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Lagrange's Identity Obtained from Product Identity

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Abstract. We present an identity of products that reduces to Lagrange's identity when a series expansion to fourth order terms are considered. Sixth and higher order terms produce other series identities.

1. Introduction

Normed division algebras require that the norm of the product is equal to the product of the norms. Lagrange's identity exhibits this equality. Due to Hurwitz theorem, it admits this interpretation only for algebras isomorphic to the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{R}^2$), quaternions ($\mathbb{R}^4$) and octonions ($\mathbb{R}^8$). If divisors of zero are allowed, many other algebraic structures in $\mathbb{R}^n$ are possible. Two such possibilities for hyperbolic numbers has been introduced by Fjelstad and Gal [1] and more recently by Catoni et al. [2]. Another approach has been presented in the context of a deformed Lorentz metric. This latter proposal is based on a transformation stemming from the product operation and magnitude definition in hyperbolic scator algebra [3]. The product identity used as a starting point here, is a consequence of the $\|\mathbf{ab}\| = \|\mathbf{a}\| \, \|\mathbf{b}\|$ equality for scator algebras.

Lagrange's identity can be proved in a variety of ways [4]. Most derivations use the identity as a starting point and prove in one way or another that the equality is true. In the present approach, Lagrange's identity is actually derived without assuming it *a priori*. The pseudo-norm of the product identity used in the derivation has the strength to imply an infinite number of identities. An example when sixth order terms are retained is shown here. The ease of the derivation has induced us to present it for complex numbers.

2. Lagrange's identity for complex numbers

Let $a_i, b_i \in \mathbb{C}$ be complex numbers and the overbar represents complex conjugate.

Proposition 1. *The product identity* $\prod_{i=1}^{n}\left(1-a_i\bar{a}_i-b_i\bar{b}_i+a_i\bar{a}_ib_i\bar{b}_i\right)$ $=\prod_{i=1}^{n}\left(1-a_i\bar{a}_i\right)\prod_{i=1}^{n}\left(1-b_i\bar{b}_i\right)$ *reduces to the complex Lagrange's identity when fourth order terms, in a series expansion, are considered.*

Proof. Expand the product on the LHS of the product identity in terms of series[1] up to fourth order

$$\begin{aligned}(2.1)\quad \prod_{i=1}^{n}\left(1-a_i\bar{a}_i-b_i\bar{b}_i+a_i\bar{a}_ib_i\bar{b}_i\right) &= 1-\sum_{i=1}^{n}\left(a_i\bar{a}_i+b_i\bar{b}_i\right)+\sum_{i=1}^{n}a_i\bar{a}_ib_i\bar{b}_i\\ &+\sum_{i<j}^{n}\left(a_i\bar{a}_ia_j\bar{a}_j+b_i\bar{b}_ib_j\bar{b}_j\right)+\sum_{i<j}^{n}\left(a_i\bar{a}_ib_j\bar{b}_j+a_j\bar{a}_jb_i\bar{b}_i\right)+\mathcal{O}^{5+}.\end{aligned}$$

The two factors on the RHS are also written in terms of series

$$\begin{aligned}\prod_{i=1}^{n}\left(1-a_i\bar{a}_i\right)\prod_{i=1}^{n}\left(1-b_i\bar{b}_i\right) &= \left(1-\sum_{i=1}^{n}a_i\bar{a}_i+\sum_{i<j}^{n}a_i\bar{a}_ia_j\bar{a}_j+\mathcal{O}^{5+}\right)\\ &\left(1-\sum_{i=1}^{n}b_i\bar{b}_i+\sum_{i<j}^{n}b_i\bar{b}_ib_j\bar{b}_j+\mathcal{O}^{5+}\right).\end{aligned}$$

The product of this expression up to fourth order is

$$\begin{aligned}(2.2)\quad \prod_{i=1}^{n}\left(1-a_i\bar{a}_i\right)\prod_{i=1}^{n}\left(1-b_i\bar{b}_i\right) &= 1-\sum_{i=1}^{n}\left(a_i\bar{a}_i+b_i\bar{b}_i\right)\\ &+\left(\sum_{i=1}^{n}a_i\bar{a}_i\right)\left(\sum_{i=1}^{n}b_i\bar{b}_i\right)+\sum_{i<j}^{n}\left(a_i\bar{a}_ia_j\bar{a}_j+b_i\bar{b}_ib_j\bar{b}_j\right)+\mathcal{O}^{5+}.\end{aligned}$$

Substitution of (2.1) and (2.2) in the product identity give

$$\sum_{i=1}^{n}a_i\bar{a}_ib_i\bar{b}_i+\sum_{i<j}^{n}\left(a_i\bar{a}_ib_j\bar{b}_j+a_j\bar{a}_jb_i\bar{b}_i\right)=\left(\sum_{i=1}^{n}a_i\bar{a}_i\right)\left(\sum_{i=1}^{n}b_i\bar{b}_i\right).$$

[1] Recall that products of the form $(1+x_i)$ can be expanded in terms of sums as

$$\prod_{i=1}^{n}\left(1+x_i\right)=1+\sum_{i=1}^{n}x_i+\sum_{i<j}^{n}x_ix_j+\mathcal{O}^{3+}(x),$$

where $\mathcal{O}^{3+}(x)$ means terms with order three or higher in x.

The product of two conjugates series can be expressed as series involving the product of conjugate terms[2], thus

$$\left(\sum_{i=1}^{n} a_i b_i\right)\left(\sum_{i=1}^{n} \overline{a_i b_i}\right) - \sum_{i<j}^{n}\left(a_i b_i \bar{a}_j \bar{b}_j + \bar{a}_i \bar{b}_i a_j b_j\right) + \sum_{i<j}^{n}\left(a_i \bar{a}_i b_j \bar{b}_j + a_j \bar{a}_j b_i \bar{b}_i\right) = \left(\sum_{i=1}^{n} a_i \bar{a}_i\right)\left(\sum_{i=1}^{n} b_i \bar{b}_i\right).$$

The terms of the last two series on the LHS are grouped as

$$a_i \bar{a}_i b_j \bar{b}_j + a_j \bar{a}_j b_i \bar{b}_i - a_i b_i \bar{a}_j \bar{b}_j - \bar{a}_i \bar{b}_i a_j b_j = \left(a_i \bar{b}_j - a_j \bar{b}_i\right)\left(\bar{a}_i b_j - \bar{a}_j b_i\right),$$

in order to obtain the complex Lagrange's identity

$$\left(\sum_{i=1}^{n} a_i b_i\right)\left(\sum_{i=1}^{n} \overline{a_i b_i}\right) + \sum_{i<j}^{n}\left(a_i \bar{b}_j - a_j \bar{b}_i\right)\left(\overline{a_i \bar{b}_j - a_j \bar{b}_i}\right) = \left(\sum_{i=1}^{n} a_i \bar{a}_i\right)\left(\sum_{i=1}^{n} b_i \bar{b}_i\right).$$

In terms of the modulii,

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 + \sum_{i<j}^{n}\left|a_i \bar{b}_j - a_j \bar{b}_i\right|^2 = \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right).$$

□

3. OTHER IDENTITIES

Proposition 2. *The non trivial identities for real numbers obtained to sixth order series expansion of the product identity* $\prod_{i=1}^{n}\left(1 - a_i^2 - b_i^2 + a_i^2 b_i^2\right) = \prod_{i=1}^{n}\left(1 - a_i^2\right)\prod_{i=1}^{n}\left(1 - b_i^2\right)$ *are*

$$\sum_{i<j}^{n}\left[a_i^2 a_j^2\left(b_i^2 + b_j^2\right)\right] + \sum_{i<j<k}^{n}\left[a_i^2 a_j^2 b_k^2 + a_i^2 b_j^2 a_k^2 + b_i^2 a_j^2 a_k^2\right] = \left(\sum_{i=1}^{n} b_i^2\right)\left(\sum_{i<j}^{n} a_i^2 a_j^2\right)$$

and its counterpart, obtained by interchanging the variables a and b .

Proof. Expand the product identity in series up to sixth order. The LHS is

$$\prod_{i=1}^{n}\left(1 - a_i^2 - b_i^2 + a_i^2 b_i^2\right) = 1 + \sum_{i=1}^{n}\left(-a_i^2 - b_i^2 + a_i^2 b_i^2\right) + \sum_{i<j}^{n}\left(-a_i^2 - b_i^2 + a_i^2 b_i^2\right)\left(-a_j^2 - b_j^2 + a_j^2 b_j^2\right) + \sum_{i<j<k}^{n}\left(-a_i^2 - b_i^2 + a_i^2 b_i^2\right)\left(-a_j^2 - b_j^2 + a_j^2 b_j^2\right)\left(-a_k^2 - b_k^2 + a_k^2 b_k^2\right) + \mathcal{O}^{7+}.$$

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[2]The conjugate series product is $\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \bar{x}_i\right) = \sum_{i=1}^{n} x_i \bar{x}_i + \sum_{i<j}^{n}\left(x_i \bar{x}_j + \bar{x}_i x_j\right)$.

Consider only the sixth order terms

$$\mathcal{O}^6\,(\mathrm{LHS}) = -\sum_{i<j}^{n}\left[a_i^2 a_j^2\left(b_i^2+b_j^2\right)+b_i^2 b_j^2\left(a_i^2+a_j^2\right)\right] - \sum_{i<j<k}^{n} a_i^2 a_j^2 a_k^2 + b_i^2 b_j^2 b_k^2$$
$$-\sum_{i<j<k}^{n}\left(a_i^2 a_j^2 b_k^2 + a_i^2 b_j^2 a_k^2 + b_i^2 a_j^2 a_k^2\right) - \sum_{i<j<k}^{n}\left(a_i^2 b_j^2 b_k^2 + b_i^2 a_j^2 b_k^2 + b_i^2 b_j^2 a_k^2\right)$$

The RHS of the product identity is similarly expanded in series up to sixth order

$$\prod_{i=1}^{n}\left(1-a_i^2\right)\prod_{i=1}^{n}\left(1-b_i^2\right) = \left(1-\sum_{i=1}^{n}a_i^2+\sum_{i<j}^{n}a_i^2a_j^2-\sum_{i<j<k}^{n}a_i^2a_j^2a_k^2+\mathcal{O}^{7+}\right)$$
$$\left(1-\sum_{i=1}^{n}b_i^2+\sum_{i<j}^{n}b_i^2b_j^2-\sum_{i<j<k}^{n}b_i^2b_j^2b_k^2+\mathcal{O}^{7+}\right),$$

and only sixth order terms retained

$$\mathcal{O}^6\,(\mathrm{RHS}) = -\sum_{i<j<k}^{n}a_i^2a_j^2a_k^2 - \sum_{i<j<k}^{n}b_i^2b_j^2b_k^2 - \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i<j}^{n}b_i^2b_j^2\right) - \left(\sum_{i=1}^{n}b_i^2\right)\left(\sum_{i<j}^{n}a_i^2a_j^2\right).$$

These two results are equated for equal powers of $a^n b^m$. The terms a^6 and b^6 give trivial identities whereas the terms involving a^4b^2 and a^2b^4 give the non trivial sixth order identities

$$\sum_{i<j}^{n}a_i^2a_j^2\left(b_i^2+b_j^2\right) + \textstyle\sum_{i<j<k}^{n}\left(a_i^2a_j^2b_k^2+a_i^2b_j^2a_k^2+b_i^2a_j^2a_k^2\right) = \left(\sum_{i=1}^{n}b_i^2\right)\left(\sum_{i<j}^{n}a_i^2a_j^2\right)$$
$$\sum_{i<j}^{n}b_i^2b_j^2\left(a_i^2+a_j^2\right) + \textstyle\sum_{i<j<k}^{n}\left(a_i^2b_j^2b_k^2+b_i^2a_j^2b_k^2+b_i^2b_j^2a_k^2\right) = \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i<j}^{n}b_i^2b_j^2\right).$$

□

4. CONCLUSIONS

Lagrange's identity for complex numbers has been obtained from a straightforward product identity. The procedure is elementary and very economical. A derivation for the reals is obviously even more succinct. In a wider context, this product identity can be seen as a consequence of the $\|\mathbf{ab}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|$ relationship for scator algebras. Since the Cauchy-Schwarz inequality is a particular case of Lagrange's identity [4], this proof is yet another way to obtain the CS inequality.

References

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