On Syndetically Hypercyclic Tuples

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Abstract

In this paper we will give some conditions for a tuple of operators or tuple of weighted shifts to be Syndetically Hypercyclic.

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1 Introduction

Let $F$ be a topological vector space (TVS) and $T_1, T_2, \ldots, T_n$ are continuous mapping on $F$, and $T = (T_1, T_2, \ldots, T_n)$ be a tuple of operators $T_1, T_2, \ldots, T_n$. The tuple $T$ is weakly mixing, if and only if, for any pair of non-empty open subsets $U, V$ in $X$, and for any syndetic sequences $\{m_{k,1}\}, \{m_{k,2}\}, \ldots, \{m_{k,n}\}$ with $\operatorname{Sup}_k (n_{k+1,j} - n_{k,j}) < \infty$ for $j = 1, 2, \ldots, n$, then there exist $m_{k,1}, m_{k,2}, \ldots, m_{k,n}$ such that $T_{1}^{m_{k,1}}T_{2}^{m_{k,2}}\ldots T_{n}^{m_{k,n}}U \cap V \neq \emptyset$, also, if and only if, it suffices in previous condition, to consider only those sequences $m_{k,1}, m_{k,2}, \ldots, m_{k,n}$ for which there is some $m_1 \geq 1, m_2 \geq 1, \ldots, m_n \geq 1$ with $m_{k,1} \in \{m_j, 2m_j\}$ for all $k$ and all $j$. Reader can see [1–10] for some information.

2 Main Results

Let $\mathcal{X}$ be a metrizable and complete topological vector space (TVS) and $T = (T_1, T_2, \ldots, T_n)$ is an $n$-tuple of operators, then we will let

$$\mathcal{F} = \{T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n} : k_i \geq 0\}$$

be the semigroup generated by $T$. For $x \in \mathcal{X}$ we take

$$\operatorname{Orb}(T, x) = \{Sx : S \in \mathcal{F}\} = \{T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}(x) : k_i \geq 0, i = 1, 2, \ldots, n\}.$$
The set \( \text{Orb}(\mathcal{T}, x) \) is called, orbit of vector \( x \) under \( \mathcal{T} \) and Tuple \( \mathcal{T} = (T_1, T_2, \ldots, T_n) \) is called hypercyclic pair if the set \( \text{Orb}(\mathcal{T}, x) \) is dense in \( \mathcal{X} \), that is

\[
\text{Orb}(\mathcal{T}, x) = \{ T_1^{k_1}T_2^{k_2} \ldots T_n^{k_n}(x) : k_i \geq 0, i = 1, 2, \ldots, n \} = \mathcal{X}.
\]

The tuple \( \mathcal{T} = (T_1, T_2, \ldots, T_n) \) is called topologically mixing if for any given open subsets \( U \) and \( V \) of \( \mathcal{X} \), there exist positive numbers \( K_1, K_2, \ldots, K_n \) such that

\[
T_1^{k_1,1}T_2^{k_2,1} \ldots T_n^{k_n,1}(U) \cap V \neq \phi , \quad \forall k_j, i \geq K_i , \quad \forall j = 1, 2, \ldots, n
\]

A sequence of operators \( \{T_n\}_{n \geq 0} \) is said to be a hypercyclic sequence on \( \mathcal{X} \) if there exists some \( x \in \mathcal{X} \) such that its orbit is dense in \( \mathcal{X} \), that is

\[
\text{Orb}(\{T_n\}_{n \geq 0}, x) = \text{Orb}(\{x, T_1x, T_2x, \ldots\}) = \mathcal{X}.
\]

In this case the vector \( x \) is called hypercyclic vector for the sequence \( \{T_n\}_{n \geq 0} \). Note that, if \( \{T_n\}_{n \geq 0} \) is a hypercyclic sequence of operators on \( \mathcal{X} \), then \( \mathcal{X} \) is necessarily separable. Also note that, the sequence \( \{T^n\} \) is a hypercyclic sequence on \( \mathcal{X} \), if and only if, the operator \( T \) is hypercyclic operator on \( \mathcal{X} \). \( T \) is said to satisfy the Hypercyclicity Criterion if it satisfies the hypothesis of below theorem.

**Theorem 2.1 (The Hypercyclicity Criterion)** Let \( \mathcal{X} \) be a separable Banach space and \( \mathcal{T} = (T_1, T_2, \ldots, T_n) \) is an \( n \)-tuple of continuous linear mappings on \( \mathcal{X} \). If there exist two dense subsets \( Y \) and \( Z \) in \( \mathcal{X} \), and strictly increasing sequences \( \{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, \ldots, \{m_{j,n}\}_{j=1}^{\infty} \) such that:

1. \( T_1^{m_{j,1}}T_2^{m_{j,2}} \ldots T_n^{m_{j,n}} \rightarrow 0 \) on \( Y \) as \( j \rightarrow \infty \),
2. There exist functions \( \{S_j : Z \rightarrow \mathcal{X}\} \) such that for every \( z \in Z \), \( S_j z \rightarrow 0 \), and \( T_1^{m_{j,1}}T_2^{m_{j,2}} \ldots T_n^{m_{j,n}}S_j z \rightarrow z \),

then \( \mathcal{T} \) is a hypercyclic \( n \)-tuple.

A strictly increasing sequence of positive integers \( \{n_k\}_k \) is said to be syndetic sequence, if \( \text{Sup}_k(n_{k+1} - n_k) < \infty \). A tuple \( \mathcal{T} = (T_1, T_2, \ldots, T_n) \) on a space \( \mathcal{X} \) is called syndetically hypercyclic if for any syndetic sequences of positive integers \( \{m_{k,1}\}_k, \{m_{k,2}\}_k, \ldots, \{m_{k,n}\}_k \)

the sequence

\[
\{T_1^{m_{k,1}}T_2^{m_{k,2}} \ldots T_n^{m_{k,n}}\}_k
\]
is hypercyclic, in other hand, there is \( x \in X \) such that \( \{T^n x : k \geq 0\} \) is dense in \( X \), that is,
\[
\{T_1^{m_{k,1}}T_2^{m_{k,2}} \ldots T_n^{m_{k,n}}(x)\} = X.
\]

Given continuous linear operators \( T_1,T_2,\ldots,T_n \) that is \( T_1,T_2,\ldots,T_n \in L(X) \), defined on a separable \( F \)-space \( X \), also suppose that \( T = (T_1,T_2,\ldots,T_n) \) be a tuple of operators \( T_1,T_2,\ldots,T_n \). Then \( T \) satisfies the Hypercyclicity Criterion if and only if for any strictly increasing sequences of positive integers \( \{m_{k,1}\}_k,\{n_{k,2}\}_k,\ldots,\{m_{k,n}\}_k \) such that
\[
\text{Sup}_k(m_{k+1,j} - m_{k,j}) < \infty
\]
for all \( j \), then the sequence \( \{T_1^{m_{k,1}}T_2^{m_{k,2}} \ldots T_n^{m_{k,n}}\}_k \) is hypercyclic. Also, for each hypercyclic vector \( x \in X \) of \( T \), there exists two strictly increasing sequence \( \{m_k\}_k,\{n_k\}_k \) such that
\[
\text{Sup}_k(m_{k+1,j} - n_{k,j}) < \infty
\]
for all \( j \), and \( \{T_1^{m_{k,1}}T_2^{m_{k,2}} \ldots T_n^{m_{k,n}}\}_k \) is somewhere dense, but not dense in \( X \), That is, the tuple \( T = (T_1,T_2,\ldots,T_n) \) and the sequence \( \{T_1^{m_{k,1}}T_2^{m_{k,2}} \ldots T_n^{m_{k,n}}\}_k \) do not share the same hypercyclic vectors. Let \( F \) be a Frechet space and \( T_1,T_2,\ldots,T_n \) are bounded linear operators on \( F \), and \( T = (T_1,T_2,\ldots,T_n) \) be a tuple of operators \( T_1,T_2,\ldots,T_n \). The space \( F \) is called topologically mixing if for any given open sets \( U \) and \( V \), there exist positive numbers \( M_1,M_2,\ldots,M_n \) such that
\[
T_1^{m_{1,1}}T_2^{m_{1,2}} \ldots T_n^{m_{1,n}}(U) \cap V \neq \emptyset \quad \forall m_{i,j} \geq M_j \quad i = 1,2,\ldots,n
\]
Notice that, If the tuple \( T \) satisfies the hypercyclic criterion for syndetic sequences, then \( T \) is topologically mixing tuple on space \( F \). Let \( V \) be a topological vector space (TVS) and \( T_1,T_2,\ldots,T_n \) are bounded linear operators on \( V \), and \( T = (T_1,T_2,\ldots,T_n) \) be a tuple of operators \( T_1,T_2,\ldots,T_n \). The tuple \( T \) is called weakly mixing if
\[
T \times T \times \ldots \times T : X \times X \times \ldots \times X \rightarrow X \times X \times \ldots \times X
\]
is topologically transitive.

**Theorem 2.2** Let \( X \) be a topological vector space (TVS) and \( T_1,T_2,\ldots,T_n \) are continuous mapping on \( X \), and \( T = (T_1,T_2,\ldots,T_n) \) be a tuple of operators \( T_1,T_2,\ldots,T_n \). Then the following are equivalent:
(i). \( T \) is weakly mixing.
(ii). For any pair of non-empty open subsets \( U,V \) in \( X \), and for any syndetic sequences \( \{m_{k,1}\},\{m_{k,2}\},\ldots,\{m_{k,n}\} \), there exist \( m'_{k,1},m'_{k,2},\ldots,m'_{k,n} \) such that
\[
T_1^{m'_{k,1}}T_2^{m'_{k,2}} \ldots T_n^{m'_{k,n}}(U) \cap (V) \neq \emptyset
\]
(iii). It suffices in (ii) to consider only those sequences \( \{m_{k,1}\}, \{m_{k,2}\}, \ldots, \{m_{k,n}\} \) for which there is some \( m_1 \geq 1, m_2 \geq 1, \ldots, m_n \geq 1 \) with

\[ m_{k,j} \in \{m_j, 2m_j\} \]

for all \( k \) and for all \( j \).

**Proof** (i) \( \rightarrow \) (ii). Given \( \{m_{k,1}\}, \{m_{k,2}\}, \ldots, \{m_{k,n}\} \) and \( U, V \) satisfying the hypothesis of condition (ii), take

\[ m_j = \text{Sup}_k \{m_{k+1,j} - m_{k,j}\} \]

for all \( j \) and the \( n \)-product map

\[
\underbrace{T \times T \ldots \times T}_{\text{n-times}}: X \times X \ldots X \rightarrow X \times X \ldots X
\]

is transitive, Then there is \( m_{k',1}, m_{k',2}, \ldots, m_{k',n} \) in \( N \) such that

\[
(T_1^{m_{k',1}}, T_2^{m_{k',2}}, \ldots, T_n^{m_{k',n}}(U)) \cap ((T_1^{m_{k',1}})^{-1}(T_2^{m_{k',2}})^{-1} \ldots (T_n^{m_{k',n}})^{-1}(V)) \neq \emptyset
\]

\[ \forall m_{k',1} = 1, 2, \ldots, m, \forall m_{k',2} = 1, 2, \ldots, m, \ldots, \forall m_{k',n} = 1, 2, \ldots, m \]

so

\[
(T_1^{m_{k',1}+m_{k',1}}, T_2^{m_{k',2}+m_{k',2}}, \ldots, T_n^{m_{k',n}+m_{k',n}}(U)) \cap (V) \neq \emptyset
\]

\[ \forall m_{k',1} = 1, 2, \ldots, m, \forall m_{k',2} = 1, 2, \ldots, m, \ldots, \forall m_{k',n} = 1, 2, \ldots, m \]

By the assumption on \( \{m_{k,1}\}, \{m_{k,2}\}, \ldots, \{m_{k,n}\} \), for all \( j \), we have

\[ \{m_{k,j} : k \in N\} \cap \{n + 1, n + 2, \ldots, n + m_j\} \neq \emptyset \]

If for all \( j \) we select \( m_{k,j} \in \{m_{k,j} : k \in N\} \cap \{n + 1, n + 2, \ldots, n + m_j\} \) then we have

\[ T_1^{m_{k,1}}T_2^{m_{k,2}}T_n^{m_{k,n}}(U) \cap (V) \neq \emptyset, \]

by this the proof of (i) \( \rightarrow \) (ii) is completed.

The case (ii) \( \rightarrow \) (iii) is trivial.

Case (iii) \( \rightarrow \) (i). Suppose that \( U, V_1, V_2 \) are non-empty open subsets of \( X \), then there are \( \{m_{k,1}\}, \{m_{k,2}\}, \ldots, \{m_{k,n}\} \) in \( N \) such that

\[ T_1^{m_{k,1}}T_2^{m_{k,2}}T_n^{m_{k,n}}U \cap V_1 \neq \emptyset \]

\[ T_1^{m_{k,1}}T_2^{m_{k,2}}T_n^{m_{k,n}}U \cap V_2 \neq \emptyset. \]

This will imply that \( T \) is weakly mixing. Since (iii) is satisfied, then we can take \( \{m_{k,1}\}, \{m_{k,2}\}, \ldots, \{m_{k,n}\} \) in \( N \) such that

\[ T_1^{m_{k,1}}T_2^{m_{k,2}}T_n^{m_{k,n}}V_1 \cap V_2 \neq \emptyset \]
By continuity, we can find $\widetilde{V}_1 \subset V_1$ open and non-empty such that

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} \widetilde{V}_1 \subset V_2.$$

Also there exist some $m_{k',1}, m_{k',2}, ..., m_{k',n}$ in $N$ such that

$$T_1^{m_{k',1} + \eta_1} T_2^{m_{k',2} + \eta_2} ... T_n^{m_{k',n} + \eta_n} U \subset \widetilde{V}_1$$

for $\eta_j = 0, m_j$ we take $m_{k,j} = m_{k',j} + \eta_j$, for all $j$, indeed we find strictly increasing sequences of positive integers $m_{k,1}, m_{k,2}, ..., m_{k,n}$ such that

$$m_{k,j} \in \{m_j, 2m_j\}$$

for all $j$,and

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} U \cap \widetilde{V}_1 = \emptyset, \forall k \in N$$

Now we have

$$T_1^{m_{k',1} + \eta_1} T_2^{m_{k',2} + \eta_2} ... T_n^{m_{k',n} + \eta_n} U \cap \widetilde{V}_1 \neq \emptyset$$

So the set

$$\emptyset \neq T_1^{m_{k',1}} T_2^{m_{k',2}} ... T_n^{m_{k',n}} (T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} U \cap \widetilde{V}_1)$$

is a subset of

$$(T_1^{m_{k',1} + \eta_1} T_2^{m_{k',2} + \eta_2} ... T_n^{m_{k',n} + \eta_n} U) \cap (T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} \widetilde{V}_1)$$

then we have

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} (U \cap (V_1) \neq \phi$$

and similarly

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} (U \cap (V_2) \neq \phi$$

now, this is the end of proof.

References


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