International Mathematical Forum, Vol. 7, 2012, no. 49, 2415 - 2430

Error Estimate of the DGFEM for Nonlinear Convection-Diffusion Problems

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Abstract

This paper is concerned with the analysis of the full discrete discontinuous Galerkin finite element method (DGFEM) applied to the space full-discretization of non-linear nonstationary convection-diffusion problem with nonlinear convection in two dimensions. General nonconforming simplicity meshes are considered and symmetric interior penalty Galerkin (SIPG)scheme is used. The error analysis is carried out for nonconforming triangular meshes. we prove the approximate solution is converges with error of order $(h^2 + \tau^2)$.

Mathematics Subject Classification: 65Nxx, 65N30

Keywords: non-linear non-stationary convection-diffusion problems, full discrete discontinuous Galerkin finite element method, optimal error estimates

1 Introduction

A number of complex problems from science and technology (aerospace engineering, turbomachinery, oil recovery, meteorology, environmental protection etc.) require to apply new efficient, robust, reliable and highly accurate numerical methods. It is necessary to develop techniques that allow to realize numerical approximations of strongle nonlinear singularly perturbed in domains with complex geometry whose solutions contain or boundary layers. A natural generalization of the finite volume and finite element methods is the discontinuous Galerkin finite element method (DGFEM). This method uses advantages of FV as well as FE methods: it is based on piecewise polynomial but discontinuous approximations, it does not require continuity on common boundaries of two neighboring elements, and the fluxes on these boundaries are evaluated with the aid of a numerical flux. The use of discontinuous functions allows a precise capturing of discontinuities and steep gradients. This system of equations, when written in conservative form contain nonlinear convective as well as viscous (diffusive) terms. Therefore a theoretical analysis of the DGFEM applied to such problems is a very important topic. It occurs that the DGFEM is very suitable for the numerical solution of nonlinear convection-diffusion problems, conservation law equations and compressible flow(cf. [2], [5], [6], [7], [11], [12], [14], [15]). In this paper the symmetric approximation of the diffusion terms is used, and therefore the method is called SIPG (symmetric interior and boundary penalty Galerkin). The class of DGFEMs seems to be one of the most promising candidates to construct high order accurate schemes for solving convection-diffusion problems. The error caused by the numerical integration was estimated and it was shown what numerical quadratures guarantee preservation of the accuracy of the method with exact integration. to obtain an optimal $L^\infty(L^2)$ error estimate in the case of SIPG version of DGFEM was the aim of the papers [8], [13] and [16]. it was shown that there using conforming simplicial meshes such an optimal error estimate can be obtained under some assumptions on the regularity of the exact solution and of the dual elliptic problem. We shall derive an optimal $L^\infty(L^2)$-error estimate for the full space-time discretization on a simplicial meshes which are nonconforming in general, and which have possibly hanging nodes (and in 3D, hanging edges). Concerning the system of triangulations we shall only suppose that it is regular and locally quasi-uniform. This paper is organized as follows. In Section 2 we present the Convection-Diffusion Equation. The discretization is shown in section 3. In section 4 we consider the spaces of discontinuous function.The properties of the numerical flux are shown in section 5.In section 6 we consider Some auxiliary restarts.The error estimate are presented in section 7. The conclusions are shown in section 8.

2 The Convection-Diffusion Equation

we consider the following nonstationary two-dimensional nonlinear convection-diffusion equation:

$$\frac{\partial u}{\partial t} - \lambda \triangle u + \nabla \cdot (\vec{b}(u).u) = f \qquad \Omega \times (0,T] = Q_T. \tag{2.1}$$

$$u|_{\partial\Omega\times(0,T)} = u_D. \tag{2.2}$$

$$u(x,0) = u^0(x) x \in \Omega \tag{2.3}$$

in the domain $\Omega \subset R^2$, with Lipschtiz-continuous boundary $\partial\Omega$ and $T > 0$. The diffusion coefficient $\lambda > 0$ is a given constant, the vector $\vec{b}(u) : Q_T \longrightarrow R^2$ is a convection coefficient $f : Q_T \longrightarrow R^2$, $u_D : \partial\Omega \times (0,T) \longrightarrow R$ and u^0 :

$\Omega \longrightarrow R$ are given functions.
Now, we define the set $Y = \{u :\mid u \mid\leq M\}$ and the coefficients of equation(2.1) satisfy the following conditions:
A(2.1) $\vec{b}(u) = (b_1(u), b_2(u)) \in W^{1,\infty}(Y) \times W^{1,\infty}(Y)$.
A(2.2) let $0 < L_b < 1$ is a constant, then$\vec{b}(u)$ is locally Lipschitz-continuous

$$|\vec{b}(u) - \vec{b}(v)| \leq L_b|u - v|, \quad \forall u, v \in R,$$

A(2,3) $u \in L^\infty(0, T; H^1(\Omega)) \bigcup L^\infty(0, T; H^2(\Omega))$ and$u_t \in L^\infty(0, T; L^\infty(\Omega))$, where u is the weak solution of problem (2.1).

3 The discretization [8].

Let $\mathcal{T}_h$ be a partition of $\bar{\Omega}$ (the closure of the domain Ω) into a finite number of closed triangles with mutually disjoint interiors. We shall call $\mathcal{T}_h$ a triangulation of Ω.We do not require the standard conforming properties of $\mathcal{T}_h$ used in the finite element method. This means that we admit the so-called hanging nodes. We shall use the following notation. By ∂e we denote the boundary of an element $e \in \mathcal{T}_h$. Let e_1, $e_2 \in \mathcal{T}_h$. We say that e_1 and e_2 are neighbors, if the set $\partial e_1 \bigcap \partial e_2$ has positive $(d-1)$- dimensional measure. We say that $\Gamma \in e$ is a face of e, if it is a maximal connected open subset either of $\partial e_1 \bigcap \partial e_2$, where e_2 is a neighbor of e_1, or of $\partial e \bigcap \partial\Omega$. By $\partial\mathcal{T}_h$ we denote the system of all faces of all elements $e \in \mathcal{T}_h$. Further, we define the set of all inner faces by

$$\partial\mathcal{T}_h^I = \{\Gamma \in \partial\mathcal{T}_h;\ \Gamma \subset \Omega\},$$

and the set of all boundary faces by

$$\partial\mathcal{T}_h^B = \{\Gamma \in \partial\mathcal{T}_h;\ \Gamma \subset \partial\Omega\}.$$

obviously, $\partial\mathcal{T}_h = \partial\mathcal{T}_h^I \bigcup \partial\mathcal{T}_h^B$. For each $\Gamma \in \partial\mathcal{T}_h$we define a unit normal vector n_Γ. We assume that for $\Gamma \in \partial\mathcal{T}_h^B$ the normal n_Γ has the same orientation as the outer normal to $\partial\Omega$. For each face $\Gamma \in \partial\mathcal{T}_h^I$ the orientation of n_Γ is arbitrary but fixed. Finally, by $d(\Gamma)$ we denote the diameter of $\Gamma \in \partial\mathcal{T}_h$.

3.1 Assumption

(**a**) $|e|$ = the area of $e \in \mathcal{T}_h$.
(**b**) Define h_e = the length of the longest side of the triangle $e \in \mathcal{T}_h$ and put $h_e = diam(e)$.
(**c**) Define ρ_e is the radius of the largest circle inscribed into e .
(**d**) there exists a constant $C_D \geq 1$ such that

$$h_e \leq C_D d(\Gamma),\ \ \Gamma \in \partial\mathcal{T}_h,\ \ \Gamma \subset e$$

$\forall e \in \mathcal{T}_h$ are neighbors, $h \in (0, h_0)$.

(**e**) there exists a constant $C_N \geq 1$ such that

$$h_{e_1} \leq C_N h_{e_2} \quad \forall e_1, e_2 \in \mathcal{T}_h \;\; are\, neighbours \;\; , h \in (0, h_0).$$

(**f**) there exists a constant $C_{\mathcal{T}} \geq 0$ such that

$$\frac{h_e}{\rho_e} \leq C_{\mathcal{T}} \quad \forall e \in \mathcal{T}_h \;, h \in (0, h_0).$$

3.2 The spaces of discontinuous function [8].

Over a triangulation $\mathcal{T}_h$ we define the broken Sobolev spaces

$$H^m(\Omega) = \{v; v|_e \in H^m(e) \;\; \forall e \in \mathcal{T}_h\},$$

equipped with the semi-norm,

$$|v|_{H^m(\Omega)} = (\sum_{e \in \mathcal{T}_h} |v|_{H^m(e)})^{\frac{1}{2}}. \tag{3.1}$$

for each face $\Gamma \in \partial\mathcal{T}_h^L$ there exist two neighbors $e_\Gamma^{(L)}, e_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset \partial e_\Gamma^{(L)} \bigcap \partial e_\Gamma^{(R)}$. We use the convention that n_Γ is the outer normal to the element $e_\Gamma^{(L)}$ and the inner normal to the element $e_\Gamma^{(R)}$. For $v \in H^e(\Omega)$ and $\Gamma \in \partial\mathcal{T}_h^L$ we introduce the following notation:
$v|_\Gamma^{(L)}$ = the trace of $v|_{e_\Gamma^{(L)}}$ on Γ, $v|_\Gamma^{(R)}$ = the trace of $v|_{e_\Gamma^{(R)}}$ on Γ,
$\langle v \rangle_\Gamma = \frac{1}{2}(v|_\Gamma^{(L)} + v|_\Gamma^{(R)})$, $[v]_\Gamma = (v|_\Gamma^{(L)} - v|_\Gamma^{(R)})$.
The value $[v]_\Gamma$ depends on the orientation of n_Γ, but the values $\langle v \rangle_\Gamma$ and $[v]_\Gamma n_\Gamma$ are independent of this orientation. Now, let $\Gamma \in \partial\mathcal{T}_h^B$ and $e_\Gamma^{(L)} \in \mathcal{T}_h$ be such an element that $\Gamma \subset e_\Gamma^{(L)} \bigcap \partial\Omega$. For $v \in H^m(\Omega)$ we set $v_\Gamma = v|_\Gamma^{(L)} = v|_\Gamma^{(R)}$ = the trace of $v|_{e_\Gamma^{(L)}}\, on \Gamma$. If $[\cdot]_\Gamma$ and $\langle \cdot \rangle_\Gamma$ appear in an integral of the form $\int_\Gamma ...\, dS$, we omit the subscript Γ and write simply $[\cdot]$ and $\langle \cdot \rangle$. For simplicity we shall use the following notation:

$$\int_{\partial\mathcal{T}_h^I} ...\, dS = \sum_{\Gamma \in \partial\mathcal{T}_h^I} \int_\Gamma ...\, dS$$

4 The semi discretization DG finite element method.

We shall introduce the SIPG (Symmetric Interior Penalty Galerkin ($\Theta = 1$, [**8**]) version of the DG approximation by multiplying the equations (2.1)-(2.3) by

v to find regular exact solution $u \in H^2(\Omega, T_h)$ such that:

$$(\frac{\partial u(t)}{\partial t}, v) + a_h(u(t), v) + b_h(u(t), v) + \lambda J_h(u(t), v) = (f(t), v) - \delta_h(v)(t), \quad (4.1)$$

$\forall v \in H^2(\Omega, T_h)$ and $u(0) = u^0$. We shall denote by $(.,.)$ the scalar product in the space $L^2(\Omega)$, i.e.

$$(u, v) = \int_\Omega uv\, dx, \quad u, v \in L^2(\Omega),$$

and define symmetric diffusion form for functions $u, v \in H^2(\Omega)$:

$$\begin{aligned} a_h(u(t), v) &= \sum_{e \in \mathcal{T}_h} \int_e \lambda \nabla u \nabla v\, dx, - \sum_{\Gamma \in \partial \mathcal{T}_h^I} \int_\Gamma \lambda \langle \nabla u \rangle .n[v]\, ds - \sum_{\Gamma \in \partial \mathcal{T}_h^I} \int_\Gamma \lambda \langle \nabla v \rangle .n[u]\, ds \\ &- \sum_{\Gamma \in \partial \mathcal{T}_h^B} \int_\Gamma \lambda \nabla u.n\, v\, ds - \sum_{\Gamma \in \partial \mathcal{T}_h^B} \int_\Gamma \lambda \nabla v.n\, u\, ds, \end{aligned}$$

We define the interior and boundary jump terms:

$$J_h(u, v) = \sum_{\Gamma \in \partial \mathcal{T}_h^B} \int_\Gamma \sigma u.\, v\, ds + \sum_{\Gamma \in \partial \mathcal{T}_h^I} \int_\Gamma \sigma [u][v]\, ds \quad (4.2)$$

and the symmetric right-hand side form:

$$\delta_h(v)(t) = \sum_{\Gamma \in \partial \mathcal{T}_h^B} \int_\Gamma \lambda v.n\, u_D\, ds - \sum_{\Gamma \in \partial \mathcal{T}_h^B} \int_\Gamma \lambda \sigma u_D\, v\, ds, \quad (4.3)$$

where the parameter σ in (4.2) and (4.3) is constant on every edge and we shall define it in remark(5.1-a) . And the form b_h approximate convective terms with the aid of a numerical flux $H(u, v, n)$, then by replacing the function u on $\Gamma \in \partial \mathcal{T}_h$ by some convex combination of the nodal values of $u|_\Gamma^{(L)}$ and $u|_\Gamma^{(R)}$ with parameter H.

$$\begin{aligned} b_h(u(t), v) &= -\sum_{e \in \mathcal{T}_h} \int_e (\vec{b}(u).u) \nabla v dx + \sum_{\Gamma \in \partial \mathcal{T}_h^I} \int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, n|_\Gamma)[v]_\Gamma ds \\ &+ \sum_{\Gamma \in \partial \mathcal{T}_h^B} \int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(L)}, n|_\Gamma) v|_\Gamma ds. \quad (4.4) \end{aligned}$$

4.1 The properties of the numerical flux[4].

We use the following assumptions:
(a) $H = H(y, z, n)$ is defined in $R^2 \times B_1$, where $B_1 = \{n \in R^d; |n| = 1\}$ is

locally Lipschitz-continuous with respect to y, z for any constant $L_H > o$ such that,

$$|H(y,z,n) - H(y^*,z^*,n)| \leq L_H(|y-y^*| + |z-z^*|) \quad \forall y, y^*, z, z^* \in R.$$

(**b**) H is consistent,

$$H(u,u,n) = \vec{b}(u)nu, \ \forall u \in R, \ n = (n_1, ..., n_d) \in B_1.$$

(**c**) H is conservative,

$$H(y,z,n) = -H(z,y,-n), \ \forall y, z \in R, \ n \in B_1.$$

5 The full discrete discontinuous Galerkin finite element scheme.

We say that $U \in C^1([0,T]; V_h)$ is a DGFE solution of the convection-diffusion problem (2.1). such that:

$$(\frac{U^{k+1} - U^k}{\tau}, v_h) + a_h(U^{k+\frac{1}{2}}, v_h) + b_h(U^k, v_h) + \lambda J_h(U^{k+\frac{1}{2}}, v_h)$$

$$= (f(U^{k+1}), v_h) - \delta_h(v_h)(t), \quad \forall v_h \in V_h \ and \ t \in (0,T) \qquad (5.1)$$

$U^0 = \prod_h u^0$, where

$$V_h = \{v; v|_e \in P^p(e), \forall e \in T_h\},$$

$P^p(e)$ denote the space of all polynomials on e of degree $\leq p$, and $\prod_h$ is interpolation operator of the initial condition u^0 onto V_h, i.e. a function defined by

$$(U^0 - u^0, v_h) = 0 \quad \forall v_h \in V_h, \qquad (5.2)$$

and $U^{k+\frac{1}{2}} = \frac{U^{k+1}+U^k}{2}$.

5.1 remark.

(**a**) Our goal will be to extend the optimal $L^\infty(L^2)-$ norm error estimate to the DGFEM applied on nonconforming meshes with hanging nodes without the limiting assumption 3.1-d, to this end, we change the definition of the weight σ similarly as in [8]:

$$\sigma|_\Gamma = \frac{2C_W}{h_{e_\Gamma^{(L)}} + h_{e_\Gamma^{(R)}}}, \ \Gamma \in \partial\mathcal{T}_h^I, \qquad \sigma|_\Gamma = \frac{C_W}{h_{e_\Gamma^{(L)}}}, \ \Gamma \in \partial\mathcal{T}_h^R, \qquad (5.3)$$

where the constant C_W satisfy under the assumption that,

$$C_W \geq 2C_M(1 + C_I)(1 + C_N), \tag{5.4}$$

Where C_M and C_I are constants.
(**b**) We consider the local quasiuniformity of the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ by assumption 3.1-e is hold.
(**c**) By c we denote a generic constant independent of $h, \lambda, ...$ which attains in general different values in differen places.
(**d**) In what follows we shall usually omit the argument t of the function u, v and $\prod_h v$ on V_h.
(**e**) We will put $h = O(\tau)$.

5.2 Properties of the diffusion terms.

For $u, v \in H^2(\Omega)$ and $\phi \in H^1(\Omega)$ let us define the form,

$$A_h(u, v) = a_h(u, v) + \lambda J_h(u, v), \tag{5.5}$$

and the norm,

$$\| \phi \|_{DG} = (\frac{1}{2}(| \phi |^2_{H^1(\Omega)} + \lambda J_h(\phi, \phi)))^{\frac{1}{2}}, \tag{5.6}$$

6 Some auxiliary restarts[8].

The derived error estimates rely on the following results:

Lemma 6.1 ***(Multiplicative trace inequality)****. There exists a constant $C_M > 0$ independent of h, e such that for all $e \in \mathcal{T}_h$, $v \in H^1(e)$ and $h \in (0, h_0)$*

$$\| v \|^2_{L^2(\partial e)} \leq C_M(\| v \|_{L^2(e)} | v |_{H^1(e)} + h_e^{-1} \| v \|^2_{L^2(e)}).$$

Lemma 6.2 ***(Inverse inequalities)****. There exists a constant $C_I > 0$ independent of h, e such that for all $e \in \mathcal{T}_h$, $v \in P^p(e)$ and $h \in (0, h_0)$*

$$| v |_{H^1(e)} \leq C_I h_e^{-1} \| v \|_{L^2(e)} .$$

7 The error estimate.

We define the projection $\Pi_h v$ is the $L^2(\Omega)$-projection of $v \in V_h$ such that:

$$\Pi_h v \in V_h, \;\; (\Pi_h v - v, v_h) = 0 \;\; \forall v_h \in V_h. \tag{7.1}$$

Theore1m 7.1 *If u and U are solutions of (4.1) and (5.1)respectively and satisfy the conditions A(2.3) then under the assumption (5.5) that,*

$$\|e_h^k\|_{L^\infty(0,T;L^2(\Omega))} \leq O(h^2+\tau^2),$$

where a constant $C>0$ is independent of h,τ.

Proof: let $w=\Pi_h u$, we set

$$e_h^k = U^k - u^k = (U^k - w^k) + (w^k - u^k) = \theta^k + \eta^k.$$

It is clear that, [10]

$$\|\eta^k\|_{L^2(\Omega)} \leq Ch^2, \quad |\eta^k|_{H^1(\Omega)} \leq Ch,$$

for all $h \in (0,h_0)$. Put $v=v_h=\theta^{k+\frac{1}{2}}$ and we subtract (4.1) from (5.1), with $D_\tau\theta^k = \frac{\theta^{k+1}-\theta^k}{\tau}$ we have,

$$\begin{aligned}(D_\tau\theta^k,\theta^{k+\frac{1}{2}}) + \lambda A_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}) &= [(f(U^{k+1}),\theta^{k+\frac{1}{2}}) - (f(u^{k+1}),\theta^{k+\frac{1}{2}})] \\ &- [(D_\tau w^k,\theta^{k+\frac{1}{2}}) - (u_t^{k+1},\theta^{k+\frac{1}{2}})] \\ &- \lambda[A_h(w^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}) - A_h(u^{k+1},\theta^{k+\frac{1}{2}})] \\ &- [b_h(U^k,\theta^{k+\frac{1}{2}}) - b_h(u^{k+1},\theta^{k+\frac{1}{2}})]. \quad (7.2)\end{aligned}$$

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$$(D_\tau\theta^k,\theta^{k+\frac{1}{2}}) = (\frac{\theta^{k+1}-\theta^k}{\tau},\frac{\theta^{k+1}+\theta^k}{2}) = \frac{1}{2}D_\tau\|\theta^k\|^2_{L^2(\Omega)}. \quad (7.3)$$

and to estimate $\lambda A_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}})$, from assumptions 3.1-e be satisfied the weight σ be defined by (5.3), under the assumption (5.4), by use the Cauchy-Schwartz inequality, (3.1), Young's inequality and the definition (4.2) we get,

$$\lambda A_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}) = \lambda|\theta^{k+\frac{1}{2}}|^2_{H^1(\Omega)} - \frac{\lambda}{\gamma}(2 \setminus (h_{e_\Gamma^{(L)}} + h_{e_\Gamma^{(R)}})) \sum_{\Gamma\in\partial\mathcal{T}_h^I}\int_\Gamma |\langle\nabla\theta^{n+\frac{1}{2}}\rangle|^2\,ds$$

$$\begin{aligned}&- \frac{\lambda}{\gamma}(1 \setminus (h_{e_\Gamma^{(L)}}) \sum_{\Gamma\in\partial\mathcal{T}_h^B}\int_\Gamma |\nabla\theta^{k+\frac{1}{2}}|^2\,ds - \gamma\lambda(2 \setminus (h_{e_\Gamma^{(L)}} + h_{e_\Gamma^{(R)}})) \sum_{\Gamma\in\partial\mathcal{T}_h^I}\int_\Gamma [\theta^{k+\frac{1}{2}}]^2\,ds \\ &- \gamma\lambda(1 \setminus (h_{e_\Gamma^{(L)}}) \sum_{\Gamma\in\partial\mathcal{T}_h^B}\int_\Gamma |\ \theta^{k+\frac{1}{2}}\ |^2\ ds + \lambda J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}) \\ &\geq \lambda\ |\ \theta^{k+\frac{1}{2}}\ |^2_{H^1(\Omega)} - \frac{\lambda(1+C_N)h_e}{2\gamma}\ |\ \theta^{k+\frac{1}{2}}\ |^2_{L^2(\partial e)} - \frac{\gamma\lambda}{C_W}J_h(\theta^{k+\frac{1}{2}},\theta^{+\frac{1}{2}}) + \lambda J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}),\end{aligned}$$

by lemmas 6.1, 6.2 , from (5.4) and (5.6) we get,

$$\begin{aligned}\lambda A_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}) &\geq \lambda \mid \theta^{k+\frac{1}{2}} \mid^2_{H^1(\Omega)} - \frac{\lambda(1+C_N)(1+C_I)h_e}{2\gamma}|\theta^{k+\frac{1}{2}}|^2_{H^1(\Omega)} \\ &- \frac{\gamma\lambda}{C_W}J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}) + \lambda J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}})\end{aligned}$$

If we set $\gamma = C_M(1+C_N)(1+C_I)$ then,

$$\lambda A_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}) \geq \lambda\|\theta^{k+\frac{1}{2}}\|^2_{DG} \tag{7.4}$$

From (7.3) and (7.4), thus we can write (7.2) in the form,

$$\frac{1}{2}D_\tau\|\theta^k\|^2_{L^2(\Omega)} + \lambda\|\theta^{k+\frac{1}{2}}\|^2_{DG} \leq \sum_{i=1}^{4} B^{(i)} \tag{7.5}$$

We start with the estimation of $B^{(1)}$ is a consequence of [9, Theorem 4.1.5 and proof of theorem 4.1.6], we use the remark 5.1-c and Young's inequalities,

$$\begin{aligned}|B^{(1)}| &= |(f(U^{k+1}),\theta^{k+\frac{1}{2}}) - (f(u^{k+1}),\theta^{k+\frac{1}{2}})| \leq C\tau\|\theta^{k+\frac{1}{2}}\|_{L^2(\Omega)} \\ &\leq C\tau^2 + C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{L^2(\Omega)} .\end{aligned} \tag{7.6}$$

To estimate $B^{(2)}$, we use [3, lemma 2.2.4],from the definition 1.2 in [1] and Young's inequalities, use the remark 5.1-c,e we get,

$$|B^{(2)}| \leq |(D_\tau w^k - u_t^{k+1},\theta^{k+\frac{1}{2}})| =| (\frac{w^{k+1}-w^k}{\tau} - u_t^{k+\frac{1}{2}} + u_t^{k+\frac{1}{2}} - u_t^{k+1},\theta^{k+\frac{1}{2}}) |,$$

let

$$u_t^{k+\frac{1}{2}} = \frac{u^{k+1}-u^k}{\tau} + \alpha^{k+1},$$

then $\parallel \alpha^{k+1} \parallel_{L^2(\Omega)} \leq C\tau^2$ and

$$| B^{(2)} | \leq | (\frac{w^{k+1}-w^k}{\tau} - \frac{u^{k+1}-u^k}{\tau},\theta^{k+\frac{1}{2}}) + (u_t^{k+\frac{1}{2}} - u_t^{k+1},\theta^{k+\frac{1}{2}}) + (\alpha^{k+1},\theta^{k+\frac{1}{2}}) | .$$

We can write,

$$B^{(2)} \leq \sum_{i=1}^{3} B^{(2i)},$$

$$\begin{aligned}| B^{(21)} | &\leq | (\frac{w^{k+1}-w^k}{\tau} - \frac{u^{k+1}-u^k}{\tau},\theta^{k+\frac{1}{2}}) | \leq \frac{1}{\tau} | (\eta^{k+1},\theta^{k+\frac{1}{2}}) | + \frac{1}{\tau} | (\eta^k,\theta^{k+\frac{1}{2}}) | \\ &\leq C\frac{h^2}{\tau} \parallel \theta^{k+\frac{1}{2}} \parallel_{L^2(\Omega)} \leq C\tau^2 + C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}
\mid B^{(22)} \mid &\leq C \parallel u_t^{k+\frac{1}{2}} - u_t^{k+1} \parallel^2_{L^2(\Omega)} + C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{L^2(\Omega)} \leq C\tau^2 + C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{L^2(\Omega)}, \\
\mid B^{(23)} \mid &\leq C \parallel \alpha^{k+1} \parallel^2_{L^2(\Omega)} + C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{L^2(\Omega)} \leq C\tau^2 + C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{L^2(\Omega)},
\end{aligned}$$

Then

$$\mid B^{(2)} \mid \leq C\tau^2 + C\|\theta^{k+\frac{1}{2}}\|^2_{L^2(\Omega)}. \tag{7.7}$$

To estimate $B^{(3)}$, we add and subtract the terms $\lambda A_h(u^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}})$, gives

$$\begin{aligned}
\mid B^{(3)} \mid &\leq \lambda \underbrace{\mid A_h(w^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}) - A_h(u^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}) \mid}_{B^{(31)}} \\
&+ \lambda \underbrace{\mid A_h(u^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}) - A_h(u^{k+1}, \theta^{k+\frac{1}{2}}) \mid}_{B^{(32)}}
\end{aligned}$$

To estimate $B^{(31)}$, we use the Cauchy-Schwartz inequality,Young's inequalities, A(2.3) and [13, lemma 4.5] we get,

$$|B^{(31)}| = \lambda \mid A_h(\eta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}) \mid \leq \lambda \mid a_h(\eta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}) \mid + \lambda \mid J_h(\eta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}) \mid,$$

scince

$$\begin{aligned}
\lambda \mid J_h(\eta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}) \mid &\leq \lambda \mid J_h(\eta^{k+\frac{1}{2}}, \eta^{k+\frac{1}{2}})^{1\backslash 2} J_h(\theta^{k+\frac{1}{2}}), \theta^{k+\frac{1}{2}})^{1\backslash 2} \mid, \\
&\leq \lambda C h |u^{k+\frac{1}{2}}|^{1\backslash 2}_{H^1(\Omega)} J_h(\theta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}})^{1\backslash 2} \\
&\leq C\tau^2 + J_h(\theta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}),
\end{aligned}$$

Now,

$$\begin{aligned}
a_h(\eta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}) &= \underbrace{\sum_{e \in \mathcal{T}_h} \int_e \lambda \nabla \eta^{k+\frac{1}{2}} \nabla \theta^{k+\frac{1}{2}} \, dx}_{(1)} \\
&- \underbrace{\sum_{\Gamma \in \partial\mathcal{T}_h^I} \int_\Gamma \lambda \langle \nabla \eta^{k+\frac{1}{2}} \rangle . n [\theta^{k+\frac{1}{2}}] \, ds - \sum_{\Gamma \in \partial\mathcal{T}_h^B} \int_\Gamma \lambda \nabla \eta^{k+\frac{1}{2}} . n \, \theta^{k+\frac{1}{2}} \, ds}_{(2)} \\
&- \underbrace{\sum_{\Gamma \in \partial\mathcal{T}_h^I} \int_\Gamma \lambda \langle \nabla \theta^{k+\frac{1}{2}} \rangle . n [\eta^{k+\frac{1}{2}}] - \sum_{\Gamma \in \partial\mathcal{T}_h^B} \int_\Gamma \lambda \nabla \theta^{k+\frac{1}{2}} . n \, \eta^{k+\frac{1}{2}} \, ds,}_{(3)}
\end{aligned}$$

immediately we obtain,

$$|(1)| \leq \lambda |\eta^{k+\frac{1}{2}}|_{H^1(\Omega)} |\theta^{k+\frac{1}{2}}|_{H^1(\Omega)} \leq C\tau^2 + \lambda |\theta^{k+\frac{1}{2}}|^2_{H^1(\Omega)}.$$

Further, by the Cauchy-Schwartz inequality, the lemmas 6.1, 6.2, the assumption 3.1-e and remark 5.1-c,e we get,

$$
\begin{aligned}
|(2)| &\leq \lambda[\sum_{\Gamma\in\partial\mathcal{T}_h^I}\int_\Gamma \frac{h_{e_\Gamma^{(L)}}+h_{e_\Gamma^{(R)}}}{2C_W}|\langle\nabla\eta^{k+\frac{1}{2}}\rangle|^2\,ds]^{1\backslash 2}\,[\sum_{\Gamma\in\partial\mathcal{T}_h^I}\int_\Gamma \frac{2C_W}{h_{e_\Gamma^{(L)}}+h_{e_\Gamma^{(R)}}}[\theta^{k+\frac{1}{2}}]^2\,ds]^{\frac{1}{2}} \\
&+ \lambda[\sum_{\Gamma\in\partial\mathcal{T}_h^B}\int_\Gamma \frac{h_{e_\Gamma^{(L)}}}{C_W}|\nabla\eta^{k+\frac{1}{2}}|^2\,ds]^{1\backslash 2}\,[\sum_{\Gamma\in\partial\mathcal{T}_h^B}\int_\Gamma \frac{C_W}{h_{e_\Gamma^{(L)}}}|\theta^{k+\frac{1}{2}}|^2\,ds]^{\frac{1}{2}}, \\
&\leq \lambda J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}})^{\frac{1}{2}}\frac{(1+C_N)^{1\backslash 2}}{\sqrt{2}(C_W)^{1\backslash 2}}(\sum_{\Gamma\in\partial\mathcal{T}_h} h_e \parallel \nabla\eta^{k+\frac{1}{2}} \parallel^2_{L^2(\partial e)})^{\frac{1}{2}}, \\
&\leq \lambda \mid \eta^{k+\frac{1}{2}} \mid_{H^1(\Omega)} J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}})^{\frac{1}{2}} \leq C\tau^2+\lambda J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}).
\end{aligned}
$$

To estimate (3), we make the same as the steps in (2) and use [8, lemma 2],

$$
\mid (3) \mid \;\leq\; \lambda \mid \theta^{k+\frac{1}{2}} \mid_{H^1(\Omega)} J_h(\eta^{k+\frac{1}{2}},\eta^{k+\frac{1}{2}})^{1\backslash 2} \leq C\tau^2 + \lambda \mid \theta^{k+\frac{1}{2}} \mid^2_{H^1(\Omega)},
$$

then

$$
\mid B^{(31)} \mid \;\leq\; C\tau^2 + 2\lambda(\mid \theta^{k+\frac{1}{2}} \mid^2_{H^1(\Omega)} + J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}})) \leq C\tau^2 + \lambda C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{DG}
$$

To estimate $B^{(32)}$, we make the same as the steps in $B^{(31)}$.
Scince $\mid u^{k+\frac{1}{2}} - u^{k+1} \mid = \frac{\tau}{2}\nabla u^k$, and

$$
C\tau\lambda \mid J_h(\nabla u^k,\theta^{k+\frac{1}{2}}) \mid \;\leq\; C\tau^2 + \lambda J_h(\theta^{k+\frac{1}{2}},\theta^{k+\frac{1}{2}}),
$$

then

$$
\begin{aligned}
&\mid B^{(32)} \mid \leq C\tau^2 + \lambda C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{DG}, \\
&\mid B^{(3)} \mid \leq C\tau^2 + \lambda C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{DG}\;. \qquad (7.8)
\end{aligned}
$$

Now, to estimate the term $B^{(4)}$ we add and subtract $b_h(\vec{b}(u^k).u^k,\theta^{k+\frac{1}{2}})$ we get,

$$
\begin{aligned}
\mid B^{(4)} \mid \;&\leq\; \underbrace{\mid b_h(\vec{b}(u^k).u^k,\theta^{k+\frac{1}{2}}) - b_h(\vec{b}(U^k).U^k,\theta^{k+\frac{1}{2}}) \mid}_{B^{(41)}} \\
&+ \underbrace{\mid b_h(\vec{b}(u^k).u^k,\theta^{k+\frac{1}{2}}) - b_h(\vec{b}(u^{k+1}).u^{k+1},\theta^{k+\frac{1}{2}}) \mid}_{B^{(42)}}
\end{aligned}
$$

To estimate $B^{(41)}$, add and subtract the term $b_h(\vec{b}(\Pi_h u^k).\Pi_h u^k,\theta^{k+\frac{1}{2}}) \quad \forall\theta^{k+\frac{1}{2}} \in V_h$, $t \in (0,T)$ and use the definition (4.4) we get,

$$
\begin{aligned}
\mid B^{(41)} \mid \;&\leq\; \underbrace{\mid b_h(\vec{b}(u^k).u^k,\theta^{k+\frac{1}{2}}) - b_h(\vec{b}(\Pi_h u^k).\Pi_h u^k,\theta^{k+\frac{1}{2}}) \mid}_{B^{(411)}} \\
&+ \underbrace{\mid b_h(\vec{b}(\Pi_h u^k).\Pi_h u^k,\theta^{k+\frac{1}{2}}) - b_h(\vec{b}(U^k).U^k,\theta^{k+\frac{1}{2}}) \mid}_{B^{(412)}}
\end{aligned}
$$

To estimate$B^{(411)}$,

$$B^{(411)} = \underbrace{-\sum_{e\in\mathcal{T}_h}\int_e(\vec{b}(u^k).u^k - \vec{b}(\Pi_h u^k).\Pi_h u^k).\nabla\theta^{k+\frac{1}{2}}\,dx}_{B_1^{(411)}}$$

$$+\underbrace{\sum_{\Gamma\in\partial\mathcal{T}_h^I}\int_\Gamma[(H^k|_\Gamma^{(L)}u^k|_\Gamma^{(L)} + H^k|_\Gamma^{(R)}u^k|_\Gamma^{(R)}) - (H^k|_\Gamma^{(L)}\Pi_h u^k|_\Gamma^{(L)} + H^k|_\Gamma^{(R)}\Pi_h u^k|_\Gamma^{(R)})].[\theta^{k+\frac{1}{2}}]_\Gamma ds}_{B_2^{(411)}}$$

$$+\underbrace{\sum_{\Gamma\in\partial\mathcal{T}_h^B}\int_\Gamma[(H^k|_\Gamma^{(L)}u^k|_\Gamma^{(L)} + H^k|_\Gamma^{(L)}u^k|_\Gamma^{(L)}) - (H^k|_\Gamma^{(L)}\Pi_h u^k|_\Gamma^{(L)} + H^k|_\Gamma^{(L)}\Pi_h u^k|_\Gamma^{(L)})].\theta^{k+\frac{1}{2}}|_\Gamma\,ds}_{B_3^{(411)}}\,.$$

To estimate $B_1^{(411)}$, add and subtract the term $\sum_{e\in\mathcal{T}_h}\int_e(\vec{b}(\Pi_h u^k).u^k.\nabla\theta^{k+\frac{1}{2}}\,dx$, use Cauchy-Schwartz inequality, Young's inequality, from the definition 1.2 in [1], the assumption A(2.2) and remark 5.1-c,e, gives

$$B_1^{(411)} \le C\tau^2 + C\lambda \mid \theta^{k+\frac{1}{2}} \mid^2_{H^1(\Omega)}\,.$$

To estimate $B_2^{(411)} + B_3^{(411)}$, we use the property of the numerical flux 4.1-a, Cauchy-Schwartz inequality, (5.3) and the assumption 3.1-e, imply that,

$$\begin{aligned}
\mid B_2^{(411)} + B_3^{(411)} \mid \;&\le\; L_H[\sum_{\Gamma\in\partial\mathcal{T}_h^I}\int_\Gamma \frac{h_{e_\Gamma^{(L)}} + h_{e_\Gamma^{(R)}}}{2C_W}(\mid u^k|_\Gamma^{(L)} - \Pi_h u^k|_\Gamma^{(L)}\mid \\
&+\; \mid u^k|_\Gamma^{(R)} - \Pi_h u^k|_\Gamma^{(R)}\mid)^2\,ds]^{1\backslash 2}[\sum_{\Gamma\in\partial\mathcal{T}_h^I}\int_\Gamma \frac{2C_W}{h_{e_\Gamma^{(L)}} + h_{e_\Gamma^{(R)}}}[\theta^{k+\frac{1}{2}}]^2_\Gamma ds]^{1\backslash 2}
\end{aligned}$$

$$+L_H[\sum_{\Gamma\in\partial\mathcal{T}_h^B}\int_\Gamma \frac{h_{e_\Gamma^{(L)}}}{C_W}(2\mid u^k|_\Gamma^{(L)} - \Pi_h u^k|_\Gamma^{(L)}\mid)^2\,ds]^{1\backslash 2}[\sum_{\Gamma\in\partial\mathcal{T}_h^B}\int_\Gamma \frac{C_W}{h_{e_\Gamma^{(L)}}}(\theta^{k+\frac{1}{2}}|_\Gamma)^2 ds]^{1\backslash 2}$$

$$\le \frac{L_H\sqrt{2}(1+C_N)^{1\backslash 2}}{C_W^{1\backslash 2}}[\sum_{\Gamma\in\partial\mathcal{T}_h} h_e \parallel u^k - \Pi_h u^k \parallel^2_{L^2(\partial e)}]^{1\backslash 2} J_h(\theta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}})^{1\backslash 2},$$

by use the lemmas(6.1, 6.2), (5.4), Young's inequality and remark 5.1-c,e we get,

$$\begin{aligned}
\sum_{\Gamma\in\partial\mathcal{T}_h} h_e \parallel \eta^k \parallel^2_{L^2(\partial e)} \;&\le\; C_M\sum_{\Gamma\in\partial\mathcal{T}_h} h_e(\parallel \eta^k \parallel_{L^2(e)} C_I h_e^{-1} \mid \eta^k \mid_{H^1(e)} + h_e^{-1} \parallel \eta^k \parallel^2_{L^2(e)}) \\
&\le\; C_M(1+C_I) \parallel \eta^k \parallel^2_{L^2(\Omega)}\,.
\end{aligned}$$

Now, we have

$$| B_2^{(411)} + B_3^{(411)} | \leq C\tau^2 + C\lambda J_h(\theta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}),$$

then

$$| B^{(411)} | \leq C\tau^2 + C\lambda \parallel \theta^{k+\frac{1}{2}} \parallel_{DG}^2 . \tag{7.9}$$

To estimate $B^{(412)}$,

$$B^{(412)} = \underbrace{-\sum_{e\in\mathcal{T}_h}\int_e (\vec{b}(\Pi_h u^k).\Pi_h u^k - \vec{b}(U^k).U^k).\nabla\theta^{k+\frac{1}{2}}\,dx}_{B_1^{(412)}}$$

$$+\underbrace{\sum_{\Gamma\in\partial\mathcal{T}_h^I}\int_\Gamma [(H^k|_\Gamma^{(L)}\Pi_h u^k|_\Gamma^{(L)} + H^k|_\Gamma^{(R)}\Pi_h u^k|_\Gamma^{(R)}) - (H^k|_\Gamma^{(L)}U^k|_\Gamma^{(L)} + H^k|_\Gamma^{(R)}U^k|_\Gamma^{(R)})].[\theta^{k+\frac{1}{2}}]_\Gamma ds}_{B_2^{(412)}}$$

$$+\underbrace{\sum_{\Gamma\in\partial\mathcal{T}_h^B}\int_\Gamma [(H^k|_\Gamma^{(L)}\Pi_h u^k|_\Gamma^{(L)} + H^k|_\Gamma^{(L)}\Pi_h u^k|_\Gamma^{(L)}) - (H^k|_\Gamma^{(L)}U^k|_\Gamma^{(L)} + H^k|_\Gamma^{(L)}U^k|_\Gamma^{(L)})].\theta^{k+\frac{1}{2}}|_\Gamma\, ds}_{B_3^{(412)}}.$$

To estimate $B_1^{(412)}$, we add and subtract the term $\sum_{e\in\mathcal{T}_h}\int_e(\vec{b}(\Pi_h u^k).U^k).\nabla\theta^{k+\frac{1}{2}}\,dx$, use Cauchy-Schwartz inequality, Young's inequality, from the definition 1.2 in [1], [3, lemma 2.4.1], the assumption A(2.2) and remark 5.1-c,e, gives

$$B_1^{(412)} \leq C\tau^2 + C\lambda \mid \theta^{k+\frac{1}{2}} \mid_{H^1(\Omega)}^2 .$$

To estimate $B_2^{(412)} + B_3^{(412)}$, we use the property of the numerical flux 4.1-a, Cauchy-Schwartz inequality, (5.3) and the assumption 3.1-e exactly same the steps of estimate for ($B_2^{(411)} + B_3^{(411)}$), imply that

$$| B_2^{(412)} + B_3^{(412)} | \leq C\tau^2 + C\lambda J_h(\theta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}),$$

Now we have

$$| B^{(412)} | \leq C\tau^2 + C\lambda \parallel \theta^{k+\frac{1}{2}} \parallel_{DG}^2 . \tag{7.10}$$

Then from (7.9) and (7.10) we get,

$$| B^{(41)} | \leq C\tau^2 + C\lambda \parallel \theta^{k+\frac{1}{2}} \parallel_{DG}^2 . \tag{7.11}$$

To estimate$B^{(42)}$, we use Green's theorem we get,

$$| B^{(42)} | \leq \underbrace{| (\vec{b}(u^{k+1}).u^{k+1}, \nabla\theta^{k+\frac{1}{2}}) - (\vec{b}(u^k).u^k, \nabla\theta^{k+\frac{1}{2}}) |}_{B^{(421)}}$$

$$+\underbrace{\mid \sum_{\Gamma\in\partial\mathcal{T}_h}\int_\Gamma \vec{b}(u^{k+1}).u^{k+1}n\theta^{k+\frac{1}{2}}\,ds - \sum_{\Gamma\in\partial\mathcal{T}_h}\int_\Gamma \vec{b}(u^{k}).u^{k}n\theta^{k+\frac{1}{2}}\,ds \mid}_{B^{(422)}}$$

To estimate $B^{(421)}$, we add and subtract $(\vec{b}(u^{k+1}).u^k, \nabla\theta^{k+\frac{1}{2}})$, by using Cauchy-Schwartz inequality, [3, lemma 2.4.1], Young's inequality and the conditions A(2.1)-A(2.3), we get

$$\begin{aligned} \mid B^{(421)} \mid &\leq \mid (\vec{b}(u^{k+1}).(u^{k+1}-u^k), \nabla\theta^{k+\frac{1}{2}}) \mid + \mid ((\vec{b}(u^{k+1})-\vec{b}(u^k)).u^k, \nabla\theta^{k+\frac{1}{2}}) \mid \\ &\leq C\tau^2 + C\lambda \mid \theta^{k+\frac{1}{2}} \mid^2_{H^1(\Omega)}, \end{aligned}$$

to estimate $B^{(422)}$, we use the property of the numerical flux 4.1-a, Cauchy-Schwartz inequality, (5.3), A(2.1)-A(2.3), [3, lemma 2.4.1], Young's inequality and the assumption 3.1-e, imply that

$$\mid B^{(422)} \mid \leq C\tau^2 + C\lambda J_h(\theta^{k+\frac{1}{2}}, \theta^{k+\frac{1}{2}}),$$

then,

$$\mid B^{(42)} \mid \leq C\tau^2 + C\lambda \parallel \theta^{k+\frac{1}{2}} \parallel^2_{DG}. \tag{7.12}$$

Now from (7.11) and (7.12),we get

$$\mid B^{(4)} \mid \leq C\tau^2 + C\lambda \parallel \theta^{k+\frac{1}{2}} \parallel^2_{DG}. \tag{7.13}$$

By substituting of equations: (7.6), (7.7), (7.8) and (7.13) in equation (7.5), we have

$$\frac{1}{2}D_\tau \parallel \theta^k \parallel^2_{L^2(\Omega)} \leq C\tau^2 + C \parallel \theta^{k+\frac{1}{2}} \parallel^2_{L^2(\Omega)},$$

then, we have

$$\parallel \theta^{k+1} \parallel^2_{L^2(\Omega)} - \parallel \theta^k \parallel^2_{L^2(\Omega)} \leq C\tau^4 + \frac{C\tau^2}{2}(\parallel \theta^{k+1} \parallel^2_{L^2(\Omega)} + \parallel \theta^k \parallel^2_{L^2(\Omega)}).$$

By using the relation [8], $(\beta+\xi)^2 \leq 2(\beta^2+\xi^2)$, gives

$$\parallel \theta^{k+1} \parallel^2 \leq \frac{C\tau^4}{(1-2C\tau^2)} + \frac{(1+2C\tau^2)}{(1-2C\tau^2)} \parallel \theta^k \parallel^2_{L^2(\Omega)}.$$

This implies that if $\tau > 0$ is small enough,

$$\parallel \theta^{k+1} \parallel^2 \leq C\tau^4 + C\tau^2 \parallel \theta^k \parallel^2_{L^2(\Omega)}.$$

Since $\parallel \theta^k(0) \parallel^2_{L^2(\Omega)} \leq C\tau^2 \mid u^0 \mid^2_{H^1(\Omega)}$ then

$$\parallel \theta^{k+1} \parallel_{L^2(\Omega)} \leq C\tau^2$$

and also

$$\parallel \eta^{k+1} \parallel_{L^2(\Omega)} \leq Ch^2,$$

hence the proof is complete.

8 Conclusion.

In this paper we derived error estimates of the full DGFEM applied to the numerical simulation of nonstationary convection-diffusion problem equipped with Dirichlet boundary condition and we show that the order of the error is of $O(h^2 + \tau^2)$ which is better than all error which derived by [3], [8].

References

[1] A.Quarteroni and A. Valli,*Numerical approximation of partial differential equations* , ISBN 3-540-57111-6. Springer-Verlag, Berlin Heidelberg, New York, 1997.

[2] B. Cockburn, G. E. Karniadakis, And C.W. Shu Eds.,*Discontinuous Galerkin Methods, Lecture Notes in Computational Science and Engineering,* Springer, Berlin,**11**(2000).

[3] H. A. Kashkool, *Upwind Finite Element Scheme For Non-linear Convection-Diffusion Problem And Applications To Numerical Simulation Reservoir,* Ph. D. thesis, College of Mathematical Sciences, Nankai University, China, 2002.

[4] L. Angermann,*Error estimates for the finite-element solution of an elliptic singularly perturbed problem* , IMA J. of Numer. Anal.,**15**, (1995) 161-196.

[5] M. F. Wheeler,*An elliptic collocation-finite element method with interior penalties,* SIAM J. Numer. Anal., **15** (1978), 152-161.

[6] M. Feistauer, J. Felcman and I.straškraba,*Mathematical and Computational Problems for Compressible Flow,* Clarendon Press, Oxford, 2003.

[7] M. Feistauer and K. Svadlenka,*Discontinuous Galerkin method of lines for solving nonstationary singularly perturbed linear problems,* J. Numer. Math., **2** (2004), 97-117.

[8] M. Feistauer, V. Dolejší, V. Kučera and V. Sobotiíková,*An Optimal $L^\infty(L^2)$-Error Estimate for the DG Approximation of a Nonlinear Nonstationary Convection-Diffusion problem on Nonconforming Meshes,* Math. Modelling and Numer. Anal. submitted, (2009), 1 -23.

[9] P.G. Ciarlet, *The finite element method for elliptic problems,* Amsterdam. North Holland, 1978.

[10] T. Ikeda,*Maximum principle in finite element models for convection-diffusion phenomena ,* Lecture Notes in Numerical and Applied Analysis **4**, North-Holland, 1983.

[11] T. Roubíček, *Nonlinear Partial Differential Equations With Applications,* Birkhuser, Basel, Boston, Berlin 2005.

[12] V. Dolejší ,*On the discontinuous Galerkin method for the numerical solution of the Navier-Stokes equations,* Internat. J. Numer. Methods Fluids, **45** (2004), 1083-1106.

[13] V. Kučera,*optimal $L^\infty(L^2)$ -error estimate for the DG method applied to nonlinear convection-diffusion problems with nonlinear diffusion,* IMA J. Numer. Anal., submitted,(2008) 1-25.

[14] V. Dolejší, *M. Feistauer, and C. Schwab, On some aspects of the discontinuous Galerkin finite element method for conservation laws,* Math. Comput. Simulation, **61** (2003), 333-346.

[15] V. Dolejší,*M. Feistauer and V. Sobotíková, Analysis discontinuous Galerkin method for nonlinear convection-diffusion problems,* Comput. Methods Appl. Mech. Engrg., **194** (2005), 2709-2733.

[16] V. Dolejší, M. Feistauer, V. Kučera and V. Sobotíková, *An optimal $L^\infty(L^2)$ -error estimate for the discontinuous Galerkin approximation for a nonlinear nonstationary convection-diffusion problem,* IMA J. Numer. Anal., **28** (2008), 496-521.

Received: April, 2012