Some Properties of Composition Operators on Weighted Hardy Spaces

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Abstract

Let \( \varphi \) be an analytic map of unit disk \( \mathbb{D} \) into itself, consider the composition operator \( C_{\varphi} \) defined by \( C_{\varphi}(f) = f \circ \varphi \) whenever \( f \) is analytic on \( \mathbb{D} \). In this paper, we discuss necessary and sufficient conditions under which a composition operator on a large class of weighted Hardy spaces is a compact.

1 Introduction and Preliminaries

An analytic self-map \( \varphi : \mathbb{D} \to \mathbb{D} \) of the open unit disk \( \mathbb{D} \) in the complex plane induces the composition operator \( C_{\varphi} \) on \( H(\mathbb{D}) \), the space of holomorphic functions on \( \mathbb{D} \), defined by \( C_{\varphi}(f) = \varphi \circ f \). A basic goal in the study of composition operators is to relate function theoretic properties of \( \varphi \) to operator theoretic properties of \( C_{\varphi} \). Here we review some results that characterize when \( \varphi \) induces a compact composition operator between various classical Banach spaces of holomorphic functions on \( \mathbb{D} \).

For \( p > 0 \), the Hardy space \( H^p(D) \) is the space of functions \( f \) that are analytic on \( \mathbb{D} \) and satisfy

\[
\lim_{r \to 1^-} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty,
\]

Denote the \( p^{th} \) root of this limit by \( \| f \|_p \). The Hard space \( H^\infty(D) \) is the set of analytic functions that are bounded in \( D \), with supremum norm \( \| f \|_\infty \) and the Bergman space \( A^p(D) \) consists of those analytic functions such that

\[
\int_D |f(z)|^p \frac{dA(z)}{\pi} < \infty
\]

Here \( dA \) denotes area measure on \( \mathbb{D} \) by, normalized so that \( A(\mathbb{D}) = 1 \), similarly \( \| f \|_p \) is the \( p^{th} \) root of this integral.
In the following the sufficient condition for compactness of composition operator \( C_\varphi \) in standard Hardy space \( H^2 \) is given.

The weighted Hardy space \( H^2(\beta) \) is a Hilbert space whose vectors are functions analytic on the unit disk and monomials \( 1, z, z^2, \ldots \) form a complete orthogonal set of non-zero vectors. If \( \|z^j\| = \beta(j) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) then \( f \in H^2(\beta) \) if and only if

\[
\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 < \infty
\]

The weighted Bergman space defined for \( \alpha > -1 \) by

\[
A^p_\alpha(D) = \{ f \text{ analytic in } D : \int_D |f(z)|^p (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} < \infty \}
\]

The linear functional \( K_x \) with \( K_x(f) = f(x) \) is called reproducing kernel. The generating function for the weighted Hardy space \( H^2(\beta) \) is the function \( k(z) = \sum_{j=0}^{\infty} \frac{z^j}{\beta(j)} \). This function is analytic on \( D \). Moreover, if \( w \in \mathbb{D} \) then \( K_w(z) = k(\bar{w}z) \) and \( \|K_w\|^2 = k(|w|^2) \), (see [1]).

Recently, there has been a great interest in studying operator theoretic properties of composition and weighted composition operators, see, for example, monographs [1] and [4], papers [2], [3], [7] and [8] as well as the reference therein.

**Theorem 1.** In a weighted Hardy space for which the series \( \sum \beta(n)^{-2} \) diverges, the normalized reproducing kernels

\[
\frac{K_\omega}{\|K_\omega\|}
\]

tends to 0 weakly as \( |\omega| \) tends to 1.

The Reproducing kernels for elements near of the \( \partial \mathbb{D} \) for weighted Hardy spaces has important consequences. For example we show that the generating function is smooth on the closed unit disk.

**Theorem 2.** If \( H^2(\beta) \) is a weighted Hardy space for which the generating function \( k \) is continuous on the closed disk then all functions in \( H^2(\beta) \) can be extended continuously to the closed disk.

**Proof.** Let \( f \in H^2(\beta), w_0 \in \partial \mathbb{D} \) we show that \( \lim_{w \to w_0} \) exists. To show this let \( z \in \mathbb{D} \)

\[
K_{w_0}(z) = \lim_{w \to w_0} = \lim_{w \to w_0} k(\bar{w}z) = k(\bar{w_0}z) < \infty
\]

and hence \( \|K_w - K_{w_0}\| \to 0 \) as \( w \to w_0 \). On the other hand \( \|K_w\|^2 = k(|w|^2) \) thus \( \|K_{w_0}\|^2 = k(|w_0|^2) \) and then
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$$\|K_w - K_{w_0}\|^2 = \|K_w\|^2 - 2\text{Re}\langle K_{w_0}, K_w \rangle + \|K_{w_0}\|^2$$

$$= \|K_w\|^2 - 2\text{Re}K_w(w_0) + \|K_{w_0}\|^2 \to 0 \text{ as } w \to w_0$$

thus

$$|f(w) - \langle f, K_{w_0} \rangle| = |\langle f, K_w - K_{w_0} \rangle| \leq \|f\| \|K_w - K_{w_0}\| \to 0$$

So

$$\lim_{w \to w_0} f(w) = \langle f, K_{w_0} \rangle = f(w_0)$$

In follow we give an example of a Hilbert space of analytic functions on \(\mathbb{D}\) consisting continuous functions on \(\overline{\mathbb{D}}\) but we can find at least one function that is not differentiable.

**Example 1.** Consider the Hardy space \(H^2(\beta)\) containing all analytic functions \(f\) on the unit disk for which

$$\|f\|^2 = \sum_{j=0}^{\infty} \beta(j)^2|a_j|^2 < \infty$$

where \(\beta(0) = 1\), \(\beta(j) = j\). It is clearly \(k(z)\) is continuous on \(\overline{\mathbb{D}}\) because

\(k(z) = \sum_{j=1}^{\infty} \frac{z^j}{j!^2} = \sum_{j=1}^{\infty} \frac{z^j}{j^2}\) by M-test weierstrass this series is uniformly convergence on \(\mathbb{D}\) thus by last Theorem each function in \(H^2(\beta)\) extends to continuous function on \(\overline{\mathbb{D}}\). The radius convergence of \(f\) is \(\geq 1\) thus \(f'(z) = \sum_{j=1}^{\infty} \frac{z^{j-1}}{j}\) is divergence, \(f\) is not differentiable at 1.

### 2 Necessary and sufficient conditions for Compactness

A linear operator on a Banach space is compact if the image of the unit ball under the operator has compact closure. This definition has a good reformation. Indeed If \(X\) is the weighted Hardy space \(H^2(\beta)\), The Hardy space \(H^p(D)\) or \(A_\alpha^p(D)\) for for \(0 < p \leq \infty\) and \(\alpha > -1\). Then \(C_\varphi\) is compact on \(X\) if and only if whenever \(\{f_n\}\) is bounded in \(X\) and \(f_n \to 0\) uniformly on compact subsets of \(D\) then \(C_\varphi f_n \to 0\) in \(X\).

**Proposition 1.** If \(\|\varphi\|_{\infty} < 1\) then \(C_\varphi\) is compact on \(H^2\).

**Proof.** Suppose \(\{f_n\}\) is a bounded sequence in \(H^2\) that converges to zero uniformly on compact subset of \(D\). By Theorem (2.1.12) it is enough to show
that \( \|f_n \circ \varphi\| \to 0 \). But even more is true: since \( \varphi(D) \) is a relatively compact subset of \( D \) and \( f_n \to 0 \) uniformly on \( \varphi(D) \), hence
\[
\|f_n \circ \varphi\| \leq \|f_n \circ \varphi\|_{\infty} = \sup_{z \in D} |(f_n \circ \varphi)(z)| \\
\leq \sup_{w \in \varphi(D)} |f_n(w)| \to 0.
\]
as desired.  

**Corollary 1.** Consider \( H^p(D) \) or \( A^p_\alpha(D) \) for \( 0 < p \leq \infty \) and \( \alpha > -1 \). If \( \overline{\varphi(D)} \subset D \) then \( C_\varphi \) is compact.

**Proof.** Suppose that \( \{f_n\} \) is a bounded sequence and \( f_n \to 0 \) uniformly on compact subsets of \( D \) since \( \varphi(D) \subset D \) is compact thus \( f_n \circ \varphi \to 0 \) uniformly and hence \( C_\varphi \) is compact.

**Corollary 2.** \( C_\varphi \) is compact on \( H^\infty(D) \) if and only if \( \overline{\varphi(D)} \subset D \).

**Proof.** Suppose that \( C_\varphi \) is compact on \( H^\infty(D) \), and \( \overline{\varphi(D)} \notin D \), so there exist \( w_0 \in \varphi(D) \) such that \( w_0 \in w_0 \) so \( |w_0| = 1 \) thus there exist a sequence \( \{z_n\} \) in \( D \) such that \( \varphi(z_n) \to w_0 \). Thus \( \|\varphi\|_{\infty} = 1 \). On the other hand, if \( f_n(z) = z^n \), \( f_n \to 0 \), uniformly on compact subsets of \( D \). But \( C_\varphi \) is compact, so by proposition? \( f_n \circ \varphi \to 0 \) in \( H^\infty(D) \), that means \( \varphi^n \to 0 \) in \( H^\infty(D) \) or \( \|\varphi^n\|_{\infty} \to 0 \). It is a contradiction since \( \|\varphi\|_{\infty} = 1 \), and consequently for every \( n \) we have \( \|\varphi^n\|_{\infty} = 1 \) so we must have \( \varphi(D) \subset D \).

Conversely suppose that \( \overline{\varphi(D)} \subset D \), and \( \{f_n\} \) is a bounded sequence in \( H^\infty(D) \) and \( f_n \to 0 \) uniformly on compact subsets of \( D \). Since \( \overline{\varphi(D)} \) is compact thus \( f_n \to 0 \) uniformly on \( \overline{\varphi(D)} \). Since \( \{f_n(w) : w \in \varphi(D)\} \subset \{f_n(w) : w \in \varphi(D)\} \) then \( \|f_n \circ \varphi\|_{\infty} \to 0 \) as \( n \to \infty \).

An equivalence necessary condition for compactness of composition operator \( C_\varphi \) on Hardy space \( H^p(D) \) or \( A^p_\alpha(D) \) with respect to Jullia-caratheodory Theorem so makes the relationship between the angular derivative \( \varphi'(x\xi) \) and the evale \( d(\xi) \) is following.

**Corollary 3.** If \( C_\varphi \) is compact on \( H^p(D) \) or \( A^p_\alpha(D) \) then \( \varphi \) has no finite angular derivative at any point of \( \partial D \).

**Theorem 3 (P.Ahem and D.Clark).** The singular inner function
\[
\varphi(z) = exp \int_{\partial D} \frac{z + \xi}{z - \xi} d\mu(\xi)
\]
for \( \mu \) singular to lebesgue measure has finite angular derivative at \( \eta \) in \( \partial D \) if and only if
\[
\int_{\partial D} \frac{d\mu(\xi)}{|\xi - \eta|^2} < \infty.
\]
Corollary 4. There exist inner functions that induce compact composition operators on $A^p_\alpha(D)$, for $\alpha > -1$ and $0 < p < \infty$.

Proof. Choose positive numbers $(\mu_n)_n$ such that $\sum_n \mu_n < \infty$, but $\sum_n \sqrt{\mu_n} = \infty$. (for example $\mu_n = \frac{1}{n^2}$). Let $(I_n)_n$ be a sequence of consecutive arcs on $\partial \mathbb{D}$ with $l(T_n) = \sqrt{\mu_n}$, and let $\xi_n$ be the center of $T_n$. Let $\mu = \sum_n \mu_n \delta_n$, where $\delta_n$ is the unit mass at $\xi_n$.

Now we show that for every $w \in \partial \mathbb{D}$, $w$ belongs to infinitely many intervals $I_n$, hence $|w - \xi_n| < \sqrt{\mu_n}$ for infinitely many $n$, so the series $\sum_n \frac{\mu_n}{|\xi_n - w|^2}$ diverges, since infinitely many terms are $> 1$. But

$$
\int_{\partial \mathbb{D}} \frac{d\mu(\xi)}{|\xi - w|^2} = \sum_n \frac{\mu_n}{|\xi_n - w|^2} = \infty.
$$

Hence by last Theorem $\varphi$ has no finite angular derivative at any point of $\partial D$ and then $C\varphi$ is compact on $A^p_\alpha(D)$.

In the following we show that the converse of corollary 3. is not necessary valid. Indeed we give an example on $H^p(D)$ such that map $\varphi$ maps $\mathbb{D}$ into itself, has no finite angular derivative and $C\varphi$ is not compact on $H^p(D)$.

Example 2. Let $\psi_0$ be an inner function as in Theorem 3. Let $a = \psi_0(0)$, consider $\psi(z) = \frac{a - \psi_0(z)}{1 - \bar{a}\psi_0(z)} \ \forall z \in \mathbb{D}$.

It is clearly $\psi(0) = 0$. Define $\varphi(z) = \frac{1 + \psi(z)}{2}$ for all $z \in \mathbb{D}$; clearly $|\varphi^*(\xi)| < 1$ almost every where.

We have $\varphi = \chi \psi_0$, where $\chi(z) = \frac{1 + z}{2}$, then $C\varphi = C\psi_0 \o C\chi$ since $\chi$ has finite angular derivative at every point of $\partial \mathbb{D}$ so $C\chi$ is no compact, so it takes the unit ball of $H^p$ into a set $A$ whose closure is not compact, but $\psi_0$ is an inner, which fixes the origin, so $C\varphi$ is an isometry of $H^p$, so $C\chi(A)$ does not have compact closure either. But $C\psi(A) = C\varphi(ballH^p)$ so $C\varphi$ is not compact operator

Now we put many conditions on weighted sequence $\beta(n)$ such that weighted Hardy space $H^2(\beta)$ become "small" or "large". Although the terminology small weighted Hardy space is vary somewhat from theorem to theorem, but one of this conditions is $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} < \infty$, on each other if weight $\beta(n)$ on $H^2(\beta)$ tends to 0 sufficiently rapidly so that $\lim_{n \to \infty} n^a \beta(n) = 0$, for all $a > 0$ $H^2(\beta)$ is called large weighted Hardy space.

Krete and MacCluer in [3] proved in such large weighted Hardy space if the symbol function $\varphi$ has angular derivative in $D$ at some point in the circle then composition operator $C\varphi$ does not map $H^1(\beta)$ into itself. In the following a dual result of Krete and MacCluer Theorem is obtained.
Theorem 4. Suppose \( \varphi : D \rightarrow D \) is analytic with \( |\varphi'(\xi)| > 1 \) for some \( \xi \in \partial D \) satisfying \( |\varphi(\xi)| = 1 \). Then \( C_\varphi \) is not compact on \( H^2(\beta) \) when ever \( \{\beta(n)\} \) satisfies
\[
\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} < \infty.
\]

Proof. Suppose \( \varphi(\xi) = \eta \in \partial D \). Let \( \psi(z) = \xi \bar{\eta} \varphi(z) \) then we have \( \psi(\xi) = \xi \bar{\eta} \varphi(\xi) = \xi \bar{\eta} = \xi \) and \( \psi'(\xi) = |\psi'(\xi)| = |\varphi'(\xi)| > 1 \). Since \( \psi \) is analytic self-map of \( D \) if \( \psi \) has no fixed point in \( D \) then by Wolff’s Theorem there is a unique point on \( \partial D \) such that \( \varphi \) has angular derivative less than or equal one. Therefore this point can not be \( \xi \). Call this point \( a \). We have
\[
C_\varphi^*(K_\xi) = K_{\psi(\xi)} = K_\xi \quad C_\psi^*(K_a) = K_{\psi(a)} = K_a,
\]
where \( K_w \) denotes the kernel function for evaluation at \( w \in \bar{D} \). Thus if \( C_\varphi \), and hence \( C_\psi \), is compact on \( H^2(\beta) \), then \( \dim \ker(C_\psi - 1) = \dim \ker(C_\psi^* - 1) \geq 2 \).

But if \( f \in \ker(C_\psi - 1) \) then \( C_\psi f = f \) and therefore \( f \circ \psi_n = f \), where \( \psi_n \) is the \( n \)th iterate of \( \psi \). Now by Denjoy-Wolff theorem and continuity of \( f \) on \( \bar{D} \) implies that \( f \) is constant.

If \( \psi \) has a fixed point in \( D \), since \( \psi \) is not an elliptic automorphism of \( D \) and by proportion (2.4.1) implies that \( f \) is constant and its contradiction with \( \dim \ker(C_\psi - 1) \geq 2 \). Thus \( C_\varphi \) can not be compact on \( H^2(\beta) \). \( \square \)

The next result applies to small spaces defined slightly more restrictively by requiring that functions in the space have derivative which extends continuously to \( \bar{D} \).

Theorem 5. Suppose \( \varphi : D \rightarrow D \) is analytic with \( |\varphi'(\xi)| = 1 \) for some \( \xi \in \partial D \) with \( |\varphi(\xi)| = 1 \). If
\[
\sum_{n=0}^{\infty} \frac{n^2}{\beta(n)^2} < \infty,
\]
then \( C_\varphi \) is not compact on \( H^2(\beta) \).

Proof. We normalize by choosing \( e^{i\theta} \) such that \( \psi = e^{i\theta} \varphi \) fixes \( \xi \), therefore \( \psi(\xi) = \xi \). By theorem (2.4.3) we have
\[
\psi'(\xi) = |\psi'(\xi)| = |e^{i\theta} \varphi'(\xi)| = |\varphi'(\xi)| = 1.
\]
Thus \( \psi'(\xi) = 1 \). As before \( C_\psi^*(K_\xi) = K_\xi \), where \( K_\xi \) is the kernel function for evaluation at \( \xi \).

If \( K^{(1)}_\xi \) is the kernel function for the evaluation of the first derivative at \( \xi \), then we have
\[
K_\xi(z) = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^2}(\xi z)^n,
\]
and

\[ K_{ξ}^{(1)}(z) = \sum_{n=1}^{∞} \frac{n}{β(n)^2} ξ^{n-1} z^n. \]

\( K_{ξ}^{(1)}(z) \) is a function in \( H^2(β) \) and for \( f \in H^2(β) \) we have

\[
< f, K_{ξ}^{(1)}(z) >_β = \sum_{n=0}^{∞} a_n \frac{n}{β(n)^2} ξ^{n-1} \frac{β(n)^2}{β(n)^2}
= \sum_{n=0}^{∞} n a_n ξ^{n-1}
= f'(ξ).
\]

Thus

\[
< f, C_ψ K_{ξ}^{(1)}>_β = < C_ψ f, K_{ξ}^{(1)}>_β
= < f \circ ψ, K_{ξ}^{(1)}>_β
= (f \circ ψ)'(ξ)
= ψ'(ξ) f'(ψ(ξ))
= ψ'(ξ) < f, K_{ψ(ξ)}^{(1)}>_β
= < f, ψ'(ξ) K_{ψ(ξ)}^{(1)}>_β
= < f, K_{ξ}^{(1)}>_β.
\]

Therefore

\[ C_ψ K_{ξ}^{(1)} = K_{ξ}^{(1)} \]

But we know that \( C_ψ K_{ξ} = K_{ξ} \), thus 1 is an eigenvalue of \( C_ψ \) with multiplicity at least 2. As in the proof of theorem (2.4.3) this show that \( C_ψ \) (and hence \( C_φ \)) cannot be compact, since \( \dim \ker(C_ψ - 1) = 1 \). □

The next result, due to \( J. \) Shapiro [17], provides a stepping stone for obtaining a necessary condition, in term of the angular derivative, for \( C_φ \) to be bounded on \( H^2(β) \).

\[ \]

In contrast to the function spaces considered in the last chapter, here we consider weighted Bergman or Hardy spaces that include functions which grow much more rapidly than functions in \( H^2(D) \) or the standard weighted Bergman spaces \( A^2_α(D) \). We will confine our attention to function spaces on the disk, and concentrate on spaces \( A^2_δ(D) \) defined for weights \( G \) positive, continuous, and
non-increasing on $(0, 1)$ with $\int_0^1 G(r) < \infty$ so that $G(|z|) dA(z)$ is a positive, circularly symmetric finite measure on $D$. Set
\[
A^2_G(D) \equiv \{ \text{fanalytic} : \int_D |f(z)|^2 G(|z|) \frac{dA(z)}{\pi} \}
\]
with
\[
||f||^2_G = \int_D |f(z)|^2 G(|z|) \frac{dA(z)}{\pi}
\]
The space $A^2_G$ is equivalent to the weighted Hardy space $H^2(\beta)$ where, $\beta(n)^2 = p_n/c$ where $\{p_n\}$ are the moments
\[
p_n = \int_0^1 r^{2n+1} G(r) dr
\]
and $c = 2 \int_0^1 G(r)r dr$, chosen so that $\beta(0) = 1$.

**Theorem 6.** *(Kriete and MacCluer)* Let $G$ be a regular fast weight. Then the following statement are equivalent.
(i) $C_\varphi$ is compact on $A^2_G$
(ii) $\lim_{|z| \to 1} G(|z|)/G(|\varphi(z)|) = 0$
(iii) $|\varphi'(\xi)| > 1$ for all $\xi$ in $\partial D$
(iv) $\beta(\varphi) > 1$

**Corollary 5.** If $\varphi(0) = 0$ and $\varphi$ is not a rotation of $D$, then $C_\varphi$ is compact on $A^2_G(D)$, where $G$ is fast and regular.

**Proof.** By Theorem 6, it is enough to show that $|\varphi'(\xi)| > 1$ for all $\xi$ in $\partial D$.

suppose $\xi$ is in the unit circle and $|\varphi'(\xi)| \leq 1$ thus $d(\xi)$ is finite and then by julia’s Lemma we conclude that $d(\xi) = 1$ that means equality holds for $0 \in D$ and then $\varphi$ is an automorphism of the disk. Thus
\[
\varphi(z) = \lambda \frac{a - z}{1 - \bar{a}z}, \quad |\lambda| = 1, \quad |a| < 1.
\]
But $0 = \varphi(0) = \lambda a$ thus $a = 0$. This is a contradiction with $\varphi$ is not a rotation.

**References**


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