Multipliers for Bounded Statistical Convergence of Double Sequences

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Abstract. Multipliers and factorizations for bounded statistically convergent sequences were studied in $\mu$-density by Connor et al. [J. Connor, K. Demirci, C. Orhan, Multipliers and factorizations for bounded statistically convergent sequences, Analysis 22 (2002), 321-333]. In this paper we get analogous results of multipliers for bounded statistically convergent double sequences.

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1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [22]. A lot of development have been made in this area after the works of Šalát [21] and Fridy [13, 14]. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [11, 13, 14, 19]. This concept was extended to the double sequences by Mursaleen and Edely [17], Akın and Altay [6] presented multidimensional analogues of the results of Fridy and Orhan [15]. Das and Bhunia [7] extended the idea of statistical convergence of a double sequence to $\mu$-statistical convergence and convergence in $\mu$-density using a two valued measure $\mu$.

The study of the multipliers of one sequence space into another is a well-established area of research and has been the object of several investigations over the last fifty years. Demirci and Orhan [9] studied the bounded multiplier space of all bounded $A$-statistically convergent sequences, and using the

In this paper we study multipliers for bounded statistical convergence of double sequences in μ-density.

2. Definitions and Notations

Throughout the paper N denotes the set of all positive integers, χA—the characteristic function of A ⊂ N, R the set of all real numbers. We often regard χA as sequence (xmn), where xmn = χA(m, n), A ⊂ N × N; note in particular, that e can be regarded as the double sequence of all 1’s.

Now, we recall the concepts of convergence, statistical and ideal convergence of the sequences (See [2, 7, 10, 11, 17, 20]).

A double sequence x = (xmn)mn∈N of real numbers is said to be bounded if there exists a positive real number M such that |xmn| < M, for all m, n ∈ N. That is

\[ \|x\|_\infty = \sup_{m,n} |xmn| < \infty. \]

A double sequence x = (xmn)mn∈N of real numbers is said to be (Pringsheim) convergent to L ∈ R, if for any ε > 0 there exists Nε ∈ N such that

|xmn − L| < ε,

whenever m, n > Nε. In this case we write

\[ P - \lim_{m,n \to \infty} xmn = L \text{ or } \lim_{m,n \to \infty} xmn = L. \]

By ℓ2∞, c2(b) and c20(b) we denote the space of all bounded, bounded convergent and bounded null double sequences, respectively.

Let K ⊂ N × N. Let Kmn be the number of (j, k) ∈ K such that j ≤ m, k ≤ n. If the sequence \{Kmn\} has a limit in Pringsheim’s sense then we say that K has double natural density and is denoted by

\[ d_2(Kmn) = \lim_{m,n \to \infty} \frac{Kmn}{m.n}. \]

A double sequence x = (xmn)mn∈N of real numbers is said to be statistically convergent to L ∈ R if for any ε > 0 we have d2(A(ε)) = 0, where A(ε) = {(m, n) ∈ N × N : |xmn − L| ≥ ε}. In this case we write

\[ st_2 - \lim_{m,n \to \infty} xmn = L. \]

Throughout the paper μ2 will denote a complete \{0, 1\} valued finite additive measure defined on an algebra Γ of subsets of N × N that contains all subsets of N × N that are contained in the union of a finite number of rows and columns.
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of \( \mathbb{N} \times \mathbb{N} \) and \( \mu_2(A) = 0 \) if \( A \) is contained in the union of a finite number of rows and columns of \( \mathbb{N} \times \mathbb{N} \).

A double sequence \( x = (x_{mn})_{m,n \in \mathbb{N}} \) of real numbers is said to be \( \mu_2 \)-statistically convergent to \( L \in \mathbb{R} \) if and only if for any \( \varepsilon > 0 \),

\[
\mu_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}) = 0.
\]

In this case we write

\[
\text{st}_{\mu_2} \lim_{m,n \to \infty} x_{mn} = L.
\]

A double sequence \( x = (x_{mn})_{m,n \in \mathbb{N}} \) of real numbers is said to be convergent to \( L \in \mathbb{R} \) in \( \mu_2 \)-density, if there exists an \( A \in \Gamma \) with \( \mu_2(A) = 1 \) such that \( x = (x_{mn})_{(m,n) \in A} \) is convergent to \( L \).

If \( C_{\mu_2} \) and \( C_{\mu_2}^* \) denote respectively the sets of all double sequences which are \( \mu_2 \)-statistically convergent and convergent in \( \mu_2 \)-density then as in [5] (see also [8]) it is easy to prove that \( C_{\mu_2}^* \) is a dense subset of \( C_{\mu_2} \) which again is closed in \( \ell_2^\infty \) (the set of all bounded double sequences of real numbers endowed with the sup metric). Further, following the methods of [5] one can easily verify that there exists a measure \( \mu_2 \) such that it is always possible to construct a double sequence \( x = (x_{mn})_{m,n \in \mathbb{N}} \) which is \( \mu_2 \)-statistically convergent but does not converge to any point in \( \mu_2 \)-density.

The measure \( \mu_2 \) is said to satisfy the condition \((APO2)\) if for every sequence \( \{A_i\}_{i \in \mathbb{N}} \) of mutually disjoint \( \mu_2 \)-null sets (i.e. \( \mu_2(A_i) = 0 \) for all \( i \in \mathbb{N} \)) there exists a countable family of sets \( \{B_i\}_{i \in \mathbb{N}} \) such that \( A_i \Delta B_i \) is included in the union of a finite number rows and columns of \( \mathbb{N} \times \mathbb{N} \) for every \( i \in \mathbb{N} \) and \( \mu_2(B_i) = 0 \), where \( B = \bigcup_{i \in \mathbb{N}} B_i \) (hence \( \mu_2(B_i) = 0 \) for every \( i \in \mathbb{N} \)).

Now, we give definition of multiplier for double sequences.

Let \( E \) and \( F \) be two double sequence spaces. A multiplier from \( E \) into \( F \) is a sequence \( u = (u_{mn})_{m,n \in \mathbb{N}} \) such that

\[
ux = (u_{mn}x_{mn}) \in F
\]

whenever \( x = (x_{mn})_{m,n \in \mathbb{N}} \in E \). The linear space of all such multipliers will be denoted by \( m(E,F) \). Bounded multipliers will be denoted by \( M(E,F) \). Hence we write

\[
M(E,F) = \ell_2^\infty \cap m(E,F).
\]

If \( E = F \), then we write \( m(E) \) and \( M(E) \) instead of \( m(E,F) \) and \( M(E,F) \), respectively.

Now we begin with quoting the lemmas due to Dündar, Altay [10] and Das, Bhunia [7] which are needed throughout the paper.

**Lemma 2.1.** [10, Theorem 3.2] If \( E \) and \( F \) are subspaces of \( \ell_2^\infty \) that contain \( c_0^2(b) \), then \( c_0^2(b) \subset m(E,F) \subset \ell_2^\infty \).

**Lemma 2.2.** [10, Lemma 3.4] \( m(c_0^2(b)) = \ell_2^\infty \).

**Lemma 2.3.** [7, Theorem 1] \( C_{\mu_2} = C_{\mu_2}^* \) if \( \mu_2 \) satisfies the condition \((APO2)\).
Lemma 2.4. [7, Theorem 2] If $C_\mu = C_\mu^*$ for a measure $\mu$, then $\mu$ has the condition (APO2).

3. Main Results

In this section, we deal with the multipliers on or into $st_{\mu_2}(b)$ and $st_{\mu_2}^0(b)$. By $st_{\mu_2}$, $st_{\mu_2}^0$, $st_{\mu_2}(b)$ and $st_{\mu_2}^0(b)$ we denote the sets of all $\mu_2$-statistically convergent double sequences, $\mu_2$-statistically null double sequences, bounded $\mu_2$-statistically convergent double sequences and bounded $\mu_2$-statistically null double sequences, respectively.

Theorem 3.1. Let $\mu_2$ be an arbitrary density. Then,

$$m(st_{\mu_2}^0(b)) = M(st_{\mu_2}^0(b)) = \ell_2^\infty.$$ 

Proof. We show that

$$m(st_{\mu_2}^0(b)) = \ell_2^\infty.$$ 

By Lemma 2.1, the inclusion

$$m(st_{\mu_2}^0(b)) \subseteq \ell_2^\infty$$

holds.

Now, we prove that

$$\ell_2^\infty \subseteq m(st_{\mu_2}^0(b)).$$

Let $u \in \ell_2^\infty$ and $z \in st_{\mu_2}^0(b)$. Then for $\varepsilon > 0$, we have

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |u_{mn}z_{mn}| \geq \varepsilon\} \subseteq \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : |z_{mn}| \geq \frac{\varepsilon}{\|u\|_\infty + 1} \right\}.$$ 

Since

$$\mu_2\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : |z_{mn}| \geq \frac{\varepsilon}{\|u\|_\infty + 1} \right\} = 0,$$

so we can write

$$\mu_2\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : |u_{mn}z_{mn}| \geq \varepsilon \right\} = 0.$$ 

Also, since $u, z \in \ell_2^\infty$ so $uz$ is bounded and hence

$$\ell_2^\infty \subseteq m(st_{\mu_2}^0(b)).$$

This completes the proof of theorem. 

Combining Lemma 2.1 and Lemma 2.2, we have the following theorem.

Theorem 3.2. Let $\mu_2$ be an arbitrary density. Then,

$$m(c_0^2(b), st_{\mu_2}^0(b)) = \ell_2^\infty.$$ 

Theorem 3.3. Let $\mu_2$ be an arbitrary density. Then,

$$c_0^2(b) \subseteq m(st_{\mu_2}(b), c^2(b)) \subseteq c^2(b).$$
Proof. For $u \in c_0^2(b)$ and $x \in \text{st}_{\mu_2}(b) \subset \ell_\infty^2$, by Lemma 2.2 since $ux \in c_0^2(b) \subset c^2(b)$, so we have $c_0^2(b) \subset m\left(st_{\mu_2}(b), c^2(b)\right)$. Let $u \in m\left(st_{\mu_2}(b), c^2(b)\right)$. Since $e \in st_{\mu_2}(b)$, so $ue = u \in c^2(b)$ and we have $m\left(st_{\mu_2}(b), c^2(b)\right) \subseteq c^2(b)$.

Thus, the proof of theorem is completed. \hfill \Box

Theorem 3.4. Let $\mu_2$ be an arbitrary density. If $c^2(b)$ is a proper subset of $st_{\mu_2}(b)$, then

$$m\left(st_{\mu_2}(b), c^2(b)\right) = c_0^2(b)$$

Proof. By Theorem 3.3, we know that $c_0^2(b) \subset m\left(st_{\mu_2}(b), c^2(b)\right)$. We show that $u \notin m\left(st_{\mu_2}(b), c^2(b)\right)$ for $u \in c^2(b) \setminus c_0^2(b)$. Then, there exists a number $l$ such that

$$\lim_{m,n \to \infty} u_{mn} = l \neq 0.$$ 

Let $z \in st_{\mu_2}(b) \setminus c^2(b)$, and, without loss of generality, suppose $z$ is $\mu_2$-statistically convergent to 1. Then, there is an $\varepsilon > 0$ such that

$$A = \{(m,n) : |z_{mn} - 1| \geq \varepsilon\}.$$ 

Note that $\mu_2(A) = 0$.

Define $x = (x_{mn})$ by

$$x_{mn} = \chi_{A^c}(m,n)$$ 

and observe that $x$ is bounded and convergent in $\mu_2$-density to 1, hence $x \in st_{\mu_2}(b)$. Also note $ux$ converges to $\ell \neq 0$ along $A^c$ and to 0 along $A$, hence $ux \notin c^2(b)$ and thus $u \notin m\left(st_{\mu_2}(b), c^2(b)\right)$. Hence, we have

$$m\left(st_{\mu_2}(b), c^2(b)\right) \subset c_0^2(b).$$ 

\hfill \Box

Theorem 3.5. Let $\mu_2$ be a density with condition (APO2). Then

$$m\left(st_{\mu_2}^0(b), c_0^2(b)\right) = \{u \in \ell_\infty^2 : u\chi_E \in c_0^2(b) \text{ for all } E \text{ such that } \mu_2(E) = 0\}.$$
Proof. Let \( K = \{ u \in \ell^2_\infty : u\chi_E \in c_0^2(b) \text{ for all } E \text{ such that } \mu_2(E) = 0 \} \). First note that if \( \mu_2(E) = 0 \), then
\[
\chi_E \in st^0_{\mu_2}(b)
\]
and hence, if \( u \in m(st^0_{\mu_2}(b), c_0^2(b)) \) then
\[
u \chi_E \in c_0^2(b)
\]
or \( u \) goes to 0 along \( E \). Thus, we have
\[
m(st^0_{\mu_2}(b), c_0^2(b)) \subseteq K.
\]
Now, let \( u \in K \) and \( x \in st^0_{\mu_2}(b) \). Then, with condition \((APO2)\) by Lemma 2.3 there is an \( A \subseteq \mathbb{N} \times \mathbb{N} \) such that
\[
x\chi_A^c \in c_0^2(b) \text{ and } \mu_2(A) = 0.
\]
With condition \((APO2)\) by Lemma 2.3, as
\[
xu = xu\chi_A^c + xu\chi_A
\]
and both terms of the right hand side are null sequences, \( xu \in c_0^2(b) \). Thus we have
\[
K \subseteq m(st^0_{\mu_2}(b), c_0^2(b)).
\]
This completes the proof of theorem.

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