p}(2) = \widetilde{p}(1) \circ \widetilde{P}$ is $\widetilde{p}(2) = \begin{bmatrix} (0.9, 0.1) & (0.8, 0.2) & (0.8, 0.2) & (0.9, 0.0) \end{bmatrix}$. Using $\widetilde{p}(2) = \widetilde{p}(0) \circ \widetilde{P}^2$ we get $\widetilde{p}(2) = \begin{bmatrix} (0.9, 0.1) & (0.8, 0.2) & (0.8, 0.2) & (0.9, 0.0) \end{bmatrix}$, which is same as the above. This illustrates theorem 3.13.

6 Conclusion

In this paper we have defined intuitionistic possibility space and in this space we define intuitionistic variable. Using these we define Intuitionistic Markov chain using intuitionistic transition possibility matrix. We analyze the path transition and future behavior of the Intuitionistic Markov chain. These concepts are illustrated through an example.

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